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HIGHER MATHEMATICS

FOR STUDENTS OF ENGINEERING AND SCIENCE

BY

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PREFACE

An experience of more than twenty years in teaching Mathematics, the greater part of which has been devoted mainly to Senior Engineering Students, has shown that there is need for a single volume dealing with those departments of Pure Mathematics which such students require.

This book has therefore been written to supply this want, and following what the author considers to be the most effective method of teaching, much of the necessary bookwork is given and proved in the form of illustrative worked examples, of which there are 190. Most of the fundamental formulae are thus established, and are numbered for reference.

Great care has been taken to incorporate a large number of actual practical calculations, and many standard works both on Engineering and Mathematics have been consulted. In some cases the branch of Applied Science with which an exercise deals is indicated in the data, but this plan has not been followed throughout, since there is a danger of the student selecting only those exercises which he thinks deal with the professional matters in which he is interested, and thus he misses the knowledge of fundamental principles which make mathematics a real working tool.

The branches of Pure Mathematics required for an Engineering degree are here dealt with in a single volume; but in many cases, especially in the chapters on the Calculus and Analytical Geometry, a much fuller treatment has been given than the majority of such syllabuses demand. The book should therefore be of service to all students reading mathematics for a degree, whether in Arts,

Science or Engineering, and many examples from University papers in these faculties have been included.

As one of the chief aims of the book is to provide a large number of exercises for practice, no attempt has been made to treat the bookwork with the utmost rigour required in modern mathematical theory; many theorems have, indeed, been merely stated and then applied. A list of books for reference has, however, been given, where complete and rigorous proofs may be found.

The general development throughout is that which experience in teaching has shown to be the most effective, and for this reason some divergence from the usual lines of treatment will be found in certain sections. Especially is this the case in the chapters on Analytical Geometry.

A large number of exercises—1337 in all—are original, whilst the others have been selected from the examination papers of the Universities of Birmingham, Bristol, Durham, Leeds, Liverpool, London, Manchester and Sheffield. For permission to reproduce these, thanks are hereby gladly accorded to the authorities of those Universities, and also to the Controller of H.M. Stationery Office for the use of certain questions taken from papers of the Board of Education.

The following abbreviations have been used to indicate the University questions taken from papers set in the Faculty of Engineering:

B.U.	University of	Birmingham.
Br.U.	,,	Bristol.
D.U.	,,	Durham.
Le.U.	,,	Leeds.
Li.U.	,,	Liverpool.
L.U.	,,	London.
M.U.	,,	Manchester.
S.U.	,,	Sheffield.

In cases where questions have been taken from papers set in other faculties, "Sc." is added to the appropriate abbreviation

given above. Questions which may be omitted on a first reading are marked with an asterisk.

To Sir Richard Gregory and Mr. John Duncan, Head of the Engineering Department of the West Ham Municipal College, the author owes a very great debt of gratitude, for it is mainly due to their valuable advice and untiring co-operation that the author's lecture notes have developed into the present form.

Thanks are also due to Mr. E. H. Madden, B.A., B.Sc., Mr. F. Sandon, M.A., Mr. W. H. Salmon, M.A., B.Sc., and Mr. A. J. V. Gale, M.A., for help in the tedious work of revising proof-sheets and for useful criticism; to Mr. E. A. Branch for drawing many of the figures; and to the printers for the excellence of their share of the work.

Although the answers have been well checked, it is scarcely to be expected that every error has been removed; the author will therefore be glad to hear of any cases of inaccuracy that may have inadvertently crept into any part of the book.

F. G. W. BROWN.

GOODMAYES,
September, 1925.

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REFERENCES

For fuller treatment of the subjects dealt with in the following pages, reference may be made to the following works which are published by Messrs. Macmillan & Co., Ltd., unless otherwise stated.

ALGEBRA.	A Treatise on Algebra, by C. Smith.
CALCULUS.	A Treatise on the Calculus, by G. A. Gibson. Differential Calculus, by Joseph Edwards. Integral Calculus (2 vols.), by Joseph Edwards. Introduction to the Theory of Fourier Series and Integrals, by Prof. H. S. Carslaw.
DYNAMICS.	Dynamics, by J. Duncan and S. G. Starling. Applied Mechanics for Engineers, by J. Duncan. Theoretical Mechanics for Students of Engineering, by Alexander Ziwet.
GEOMETRY.	A Treatise on Conic Sections, by C. Smith. A Treatise on Solid Geometry, by C. Smith. Elements of Coordinate Geometry, by S. L. Loney.
TRIGONOMETRY.	Plane Trigonometry, by Prof. H. S. Carslaw. A Treatise on Plane Trigonometry, by Prof. E. W. Hobson. (Camb. Univ. Press.) Spherical Trigonometry, by W. W. Lane. Treatise on Spherical Trigonometry, by W. J. M'Clelland and T. Preston.
GENERAL.	Practical Mathematical Analysis, by H. von Sanden, translated by Prof. Levy. (Methuen & Co., Ltd.) Logarithms and Other Tables for Schools, by Frank Castle. Five-Figure Logarithmic and other Tables, by Frank Castle. The Calculus of Observations, by E. T. Whittaker and G. Robinson. (Blackie & Sons.)

CHAPTER I

BINOMIAL THEOREM. PARTIAL FRACTIONS. SIMPLE SERIES

1. The Binomial Theorem. This important theorem may be stated generally as follows :

$$\left.\begin{aligned}(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 \\ + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}x^4 + \dots, \\ \text{the } (r+1)\text{th term being } \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}}\,.\,x^r.\end{aligned}\right\} \dots (1)$$

This is true for all values of n, integral or fractional, positive or negative, provided that $-1 < x < 1$, when n is not a positive integer. It should be carefully observed that when n is a positive integer the series terminates with the $(n+1)$th term, but when n is not a positive integer, the series consists of an infinite number of terms, hence the restriction on the value of x, as will be seen from § 4. A proof of the theorem by induction is given in Ex. 1, for the case when n is a positive integer.

Ex. 1. *Prove the binomial theorem when n is a positive integer; hence, expand* $(2+x)^7$ *and* $(1-x+x^2)^4$.

Shew also that the term independent of x in the expansion of $\left(x - \frac{4}{x^2}\right)^6$ *is* $2\underline{|5}$.

(i) Proof of binomial theorem when n is a positive integer.

In (1) put $n=1, 2, 3$ respectively, then

$$\begin{aligned}(1+x) &= 1 + x,\\ (1+x)^2 &= 1 + 2x + x^2,\\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3,\end{aligned}$$

which are known to be true by direct multiplication.

Multiply (1) throughout by $(1+x)$, then

$$(1+x)^{n+1} = (1+x)^n(1+x) = 1 + (n+1)x + \left\{\frac{n(n-1)}{\underline{|2}} + n\right\} x^2$$
$$+ \left\{\frac{n(n-1)(n-2)}{\underline{|3}} + \frac{n(n-1)}{\underline{|2}}\right\} x^3 + \dots$$
$$= 1 + (n+1)x + \frac{(n+1)n}{\underline{|2}} x^2 + \frac{(n+1)n(n-1)}{\underline{|3}} \cdot x^3 + \dots .$$

This expression so far is precisely the same as the right-hand side of (1), except that $n+1$ replaces n. To test the general term, it should be observed that the coefficient of x^r in the product of $(1+x)^n$ and $(1+x)$, *i.e.* in $(1+x)^{n+1}$, is equal to the sum of the coefficients of x^{r-1} and x^r in $(1+x)^n$; this is

$$\frac{n(n-1)(n-2)\dots(n-r+2)}{\underline{|r-1}} + \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}}$$
$$= \frac{(n+1)n(n-1)\dots(n-r+2)}{\underline{|r}},$$

which is of exactly the same form as the $(r+1)$th term in $(1+x)^n$, one being derivable from the other by writing $n+1$ in place of n.

Hence, if the theorem is true for the index n, it is true for $n+1$; and as it is known to be true in the first three cases, it must be true for all integral values of n.

(ii) To put $(2+x)^7$ in standard form, take out the factor 2^7; then

$$(2+x)^7 = 2^7\left(1+\frac{x}{2}\right)^7$$
$$= 2^7\left\{1 + 7\cdot\frac{x}{2} + \frac{7\cdot6}{1\cdot2}\cdot\left(\frac{x}{2}\right)^2 + \frac{7\cdot6\cdot5}{1\cdot2\cdot3}\cdot\left(\frac{x}{2}\right)^3 + \frac{7\cdot6\cdot5\cdot4}{1\cdot2\cdot3\cdot4}\cdot\left(\frac{x}{2}\right)^4\right.$$
$$+ \frac{7\cdot6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4\cdot5}\left(\frac{x}{2}\right)^5 + \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4\cdot5\cdot6}\left(\frac{x}{2}\right)^6$$
$$\left. + \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\left(\frac{x}{2}\right)^7\right\}$$
$$= 2^7\{1 + \tfrac{7}{2}x + \tfrac{21}{4}x^2 + \tfrac{35}{8}x^3 + \tfrac{35}{16}x^4 + \tfrac{21}{32}x^5 + \tfrac{7}{64}x^6 + \tfrac{1}{128}x^7\}$$
$$= \mathbf{128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7.}$$

(iii) To put $(1-x+x^2)^4$ in standard form, write u for $x-x^2$, then

$$(1-x+x^2)^4=(1-u)^4=1+4(-u)+\frac{4\,.\,3}{1\,.\,2}(-u)^2+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}(-u)^3$$
$$+\frac{4\,.\,3\,.\,2\,.\,1}{1\,.\,2\,.\,3\,.\,4}(-u)^4$$
$$=1-4u+6u^2-4u^3+u^4$$
$$=1-4(x-x^2)+6(x-x^2)^2-4(x-x^2)^3+(x-x^2)^4$$
$$=1-4(x-x^2)+6(x^2-2x^3+x^4)-4(x^3-3x^4+3x^5-x^6)$$
$$+x^4-4x^5+6x^6-4x^7+x^8$$
$$=\mathbf{1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8}.$$

(iv) Putting the expression in standard form,

$$\left(x-\frac{4}{x^2}\right)^6=x^6\left(1-\frac{4}{x^3}\right)^6;$$

hence the term independent of x is that involving $1/x^6$ in the expansion of the binomial; this is obviously the third, so that

$$\text{3rd term in expansion of }\left(1-\frac{4}{x^3}\right)^6=\frac{6\,.\,5}{1\,.\,2}\left(-\frac{4}{x^3}\right)^2=2\,.\,\underline{|5}\,/x^6,$$

$$\therefore \quad ,, \quad ,, \quad ,, \quad x^6\left(1-\frac{4}{x^3}\right)^6=2\,\underline{|5}.$$

2. Fractional and Negative Indices. A proof of the binomial theorem when n is not a positive integer is beyond the scope of this book. It will therefore be assumed that the theorem is true for all values of n.

Ex. 2. *Expand $(1+x)^{-n}$ as far as x^4, and shew that when x is small $(1+x)^{-n}$ may be taken as nearly equal to $1-nx$. What is the percentage error in this when $n=2$ and $x=0{\cdot}01$?*

From (1),

$$(1+x)^{-n}=1-nx+\frac{-n(-n-1)}{\underline{|2}}x^2+\frac{-n(-n-1)(-n-2)}{\underline{|3}}x^3$$
$$+\frac{-n(-n-1)(-n-2)(-n-3)}{\underline{|4}}x^4-\dots$$
$$=\mathbf{1-nx+\frac{n(n+1)}{\underline{|2}}x^2-\frac{n(n+1)(n+2)}{\underline{|3}}x^3}$$
$$\mathbf{+\frac{n(n+1)(n+2)(n+3)}{\underline{|4}}x^4-\dots}.$$

When x is very small compared with unity, x^2 and higher powers of x will be much smaller, hence approximately

$$(1+x)^{-n} = 1 - nx.$$

When $x = 0{\cdot}01$, $n = 2$; $(1+x)^{-n} = (1{\cdot}01)^{-2} = 0{\cdot}9804$, and

$$1 - nx = 1 - 0{\cdot}02 = 0{\cdot}98,$$

$$\therefore \text{ percentage error} = +0{\cdot}0004 \times 100 \div 0{\cdot}9804 = 0{\cdot}0408.$$

3. Approximate Expansion. In practical calculation, the following approximations are usually sufficient, when x and y are very small compared with unity:

$$\left.\begin{aligned} (1 \pm x)^n &= 1 \pm nx, \quad n \text{ positive or negative,} \\ (1 \pm x)(1 \pm y) &= 1 \pm x \pm y. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Ex. 3. *If x be so small that its cube and higher powers may be neglected, prove that*

$$\frac{(1-x)^{-\frac{5}{2}} + (16+8x)^{\frac{1}{2}}}{(1+x)^{-\frac{1}{2}} + (2+x)^2} = 1 + \tfrac{23}{40}x^2. \qquad \text{(L.U.)}$$

Taking each binomial in turn, and expanding as far as x^2

$$(1-x)^{-\frac{5}{2}} = 1 + (-\tfrac{5}{2})(-x) + \frac{(-\frac{5}{2})(-\frac{5}{2}-1)}{\lfloor 2} \cdot (-x)^2 + \ldots$$

$$= 1 + \tfrac{5}{2}x + \tfrac{35}{8}x^2 + \ldots,$$

$$(16+8x)^{\frac{1}{2}} = 16^{\frac{1}{2}}\left(1 + \frac{x}{2}\right)^{\frac{1}{2}} = 4(1 + \tfrac{1}{4}x - \tfrac{1}{32}x^2 + \ldots),$$

$$(1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \ldots,$$

$$(2+x)^2 = 4 + 4x + x^2.$$

Hence, denoting the fraction by F,

$$F = \frac{1 + \frac{5}{2}x + \frac{35}{8}x^2 + \ldots + 4 + x - \frac{1}{8}x^2 + \ldots}{1 - \frac{1}{2}x + \frac{3}{8}x^2 - \ldots + 4 + 4x + x^2} = \frac{5 + \frac{7}{2}x + \frac{17}{4}x^2 + \ldots}{5 + \frac{7}{2}x + \frac{11}{8}x^2 + \ldots}.$$

The desired result may now be obtained by division, but it is better to make another application of the binomial theorem, as will be seen in Chap. VII; thus, taking 5 out of the denominator,

$$F = \tfrac{1}{5}(5 + \tfrac{7}{2}x + \tfrac{17}{4}x^2 + \ldots)(1 + \tfrac{7}{10}x + \tfrac{11}{40}x^2 + \ldots)^{-1}$$

$$= \tfrac{1}{5}(5 + \tfrac{7}{2}x + \tfrac{17}{4}x^2 + \ldots)\{1 - (\tfrac{7}{10}x + \tfrac{11}{40}x^2) + (\tfrac{7}{10}x + \tfrac{11}{40}x^2)^2 + \ldots\}$$

$$= \tfrac{1}{5}(5 + \tfrac{7}{2}x + \tfrac{17}{4}x^2 + \ldots)(1 - (\tfrac{7}{10}x + \tfrac{43}{200}x^2) + \ldots)$$

$$= \tfrac{1}{5}(5 + \tfrac{115}{40}x^2 + \ldots) = 1 + \tfrac{23}{40}x^2 + \ldots .$$

Ex. 4. *Prove that the series*

$$1+\tfrac{21}{8}+\tfrac{21}{8}\cdot\tfrac{27}{16}+\tfrac{21}{8}\cdot\tfrac{27}{16}\cdot\tfrac{33}{24}+\tfrac{21}{8}\cdot\tfrac{27}{16}\cdot\tfrac{33}{24}\cdot\tfrac{39}{32}+\dots$$

is a binomial series, and find its value. (L.U., Sc.)

Suppose the first three terms to be the expansion of $(1+x)^n$,

then $$nx=\tfrac{21}{8}, \text{ and } \frac{n(n-1)}{\lfloor 2}\cdot x^2=\tfrac{21}{8}\cdot\tfrac{27}{16}.$$

Substituting the first in the second, then $(\tfrac{21}{8})^2-\tfrac{21}{8}x=\tfrac{21}{4}\cdot\tfrac{27}{16}$, from which $x=-\tfrac{3}{4}$; inserting this value in the first equation,

$$n=-\tfrac{7}{2}.$$

Now $$(1-\tfrac{3}{4})^{-\frac{7}{2}}=1+(-\tfrac{7}{2})(-\tfrac{3}{4})+\frac{-\frac{7}{2}(-\frac{7}{2}-1)}{\lfloor 2}(-\tfrac{3}{4})^2$$
$$+\frac{-\frac{7}{2}(-\frac{7}{2}-1)(-\frac{7}{2}-2)}{\lfloor 3}(-\tfrac{3}{4})^3$$
$$+\frac{-\frac{7}{2}(-\frac{7}{2}-1)(-\frac{7}{2}-2)(-\frac{7}{2}-3)}{\lfloor 4}(-\tfrac{3}{4})^4+\dots$$
$$=1+\tfrac{21}{8}+\tfrac{21}{8}\cdot\tfrac{27}{16}+\tfrac{21}{8}\cdot\tfrac{27}{16}\cdot\tfrac{33}{24}+\tfrac{21}{8}\cdot\tfrac{27}{16}\cdot\tfrac{33}{24}\cdot\tfrac{39}{32}+\dots,$$

which shews that the given series is the expansion of $(1-\tfrac{3}{4})^{-\frac{7}{2}}$, and its value is therefore $(\tfrac{1}{4})^{-\frac{7}{2}}=4^{\frac{7}{2}}=128$.

4. Convergency. By the binomial theorem,

$$1/(1-x)=(1-x)^{-1}=1+x+x^2+x^3+\dots+x^r+\dots.$$

Put $x=5$, then $-0{\cdot}25=1+5+25+125+625+\dots$, which is obviously untrue, since the greater the number of terms taken in the series, the greater their sum will be, so that ultimately, as the number of terms tends to be infinite, the sum will tend to infinity. Such a series is said to be **divergent**, and is useless for practical calculation.

Now let $x=0{\cdot}2$, then
$$1{\cdot}25=1+0{\cdot}2+0{\cdot}04+0{\cdot}008+0{\cdot}0016+0{\cdot}00032+\dots$$
$$=1{\cdot}24992, \text{ taking six terms only.}$$

By taking more terms it will readily be seen that the sum gradually tends to the value 1·25; such a series is said to be **convergent**,

and for purposes of calculation, it is essential that the series employed be convergent for the required range of numerical values.

An infinite series is thus convergent when the sum of n terms tends to a definite value as n is indefinitely increased.

5. D'Alembert's Ratio Test for Convergence. This is one of the most effective tests for convergency, and may be stated as follows: If $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ be an infinite series, then it will be convergent or divergent according as the $\underset{n\to\infty}{\text{Lt}}(u_{n+1}/u_n)$ is numerically less or greater than unity. Should this limit be unity, some further test is needed.

Ex. 5. *Find the range of values for which each of the following is valid,* i.e. *convergent, and therefore legitimate for purposes of calculation.*

(*a*) *The expansion of* $1/(a - bx)^p$, *where* a, b, p *are positive.*

(*b*) *The series whose* n*th term is* $x^{n-1}/\lfloor n-1$.

Hence, from (*a*), *indicate how* $(P/Q)^n$ *may be calculated when* P *and* Q *are positive and nearly equal, and illustrate the method when* $P = 13{\cdot}64$, $Q = 13{\cdot}75$ and $n = \frac{1}{4}$.

(*a*) $1/(a - bx)^p = \left(1 - \frac{b}{a}x\right)^{-p} a^p$.

Write μ for b/a which is positive, then if u_{n+1}, u_n be the $(n+1)$th and nth terms respectively in the expansion of $(1 - \mu x)^{-p}$,

$$u_{n+1} = \frac{-p(-p-1)(-p-2)\ldots(-p-n+1)}{\lfloor n} \cdot (-\mu x)^n,$$

and
$$u_n = \frac{-p(-p-1)(-p-2)\ldots(-p-n+2)}{\lfloor n-1} \cdot (-\mu x)^{n-1};$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{-p-n+1}{n}(-\mu x) = \left\{\frac{p-1}{n} + 1\right\}\mu x;$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{p-1}{n} + 1\right)\mu x = \mu x.$$

Hence, for convergency, $\mu x < 1$, or $x < \dfrac{1}{\mu}$, *i.e.* $x < \dfrac{a}{b}$.

Similarly for $(a+bx)^{-p}$, $\underset{n\to\infty}{\mathrm{Lt}}\, u_{n+1}/u_n = -\mu x$. The sign, however, may be neglected, since the limit of the ratio must be numerically less than unity; hence the expansion of $(a \pm bx)^{-p}$ will be valid as long as $x < \dfrac{a}{b}$

(*b*) Since $u_n = x^{n-1}/\underline{|n-1}$, $\quad \therefore\ u_{n+1} = x^n/\underline{|n}$;

$$\therefore\ \underset{n\to\infty}{\mathrm{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\mathrm{Lt}} \frac{x}{n} = 0, \text{ for all finite values of } x.$$

$\therefore$ The series is convergent for all finite values of x.

Since P and Q are nearly equal, let $P = Q \pm \mu$, according as $P <$ or $> Q$, then $P/Q = (Q \pm \mu)/Q = 1 \pm \mu/Q$.

Now $\mu < Q$, so that $\mu/Q < 1$, hence the expansion of

$$(P/Q)^n = (1 \pm \mu/Q)^n$$

is valid, and may be employed for the calculation; thus, with the given values,

$$\begin{aligned}(P/Q)^n &= (13{\cdot}64/13{\cdot}75)^{\frac14} = (1-0{\cdot}11/13{\cdot}75)^{\frac14} = (1-0{\cdot}008)^{\frac14}\\ &= 1 - \tfrac{1}{4}(0{\cdot}008) - \tfrac{3}{32}(0{\cdot}008)^2 - \tfrac{7}{128}(0{\cdot}008)^3 - \dots\\ &= 1 - 0{\cdot}002 - 0{\cdot}000006 - 0{\cdot}000000028 - \dots\\ &= 1 - 0{\cdot}002006\ \dots = \mathbf{0{\cdot}997994 \text{ to six figures.}}\end{aligned}$$

6. Partial Fractions. Let $f(x)$ be a rational algebraic fraction which may be written in the form $\phi(x)/\psi(x)$; then, if (i) $\phi(x)$ is of a lower degree in x than $\psi(x)$, and (ii) $\psi(x)$ is resolvable into real factors, $f(x)$ may be expressed as an algebraic sum of simpler fractions, each having as its denominator one of the factors of $\psi(x)$. For a prime factor of $\psi(x)$ of the nth degree in x, the numerator will, in general, be of the $(n-1)$th degree in x.

When $\phi(x)$ is initially of the same or higher degree in x than $\psi(x)$, it must first be reduced by division.

This process is known as splitting a given fraction into its **partial** fractions, and is of great importance in Integration. (See § 38, p. 111.) It is also of use in expanding algebraic fractions in series, as is illustrated in Ex. 6.

Ex. 6. *Resolve into partial fractions :*

$$(a)\ \frac{4x^2-12x+13}{4x^2-16x+15},\quad (b)\ \frac{3(4x^3+x^2+5)}{x(2x^3+1)(x^2+3)}.$$

(*a*) Since the numerator is of the same degree in x as the denominator,

∴ by division,

$$\frac{4x^2-12x+13}{4x^2-16x+15}=1+\frac{4x-2}{4x^2-16x+15}=1+\frac{4x-2}{(2x-5)(2x-3)}.$$

Now suppose

$$\frac{4x-2}{(2x-5)(2x-3)}=\frac{A}{2x-5}+\frac{B}{2x-3},$$

where A and B are constants to be determined, there being no x in the numerators because the denominators are of the first degree in x.

Clearing off fractions :

$$4x-2=A(2x-3)+B(2x-5).$$

This is identically true for all values of x ; to find the constants, it is therefore only necessary to give x any two values in order to obtain two equations in A and B ; thus suppose $x=0$, and $x=1$, then $3A+5B=2$, $A+3B=-2$, giving $A=4$, $B=-2$.

It is shorter, however, to give x those values which make each of the factors of the denominators zero in turn ; thus, if

$$x=1{\cdot}5\,;\quad -2B=4,\quad \text{or}\quad B=-2,\ \text{and if}$$
$$x=2{\cdot}5\,;\quad 2A=8,\quad \text{or}\quad A=4.$$

When the factors are not all linear, it may be very difficult to apply this method, and in such a case A and B must be chosen by making corresponding coefficients on either side of the identity equal ; thus

$$4x-2=2(A+B)x-(3A+5B),$$

so that to fulfil this condition

$$2(A+B)=4\,;\quad 3A+5B=2,$$

from which $A=4$, $B=-2$ as before.

It may, indeed, often happen that some combination of all the above methods may be necessary.

Hence
$$\frac{4x^2-12x+13}{4x^2-16x+15}=1+\frac{4}{2x-5}-\frac{2}{2x-3}.$$

(*b*) In this case, the prime factors of the denominator are x, $2x^3+1$, and x^2+2; hence the corresponding numerators must be of degrees 0, 2, 1 in x.

Let $$\frac{3(4x^3+x^2+5)}{x(2x^3+1)(x^2+3)}=\frac{Ax^2+Bx+C}{2x^3+1}+\frac{Dx+E}{x^2+3}+\frac{F}{x},$$
where A, B, C, D, E, F are constants to be determined.

Clearing off fractions:
$$\begin{aligned}12x^3+3x^2+15&=(Ax^2+Bx+C)(x^2+3)x+(Dx+E)(2x^3+1)x\\&\qquad+F(2x^3+1)(x^2+3)\\&=(A+2D+2F)x^5+(B+2E)x^4+(C+3A+6F)x^3\\&\qquad+(3B+D+F)x^2+(3C+E)x+3F.\end{aligned}$$

Since this is identically true for all values of x,
$$\begin{array}{ll}\therefore\ 3F=15, & \therefore\ F=5.\\ 3C+E=0, & \\ 3B+D+F=3, & \therefore\ 3B=-2-D.\\ C+3A+6F=12, & \therefore\ C=-18-3A.\\ B+2E=0, & \\ A+2D+2F=0, & \therefore\ A=-10-2D.\end{array}$$

The first equation gives the value of F directly; to solve the remaining five equations, insert the value of F and add them; then
$$4(A+B+C)+3(D+E)=-30.$$

Expressing A, B, C, E in terms of D,
$$\therefore\ -40-8D-\tfrac{4}{3}(2+D)+48+24D+3D-108-54D=-30;$$
$$\therefore\ -\tfrac{109}{3}D=\tfrac{218}{3},$$
so that $D=-2$, and $A=-6$, $B=0$, $C=0$, $E=0$.
$$\therefore\ \frac{3(4x^3+x^2+5)}{x(2x^3+1)(x^2+3)}=\frac{5}{x}-\frac{2x}{x^3+3}-\frac{6x^2}{2x^3+1}.$$

Ex 7. *Resolve* $\dfrac{43x+13}{(4x^2+3)(5-3x)}$ *into partial fractions; state the condition that the fractions can be expanded by the binomial theorem, and, assuming this condition is satisfied, find the expansion as far as* x^2.

Let $$\frac{43x+13}{(4x^2+3)(5-3x)}=\frac{Ax+B}{4x^2+3}+\frac{C}{5-3x};$$
$$\begin{aligned}\therefore\ 43x+13&=(5-3x)(Ax+B)+(4x^2+3)\,.\,C\\&=(4C-3A)x^2+(5A-3B)x+5B+3C.\end{aligned}$$

Hence, $\quad 4C - 3A = 0, \quad 5A - 3B = 43, \quad 5B + 3C = 13,$
which, on solving, give: $\quad A = 8, \quad B = -1, \quad C = 6.$

$$\therefore \frac{43x + 13}{(4x^2 + 3)(5 - 3x)} = \frac{8x - 1}{4x^2 + 3} + \frac{6}{5 - 3x}.$$

Now $\dfrac{8x - 1}{4x^2 + 3} = \frac{1}{3}(8x - 1)(1 + \frac{4}{3}x^2)^{-1} = \frac{1}{3}(8x - 1)(1 - \frac{4}{3}x^2 + \ldots),$

and $\quad \dfrac{6}{5 - 3x} = \frac{6}{5}(1 - \frac{3}{5}x)^{-1} = \frac{6}{5}(1 + \frac{3}{5}x + \frac{9}{25} \cdot x^2 + \ldots).$

By § 4, the series for $(1 + \frac{4}{3}x^2)^{-1}$ is valid if $\frac{4}{3}x^2 < 1$, *i.e.* $x < \sqrt{\frac{3}{4}}$, or $x < 0{\cdot}866$, and the series for $(1 + \frac{3}{5}x)^{-1}$ is valid if $\frac{3}{5}x < 1$, or $x < 1{\cdot}67$; hence both series will be valid if $x < 0{\cdot}866$; assuming this condition fulfilled, the given fraction

$$= \tfrac{1}{3}(8x - 1)(1 - \tfrac{4}{3}x^2 + \ldots) + \tfrac{6}{5}(1 + \tfrac{3}{5}x + \tfrac{9}{25}x^2 + \ldots)$$
$$= \tfrac{13}{15} + \tfrac{254}{75}x + \tfrac{986}{1125}x^2 + \ldots.$$

The expansion may also be obtained without first resolving into partial fractions, by a direct application of the method of undetermined coefficients; thus, let

$$(43x + 13)/(4x^2 + 3)(5 - 3x) = A + Bx + Cx^2 + Dx^3 + \ldots,$$

where A, B, C, D are coefficients independent of x, whose values are to be determined.

Multiplying throughout by $(4x^2 + 3)(5 - 3x)$,

$$43x + 13 = (4x^2 + 3)(5 - 3x)(A + Bx + Cx^2 + Dx^3 + \ldots)$$
$$= 15A + (15B - 9A)x + (15C - 9B + 20A)x^2 + \ldots.$$

This must be identically true for all values of x; hence, equating corresponding coefficients:

$$15A = 13, \quad 15B - 9A = 43, \quad 15C - 9B + 20A = 0.$$

Solving for A, B, C:

$$A = \tfrac{13}{15}, \quad B = \tfrac{254}{75}, \quad C = \tfrac{986}{1125};$$

hence the first three terms of the expansion are

$$\tfrac{13}{15} + \tfrac{254}{75}x + \tfrac{986}{1125}x^2 + \ldots$$

as before.

7. Summation of Simple Series. Many algebraical series may be summed by the methods of undetermined coefficients and partial fractions, the former being most useful when the terms are whole numbers, and the latter when the terms are fractional.

The following important results are quoted for reference.

(i) Arithmetical Progression.

$$\left.\begin{array}{l} \mathbf{a+(a+d)+(a+2d)+\ldots+a+(n-1)d=\frac{n}{2}\{2a+(n-1)d\}.} \\ \text{When } \mathbf{a=d=1,} \\ \mathbf{S_1=1+2+3+\ldots+n=\frac{n}{2}(n+1).} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\mathbf{(3)}$$

(ii) Geometrical Progression.

$$\left.\begin{array}{l} \mathbf{a+ar+ar^2+\ldots+ar^{n-1}=a\frac{(1-r^n)}{1-r}.} \\ \textbf{When } \mathbf{-1<r<1,} \\ \mathbf{a+ar+ar^2+\ldots \text{ to infinity}=\frac{a}{1-r}.} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\mathbf{(4)}$$

It should be noted that this infinite G.P. is really the binomial expansion of $a(1-r)^{-1}$; hence the restriction on the value of r.

Ex. 8. *Sum to n terms each of the series*

(*a*) $1^2+2^2+3^2+\ldots$,

(*b*) $1^3+2^3+3^3+\ldots$,

(*c*) $1\,.\,3\,.\,5+2\,.\,4\,.\,6+3\,.\,5\,.\,7+\ldots$.

(*a*) Denote this sum by S_2; it will obviously be a function of n. Assume that
$$S_2=A_0+A_1n+A_2n^2+A_3n^3+\ldots,$$
where the A's are numerical coefficients to be determined.

Change n into $n+1$, then
$$S_2+(n+1)^2=A_0+A_1(n+1)+A_2(n+1)^2+A_3(n+1)^3+\ldots.$$
$\therefore$ by subtraction,
$$(n+1)^2=A_1+A_2(2n+1)+A_3(3n^2+3n+1).$$
This must be identically true, so that all the coefficients after A_3 vanish, since the highest power of n on the left-hand side is n^2; hence, equating corresponding coefficients,
$$3A_3=1;\quad 2A_2+3A_3=2;\quad A_1+A_2+A_3=1,$$
from which
$$A_3=\tfrac{1}{3},\quad A_2=\tfrac{1}{2},\quad A_1=\tfrac{1}{6}.$$
When $n=1$, $S_2=1$; $\therefore$ $A_0+A_1+A_2+A_3=1$, giving $A_0=0$; hence
$$S_2=\tfrac{1}{6}n+\tfrac{1}{2}n^2+\tfrac{1}{3}n^3=\tfrac{1}{6}n(2n^2+3n^2+1),$$
so that
$$\mathbf{S_2=\sum_1^n r^2=\tfrac{1}{6}n(n+1)(2n+1).} \quad \ldots\ldots\ldots\ldots\mathbf{(5a)}$$

(*b*) Let S_3 denote this sum, and suppose

$$S_3 = A_0 + A_1 n + A_2 n^2 + A_3 n^3 + A_4 n^4 + \dots ;$$

then proceeding exactly as in (*a*), the following values are readily obtained :

$$A_0 = A_1 = 0, \quad A_2 = A_4 = \tfrac{1}{4}; \quad A_3 = \tfrac{1}{2};$$

$$\therefore \ S_3 = \tfrac{1}{4}n^2 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^4 = \tfrac{1}{4}n^2(n+1)^2,$$

hence

$$\mathbf{S_3} = \sum_1^{\mathbf{n}} \mathbf{r^3} = \tfrac{1}{4}\mathbf{n^2(n+1)^2} = \mathbf{S_1}^2. \quad \dots\dots\dots\dots\dots\dots\dots\dots(\mathbf{5b})$$

(*c*) The *r*th term of this series is $r(r+2)(r+4) = r^3 + 6r^2 + 8r$.

$$\therefore \text{ Sum of } n \text{ terms} = \sum_1^n (r^3 + 6r^2 + 8r) = S_3 + 6S_2 + 8S_1$$

$$= \tfrac{1}{4}n^2(n+1)^2 + n(n+1)(2n+1) + 4n(n+1)$$

$$= \tfrac{1}{4}\mathbf{n(n+1)(n+4)(n+5)}.$$

Ex. 9. *Prove, by summing the series in each case, that*

$$(a) \ \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots \text{ to } n \text{ terms} = \frac{n}{2n+1},$$

and (*b*) $1 - x + 2x^2 - 3x^3 + 4x^4 - \dots$ *to infinity* $= \dfrac{1 + x + x^2}{(1+x)^2}$,

provided $x < 1$.

(*a*) The *n*th term $= \dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}\left(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right);$

$$\therefore \ \text{1st} \ ,, \ = \tfrac{1}{2}(1 - \tfrac{1}{3}),$$

$$\text{2nd} \ ,, \ = \tfrac{1}{2}(\tfrac{1}{3} - \tfrac{1}{5}),$$

$$\text{3rd} \ ,, \ = \tfrac{1}{2}(\tfrac{1}{5} - \tfrac{1}{7}),$$

....

$$n\text{th} \ ,, \ = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

Hence, on adding,

$$\text{sum to } n \text{ terms} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{\mathbf{n}}{\mathbf{2n+1}}.$$

(*b*) Let S denote the sum, then

$$S = 1 - x + 2x^2 - 3x^3 + 4x^4 - \dots,$$

and

$$xS = \quad x - x^2 + 2x^3 - 3x^4 + \dots .$$

$\therefore$ By addition,

$$
\begin{aligned}
(1+x)S &= 1 + x^2 - x^3 + x^4 - \dots \\
&= (1 - x + x^2 - x^3 + x^4 - \dots) + x \\
&= (1+x)^{-1} + x, \text{ by (3), since } x < 1, \\
&= (1 + x + x^2)/(1+x).
\end{aligned}
$$

$$\therefore \mathbf{S = (1 + x + x^2)/(1+x)^2}.$$

This result may readily be verified by expanding $(1+x)^{-2}$ by the binomial theorem, and multiplying the series by $(1 + x + x^2)$, the product being the given series.

EXERCISES 1.

1. Find the middle term in the expansion of $\left(2x - \dfrac{3}{x}\right)^{20}$.

2. Expand $(1 + 2x + \frac{1}{2}x^2)^8$ as far as x^4.

3. Find the coefficient of x^7 in the expansion of $(1 + x + x^2 + x^3)^8$.

4. Determine the coefficient of x^5 in the expansion of $(1 + x - 2x^2)^{10}$.

5. Use the binomial theorem to expand $(1 - px + qx^2)^{\frac{1}{2}}$ in the form $1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$. Prove that $a_4 + a_1a_3 + \frac{1}{2}a_2{}^2 = 0$. (Le. U., Sc.)

6. Explain how the value of the expression $(p/q)^n$ can be determined to any desired degree of accuracy by the binomial theorem, if p and q are nearly equal.

Evaluate $\sqrt{\dfrac{32{\cdot}13}{32{\cdot}20}}$ correct to seven figures. (L.U.)

7. Shew by the binomial theorem that

$$\left(\tfrac{9}{10}\right)^{\frac{4}{5}} = 0{\cdot}9191662.$$

8. Shew that, in the expansion of $(a+b+c)^n$, there will be three terms whose sum is

$$n(n-1)abc(a^{n-3} + b^{n-3} + c^{n-3}).$$

9. If x be so small that its cube and higher powers may be neglected, prove that

$$\frac{(1-5x)^{\frac{2}{5}} + (1-2x)^{-\frac{1}{4}}}{(1-3x)^{\frac{4}{5}}} = 2 + \tfrac{89}{30}x + \tfrac{15133}{1800}x^2.$$

10. Verify that, if x is large compared with y, the two expressions $x - y^2/2x$, $x - y^2/2x - y^4/4x^3$ are approximations to $\sqrt{x^2 - y^2}$, proving that the first expression is greater than, and the second less than $\sqrt{x^2 - y^2}$.

By using the first approximation calculate $\sqrt{563 \times 567}$, and give an estimate of the magnitude of the error. (B.U.)

11. Prove that, when x is small, an approximate expression for

$$\frac{(1-x)^{-\frac{3}{2}}-(1+x)^{\frac{3}{2}}}{6+9x} \text{ is } \tfrac{1}{4}x^2+\tfrac{13}{32}x^4. \qquad \text{(B.U.)}$$

12. Prove that

$$\frac{1+x}{(1-x)^4(1+x+x^2)}=1+4x+9x^2+17x^3+\dots .$$

13. Assuming that the series $1-6x+20x^2-56x^3+\dots$ is identical with the expansion of $(1+cx)/(1+ax+bx^2)$, determine a, b, c. (Le. U., Sc.)

14. Prove that the series

$$1-\tfrac{1}{2}+\tfrac{1}{2}\cdot\tfrac{3}{4}-\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}+\dots$$

is a binomial series; prove that it is convergent, and find its value. (L.U.)

15. Prove that $32=1+\frac{15}{8}+\frac{15}{8}\cdot\frac{21}{16}+\frac{15}{8}\cdot\frac{21}{16}\cdot\frac{27}{24}+\dots$. (L.U., Sc.)

16. Shew from the binomial theorem that the series

$$1-\tfrac{1}{8}-\tfrac{1}{64}-\tfrac{5}{1536}-\dots$$

represents the cube root of 0·625. Find the next term and calculate the cube root to five significant figures.

17. Obtain the first four terms of the expansion of $(N^5+a)^{\frac{1}{5}}$ by the binomial theorem, where $a<N^5$; hence shew that $N\cdot\dfrac{5N^5+3a}{5N^5+2a}$ is an approximate value of $(N^5+a)^{\frac{1}{5}}$.

Deduce that the difference between $(100100)^{\frac{1}{5}}$ and $\frac{50030}{5002}$ is of the order $1/10^{10}$. (L.U.)

18. Identify the series $1+\frac{5}{6}x+\frac{5}{6}\cdot\frac{7}{12}x^2+\frac{5}{6}\cdot\frac{7}{12}\cdot\frac{9}{18}x^3+\dots$ as a binomial expansion, and find its sum to infinity when $x=\frac{3}{5}$. (L.U., Sc.)

19. Sum the series $1+\frac{7}{18}+\frac{7}{18}\cdot\frac{9}{36}+\frac{7}{18}\cdot\frac{9}{36}\cdot\frac{11}{54}+\dots$.

If $(1+x)^n=A_0+A_1x+A_2x^2+\dots+A_nx^n$, prove the following properties of the coefficients:

20. $A_0+A_1+A_2+\dots+A_n=2^n$.

21. $A_0+A_2+A_4+\dots=A_1+A_3+A_5+\dots$.

22. $A_0^2+A_1^2+A_2^2+\dots+A_n^2=|\underline{2n}/(|\underline{n})^2$.

23. $A_0A_1+A_1A_2+A_2A_3+\dots+A_{n-1}A_n=|\underline{2n}/(|\underline{n-1}\,.\,|\underline{n+1})$.

24. Expand $\dfrac{x}{1-x-6x^2}$ in ascending powers of x, and find the sum of the first n coefficients. (L.U., Sc.)

25. Resolve $\dfrac{3-x}{(1-2x)(1+x^2)}$ into partial fractions.
State the condition that the fractions can be expanded by the binomial theorem, in a series of ascending powers of x, and assuming this condition fulfilled, write down the expansion as far as the term involving x^4. (L.U.)

26. Shew that $\dfrac{(1+x+x^2)(1+x)^2}{1-x+x^2} = 1+4x+7x^2+6x^3$,

if powers beyond x^3 are neglected. (L.U.)

27. Resolve $\dfrac{14x+19}{(4x-3)(3x^2-2)}$ into partial fractions; hence expand the fraction as far as x^3, and state the condition that this expansion should be valid.

28. Express the fraction $\dfrac{x}{(1+2x)^2(1-3x)}$ as the sum of partial fractions. Deduce the series for the given fraction in ascending powers of x, and state the value of x for which the series is convergent. (L.U., Sc.)

29. Express $\dfrac{2-3x}{(4-x)^2(4+x^2)}$ in partial fractions. Under what conditions may this function be expanded in a series of ascending powers of x? Find the coefficient of x^{10} in the expansion. (L.U.)

30. Shew that the sum of the first n even numbers is equal to $\left(1+\dfrac{1}{n}\right)$ times the sum of the first n odd numbers.

31. Prove that the sum of the first n odd integers is n^2, and that the sum of the squares of the first n odd integers is $\frac{1}{3}n(4n^2-1)$. (M.U., Sc.)

32. Find the sum of n terms of the series

$$1^2+2^2+3^2+\dots.$$

A pile of shot is arranged in an incomplete pyramid whose base is an equilateral triangle. If each side of the base contains 30 shot and each side of the top layer contains 10 shot, find the number of shot in the pile. (L.U.)

33. Sum the series $1^4+2^4+3^4+\dots+n^4$.

Sum each of the following series :

34. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots$ to n terms.

35. Prove that the sum of n terms of the series

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots$$

is $n(n+1)(2n+7)/6$. (S.U., Sc.)

36. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \dots$ to n terms. (L.U.)

37. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots$ to n terms. (L.U.)

38. Prove that the sum of n terms of the series

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots$$

is $\frac{1}{4}n(3n+5)/(n+1)(n+2)$, and that when n increases indefinitely the sum approaches the value $3/4$. (S.U., Sc.)

39. $\frac{1}{5 \cdot 2} + \frac{1}{8 \cdot 5} + \frac{1}{11 \cdot 8} + \dots$ to n terms. (L.U., Sc.)

40. $1 + x(1+x) + x^2(1+x+x^2) + x^3(1+x+x^2+x^3) + \dots$ to infinity, when $x < 1$.

41. Shew that

$$1 + 2x + 3x^2 + 4x^3 + \dots \text{ to } n \text{ terms} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2},$$

and verify it when $n=3$.

CHAPTER II

DETERMINANTS AND THEIR APPLICATIONS

8. Solution of Linear Simultaneous Equations. The solution of the system of linear equations:

$$ax+by+c=0,$$
$$px+qy+r=0,$$

is easily found to be

$$x=(br-qc)/(aq-bp)\ ; \quad y=(cp-ar)/(aq-bp),$$

which may be written

$$\frac{x}{br-cq}=\frac{y}{cp-ar}=\frac{1}{aq-bp},$$

and the denominators, being of the same form, may each be conveniently represented symbolically, thus:

$$aq-bp\equiv\begin{vmatrix} a & b \\ p & q \end{vmatrix}.$$

The right-hand member of this identical equation is called **a determinant,** whilst the left-hand member is known as the **development** or **expansion** of the determinant. The numbers a, b, p, q are called **elements,** and since each term in the expansion is the product of *two* elements, the determinant is said to be of the *Second Order*.

The expansion of a determinant of the second order is therefore formed by taking the product of the elements on the diagonal passing downwards from left to right and subtracting from it the product of the elements on the other diagonal.

Representing the denominators of x and y in a similar manner, the solution of the equations

$$ax + by + c = 0,$$
$$px + qy + r = 0,$$

is given by

$$\frac{x}{\begin{vmatrix} b & c \\ q & r \end{vmatrix}} = \frac{y}{\begin{vmatrix} c & a \\ r & p \end{vmatrix}} = \frac{1}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}}, \quad \ldots\ldots\ldots\ldots(6)$$

in which each denominator is formed by writing down the coefficients in cyclic order in two corresponding rows, when the terms involving the numeration are deleted.

The principle of cyclic order implies that b follows a, c follows b, and a follows c, as in moving round a circle upon the circumference of which the coefficients a, b, c are placed.

Ex. 1. *Expand the determinants* :

$$(a)\ \begin{vmatrix} 8 & 5 \\ 9 & 6 \end{vmatrix} \qquad (b)\ \begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix}$$

and (*c*) *solve the equation*

$$\begin{vmatrix} 10x & 5 \\ -9 & 2x \end{vmatrix} = \begin{vmatrix} x & -3 \\ 17 & 7 \end{vmatrix}.$$

$$(a)\ \begin{vmatrix} 8 & 5 \\ 9 & 6 \end{vmatrix} = 8\,.\,6 - 5\,.\,9 = 48 - 45 = 3.$$

$$(b)\ \begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix} = (a+b)^2 - 4ab = (a-b)^2.$$

(*c*) The equation

$$\begin{vmatrix} 10x & 5 \\ -9 & 2x \end{vmatrix} = \begin{vmatrix} x & -3 \\ 17 & 7 \end{vmatrix} \text{ becomes, on expansion,}$$

$$20x^2 + 45 = 7x + 51,$$

or $$20x^2 - 7x - 6 = 0;$$

i.e. $$(4x-3)(5x+2) = 0,$$

giving $$x = 0{\cdot}75 \text{ or } -0{\cdot}4.$$

Ex. 2. *Solve the equation*

$$8x - y = -25, \quad 5x + 2y = 29$$

Arranging the equations in proper order,

$$8x - \ \ y + 25 = 0,$$
$$5x + 2y - 29 = 0;$$

$$\therefore \frac{x}{\begin{vmatrix} -1 & 25 \\ 2 & -29 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 25 & 8 \\ -29 & 5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 8 & -1 \\ 5 & 2 \end{vmatrix}},$$

or
$$\frac{x}{-21} = \frac{y}{357} = \frac{1}{21}.$$

$$\mathbf{x = -1, \quad y = 17.}$$

9. Equations in Three Unknowns. Consider the simultaneous linear system:

$$ax + by + cz + d = 0,$$
$$ex + fy + gz + h = 0,$$
$$px + qy + rz + s = 0.$$

Solving these by the ordinary process of elimination,

$$x = (bhr + cfs + dgq - bgs - chq - dfr)/H,$$
$$y = (chp + der + ags - ahr - ces - dgp)/H,$$
$$z = (bes + dfp + ahq - afs - bhq - deq)/H,$$

where
$$\begin{aligned} H &= afr + bgp + ceq - cfp - agq - ber \\ &= a(fr - gq) + b(gp - er) + c(eq - fp) \\ &= a\begin{vmatrix} f & g \\ q & r \end{vmatrix} + b\begin{vmatrix} g & e \\ r & p \end{vmatrix} + c\begin{vmatrix} e & f \\ p & q \end{vmatrix}. \end{aligned}$$

Following the analogy of § 8, however, H may be represented symbolically by the determinant

$$\begin{vmatrix} a & b & c \\ e & f & g \\ p & q & r \end{vmatrix},$$

which is of the *Third Order*, so that

$$\begin{vmatrix} a & b & c \\ e & f & g \\ p & q & r \end{vmatrix} = a\begin{vmatrix} f & g \\ q & r \end{vmatrix} + b\begin{vmatrix} g & e \\ r & p \end{vmatrix} + c\begin{vmatrix} e & f \\ p & q \end{vmatrix}.$$

Thus the coefficient of a is the determinant formed by deleting the row and the column in which a lies. Similarly for b and c, provided the elements on each row are taken in cyclic order. Each of these second order determinants is called the *minor* of the element which multiplies it.

Taking now the numerator of the expression giving the value of x,

$$bhr + cfs + dgq - bgs - chq - dfr = -b(gs - hr) - c(hq - fs) - d(fr - gq)$$

$$= -b\begin{vmatrix} g & h \\ r & s \end{vmatrix} - c\begin{vmatrix} h & f \\ s & q \end{vmatrix} - d\begin{vmatrix} f & g \\ q & r \end{vmatrix} = -\begin{vmatrix} b & c & d \\ f & g & h \\ q & r & s \end{vmatrix}.$$

Similarly the numerators of the values of y and z may be expressed as determinants; hence the solution of the equations,

$$ax + by + cz + d = 0, \quad ex + fy + gz + h = 0, \quad px + qy + rz + s = 0,$$

may be written in the form

$$\frac{-x}{\begin{vmatrix} b & c & d \\ f & g & h \\ q & r & s \end{vmatrix}} = \frac{y}{\begin{vmatrix} c & d & a \\ g & h & e \\ r & s & p \end{vmatrix}} = \frac{-z}{\begin{vmatrix} d & a & b \\ h & e & f \\ s & p & q \end{vmatrix}} = \frac{1}{\begin{vmatrix} a & b & c \\ e & f & g \\ p & q & r \end{vmatrix}}, \qquad (7)$$

the denominators being formed in precisely the same manner as in the case of the equations of § 8; the signs, however, are alternately negative and positive in order to preserve cyclic order.

A determinant of the third order may be easily expanded by the **Rule of Sarrus**. Thus repeating the first two columns of the determinant on the left, the expansion may be written down by taking the algebraic sum of the products formed by the elements on each of the six diagonals shown, the products taken downwards being positive, whilst those taken upwards are negative.

$$-pfc - qga - reb$$

$$\begin{vmatrix} a & b & c \\ e & f & g \\ p & q & r \end{vmatrix} \begin{matrix} a & b \\ e & f \\ p & q \end{matrix}$$

$$+afr + bgp + ceq.$$

Ex. 3. *Expand each of the determinants :*

$$(a)\ \begin{vmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{vmatrix}, \quad (b)\ \begin{vmatrix} 1 & \sin\theta & 2 \\ \sin\theta & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{vmatrix}.$$

$$(a)\ \begin{vmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 8 \\ 4 & 9 \end{vmatrix} + 6\begin{vmatrix} 8 & 2 \\ 9 & 3 \end{vmatrix} + 7\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$$

$$= 13 + 36 - 49 = \mathbf{0}.$$

Or, expanding by the rule of Sarrus, the given determinant

$$= 1.5.9 + 6.8.3 + 7.2.4 - 3.5.7 - 4.8.1 - 9.2.6$$
$$= 45 + 144 + 56 - 105 - 32 - 108 = \mathbf{0}.$$

(*b*) Similarly

$$\begin{vmatrix} 1 & \sin\theta & 2 \\ \sin\theta & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{vmatrix} = 1 + 0 + \underline{2\sin\theta\cos\theta} - 0 - \cos^2\theta - \sin^2\theta$$

$$= 2\sin\theta\cos\theta = \mathbf{\sin 2\theta}.$$ X

Ex. 4. *Solve by determinants the equations*

$$4x + y + 2z - 15 = 0,$$
$$3x - 2y - z - 7 = 0,$$
$$7x + 3y - 3z + 10 = 0.$$

Since the equations are written in proper order, the solution may, by (7), be written down in symbolic form, thus :

$$\frac{-x}{\begin{vmatrix} 1 & 2 & -15 \\ -2 & -1 & -7 \\ 3 & -3 & 10 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & -15 & 4 \\ -1 & -7 & 3 \\ -3 & 10 & 7 \end{vmatrix}} = \frac{-z}{\begin{vmatrix} -15 & 4 & 1 \\ -7 & 3 & -2 \\ 10 & 7 & 3 \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} 4 & 1 & 2 \\ 3 & -2 & -1 \\ 7 & 3 & -3 \end{vmatrix}},$$

which becomes, on expanding the determinants :

$$\frac{x}{168} = \frac{y}{-252} = \frac{z}{420} = \frac{1}{84},$$

giving, $\mathbf{x = 2, \quad y = -3, \quad z = 5.}$

10. General Solution for n unknowns. The method of solution by determinants is applicable in precisely the same way to any system of homogeneous linear equations. Let

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n + k_1 = 0,$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n + k_2 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n + k_n = 0,$$

be a system of n homogeneous linear equations, then the solution may be written in the form

$$\frac{x_1}{(-1)^n D_1} = \frac{x_2}{D_2} = \frac{x_3}{(-1)^n D_3} = \frac{x_4}{D_4} = \ldots = \frac{1}{D_0}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where D_0 is the determinant of the coefficients, when the column of absolute terms is deleted, and D_r $(r = 1, 2, 3, \ldots n)$ is the determinant formed by the coefficients when the column of terms involving x_r is deleted, and each row is written in cyclic order. Each of the determinants will be of the nth order, and it becomes necessary to examine briefly the principles upon which a determinant of a higher order than the third may be expanded.

Ex. 5. *Cite the chief properties of the determinant*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \qquad \text{(L.U.)}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

$$= a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Hence (i) **The value of a determinant is unaltered by changing columns into rows, and rows into columns.**

The above expansion may also be written

$$-\{a_2(b_1c_3-b_3c_1)+a_1(b_3c_2-b_2c_3)+a_3(b_2c_1-b_1c_2)\}$$
$$=-\begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix},$$

so that (ii) **The sign of a determinant is changed if two rows or columns are interchanged.**

Again, from the above expansion, it is readily seen that if $a_3=a_2$, $b_3=b_2$ and $c_3=c_2$, the value becomes zero, so that

$$\begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_2 \end{vmatrix}=0.$$

Hence (iii) **When two rows or columns are identical, the determinant vanishes.**

Now let a_1, b_1, c_1, be replaced by ma_1, mb_1, mc_1 respectively, then it readily follows from the expanded form, that

$$\begin{vmatrix} ma_1 & a_2 & a_3 \\ mb_1 & b_2 & b_3 \\ mc_1 & c_2 & c_3 \end{vmatrix}=m\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Hence (iv) **The effect of multiplying each element of a row or column by a given factor is to multiply the determinant by that factor.**

Further, let $a_1=\alpha_1+p$, $b_1=\beta_1+q$, $c_1=\gamma_1+r$, then the expanded form becomes

$$(\alpha_1+p)(b_2c_3-b_3c_2)+a_2(b_3\gamma_1+b_3r-\beta_1c_3-c_3q)$$
$$+a_3(\beta_1c_2+c_2q-b_2\gamma_1-b_2r)$$
$$=\{\alpha_1(b_2c_3-b_3c_2)+a_2(b_3\gamma_1-\beta_1c_3)+a_3(\beta_1c_2-b_2\gamma_1)\}$$
$$+\{p(b_2c_3-b_3c_2)+a_2(b_3r-c_3q)+a_3(c_2q-b_2r)\},$$

which is the sum of the expansions of two determinants of the third order; hence

$$\begin{vmatrix} \alpha_1+p & a_2 & a_3 \\ \beta_1+q & b_2 & b_3 \\ \gamma_1+r & c_2 & c_3 \end{vmatrix}=\begin{vmatrix} \alpha_1 & a_2 & a_3 \\ \beta_1 & b_2 & b_3 \\ \gamma_1 & c_2 & c_3 \end{vmatrix}+\begin{vmatrix} p & a_2 & a_3 \\ q & b_2 & b_3 \\ r & c_2 & c_3 \end{vmatrix},$$

so that

(v) **When each element of any row or column consists of the algebraic sum of two or more terms, the determinant is equal to the sum of two or more determinants, each of whose elements consists of a single term.**

From this property and from (iv), it is easily seen that

$$\begin{vmatrix} a_1+pa_2-qa_3 & a_2 & a_3 \\ b_1+pb_2-qb_3 & b_2 & b_3 \\ c_1+pc_2-qc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + p\begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix} - q\begin{vmatrix} a_3 & a_2 & a_3 \\ b_3 & b_2 & b_3 \\ c_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

since the last two determinants vanish by (iii).

Hence (vi) **When equimultiples of the elements of one or more columns or rows are added algebraically to the corresponding elements of another column or row, the value of the determinant is unchanged.**

11. General System in n Unknowns. The properties established in § 10 for a determinant of the third order are quite general and apply to determinants of all orders. For a determinant of the nth order, expansion may be effected by minors. Thus let

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & a_{n3} \dots a_{nn} \end{vmatrix};$$

then a first expansion is given by

$$\mathbf{D = a_{11}A_1 - a_{12}A_2 + a_{13}A_3 - \dots + (-1)^{n-1}a_{1n}\ A_n}, \qquad \dots\dots\dots\dots (9)$$

where A_r $(r=1, 2, 3, \dots n)$ is the minor of a_{1r}, a determinant formed by deleting the row and column containing a_{1r}, and writing the rows in cyclic order. Each of the minors is of the $(n-1)$th order, whilst D is of the nth order. In practice, a combination of this rule and the six properties established above is employed, as the next example will shew.

Ex. 6. *Solve the equations*

$$\begin{aligned} 16x - 12y + 7z + 15w + 3 &= 0,\\ 22x - 3y + 3z + 11w + 2 &= 0,\\ 76x + 10y - 11z + 16w + 4 &= 0,\\ -6x - 38y + 10z + 19w + 5 &= 0, \end{aligned}$$

by first finding x, and using this value to reduce the system to three simultaneous equations.

The value of x is given by

$$\frac{x}{D_1} = \frac{1}{D_0},$$

where

$$D_1 = \begin{vmatrix} -12 & 7 & 15 & 3 \\ -3 & 3 & 11 & 2 \\ 10 & -11 & 16 & 4 \\ -38 & 10 & 19 & 5 \end{vmatrix} \text{ and } D_0 = \begin{vmatrix} 16 & -12 & 7 & 15 \\ 22 & -3 & 3 & 11 \\ 76 & 10 & -11 & 16 \\ -6 & -38 & 10 & 19 \end{vmatrix}.$$

Taking the former,

$$D_1 = \begin{vmatrix} -12+14 & 7 & 15-15 & 3 \\ -3+6 & 3 & 11-10 & 2 \\ 10-22 & -11 & 16-20 & 4 \\ -38+20 & 10 & 19-25 & 5 \end{vmatrix} \text{ by (vi)}$$

$$= \begin{vmatrix} 2 & 7 & 0 & 3 \\ 3 & 3 & 1 & 2 \\ -12 & -11 & -4 & 4 \\ -18 & 10 & -6 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 3 \\ 3 & -5 & 1 & 2 \\ -12 & 9 & -4 & 4 \\ -18 & 41 & -6 & 5 \end{vmatrix}$$

on taking the sum of twice the first column and the fourth column from the second column:

$$= 2\begin{vmatrix} -5 & 1 & 2 \\ 9 & -4 & 4 \\ 41 & -6 & 5 \end{vmatrix} - 3\begin{vmatrix} 3 & -5 & 1 \\ -12 & 9 & -4 \\ -18 & 41 & -6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 1 & 0 \\ 13 & -4 & 12 \\ 45 & -6 & 17 \end{vmatrix} - 9\begin{vmatrix} 1 & -5 & 1 \\ -4 & 9 & -4 \\ -6 & 41 & -6 \end{vmatrix} = 2\begin{vmatrix} 12 & 13 \\ 17 & 45 \end{vmatrix} \text{ by (iii)}$$

$$= 2\begin{vmatrix} 12 & 1 \\ 17 & 28 \end{vmatrix} = 638.$$

And $D_0 = 2\begin{vmatrix} 8 & -12 & 7 & 15 \\ 11 & -3 & 3 & 11 \\ 38 & 10 & -11 & 16 \\ -3 & -38 & 10 & 19 \end{vmatrix} = 2\begin{vmatrix} 15 & 3 & -5 & 15 \\ 14 & 8 & 0 & 11 \\ 27 & 26 & -1 & 16 \\ 7 & -19 & -28 & 19 \end{vmatrix}$;

on adding columns 1 and 3, 2 and 4, and 2 and 3,

$$= \tfrac{2}{3}\begin{vmatrix} 15 & 3 & -15 & 15 \\ 14 & 8 & 0 & 11 \\ 27 & 26 & -3 & 16 \\ 7 & -19 & -84 & 19 \end{vmatrix} = \tfrac{2}{3}\begin{vmatrix} 0 & 3 & 0 & 0 \\ -26 & 8 & 14 & 11 \\ -103 & 26 & 24 & 13 \\ 102 & -19 & -77 & -65 \end{vmatrix};$$

on subtracting 5 times column 2 from column 1, and adding columns 1 and 3, and 3 and 4,

$$= -2\begin{vmatrix} 14 & 11 & -26 \\ 24 & 13 & -103 \\ -77 & -65 & 102 \end{vmatrix} = -2\begin{vmatrix} 3 & 11 & -1 \\ 11 & 13 & -66 \\ -12 & -65 & -10 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 2 & 1 \\ 11 & -20 & 66 \\ -12 & -29 & 40 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 & 1 \\ -35 & -152 & 66 \\ -23 & -109 & 40 \end{vmatrix}$$

$$= 2\begin{vmatrix} 35 & 152 \\ 23 & 109 \end{vmatrix} = 2\begin{vmatrix} 35 & 12 \\ 23 & 17 \end{vmatrix} = 2\begin{vmatrix} -1 & 12 \\ -28 & 17 \end{vmatrix} -638.$$

Hence, $\qquad \mathbf{x = D_1/D_0 = 1.}$

Inserting this value in the first, second and fourth equations, the system becomes

$$\begin{aligned} -12y + 7z + 15w + 19 &= 0, \\ -3y + 3z + 11w + 24 &= 0, \\ -38y + 10z + 19w - 1 &= 0, \end{aligned}$$

the solution of which is given by

$$\frac{-y}{\begin{vmatrix} 7 & 15 & 19 \\ 3 & 11 & 24 \\ 10 & 19 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 15 & 19 & -12 \\ 11 & 24 & -3 \\ 19 & -1 & -38 \end{vmatrix}}$$

$$= \frac{-w}{\begin{vmatrix} 19 & -12 & 7 \\ 24 & -3 & 3 \\ -1 & -38 & 10 \end{vmatrix}} = \frac{1}{\begin{vmatrix} -12 & 7 & 15 \\ -3 & 3 & 11 \\ -38 & 10 & 19 \end{vmatrix}}.$$

Now

$$\begin{vmatrix} 7 & 15 & 19 \\ 3 & 11 & 24 \\ 10 & 19 & -1 \end{vmatrix} = \begin{vmatrix} 7 & 1 & 4 \\ 3 & 5 & 13 \\ 10 & -1 & -20 \end{vmatrix} = \begin{vmatrix} 7 & 1 & 0 \\ 3 & 5 & -7 \\ 10 & -1 & -16 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 \\ -32 & 5 & -7 \\ 17 & 1 & -16 \end{vmatrix} = \begin{vmatrix} -7 & -32 \\ -16 & 17 \end{vmatrix} = 631.$$

By similarly developing the other determinants,

$$\frac{y}{631} = \frac{z}{-1262} = \frac{w}{1893} = \frac{1}{-631},$$

so that $$y = -1, \quad z = 2, \quad w = -3.$$

These values also satisfy the third equation, so that the complete solution has been found.

12. Elimination. When a system of linear homogeneous equations contains more equations than there are unknowns, it is not, in general, possible to find values of the unknowns which will simultaneously satisfy the system when all the equations are independent.

When a complete set of values does simultaneously satisfy a system of $(m+n)$ equations in n unknowns, then m of the equations are not independent, and the system is said to be *consistent*.

Let
$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0.$$
$$a_3x + b_3y + c_3z + d_3 = 0,$$
$$a_4x + b_4y + c_4z + d_4 = 0,$$

be a consistent system, then from the last three equations:

$$\frac{-x}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \\ c_4 & d_4 & a_4 \end{vmatrix}} = \frac{-z}{\begin{vmatrix} d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \\ d_4 & a_4 & b_4 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}.$$

Inserting these values of x, y, z in the first equation, and multiplying throughout by the common denominator with its sign changed,

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \\ c_4 & d_4 & a_4 \end{vmatrix} + c_1\begin{vmatrix} d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \\ d_4 & a_4 & b_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0,$$

i.e.

$$\begin{vmatrix} \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{d}_1 \\ \mathbf{a}_2 & \mathbf{b}_2 & \mathbf{c}_2 & \mathbf{d}_2 \\ \mathbf{a}_3 & \mathbf{b}_3 & \mathbf{c}_3 & \mathbf{d}_3 \\ \mathbf{a}_4 & \mathbf{b}_4 & \mathbf{c}_4 & \mathbf{d}_4 \end{vmatrix} = 0. \qquad \ldots\ldots\ldots\ldots (10)$$

This is the condition that the system should be consistent, hence:

A system of $(n+1)$ homogeneous linear equations in n unknowns is consistent when the determinant of the coefficients is zero.

The condition (10) is also called the **Eliminant** of x, y, z in the given system.

Ex. 7. *Express in the form of a determinant the condition that the three equations*

$$a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0, \quad a_3x + b_3y + c_3z = 0,$$

should be consistent.

If
$$\frac{5x - 2x}{x} = \frac{5y + z}{y} = \frac{-2x + y + z}{z} = k,$$

obtain an equation from which to find k.

Solve this equation and find the mutual ratios of x, y, z corresponding to the several solutions. (L.U.)

The given system is not quite in the same form as that given in § 12, but if the unknowns be considered as ratios of x, y, z, formed by dividing each equation throughout by one of these quantities, it immediately assumes that form; thus, let

$$u = x/z \quad \text{and} \quad v = y/z,$$

then the system becomes, assuming z is not zero,

$$a_1u+b_1v+c_1=0,$$
$$a_2u+b_2v+c_2=0,$$
$$a_3u+b_3v+c_3=0;$$

hence, by (10), the condition that the system should be consistent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

This is the eliminant of x, y, z in the given system.

In the second system, clear off fractions and change x, y, z into u and v, as above, then,

$$\begin{aligned} (5-u)u \qquad\quad -2&=0,\\ (5-k)v+1&=0,\\ -2u+v+1-k&=0; \end{aligned}$$

hence, if these equations are consistent,

$$\begin{vmatrix} 5-k & 0 & -2 \\ 0 & 5-k & 1 \\ -2 & 1 & 1-k \end{vmatrix}=0,$$

which, on expansion, gives

$$k(5-k)(k-6)=0;$$

$$\therefore\ k=0,\quad 5 \text{ or } 6.$$

Solving the equations for these values of k gives :

(i) $k=0$; $\qquad u=x/z=0{\cdot}4$; $\quad v=y/z=-0{\cdot}2$

and $\qquad x/y=u/v=-2.$

(ii) $k=5$; $\qquad u=\infty$; $\quad v=\infty$;

this shows that $z=0$.

To find x/y, divide the equation

$$-2x+y+(1-k)z=0$$

by y; then, since $z=0$, $\qquad x/y=0{\cdot}5.$

(iii) $k=6$, $\qquad u=-2$, $\quad v=1$, $\quad$ and $\quad x/y=-2.$

Ex. 8. *Determine the values of a such that the system $2x+z=4$, $4x-ay=20$, $5x+y+2z=7$, $144x+128y+az+96=0$, is consistent.*

Writing the given equations in standard form :

$$\begin{aligned} 2x \qquad\quad + z - 4 &= 0,\\ 4x - \quad ay \qquad - 20 &= 0,\\ 5x + \quad y + 2z - 7 &= 0,\\ 144x + 128y + az + 96 &= 0, \end{aligned}$$

the condition for consistency is $D=0$, where

$$D = \begin{vmatrix} 2 & 0 & 1 & -4 \\ 4 & -a & 0 & -20 \\ 5 & 1 & 2 & -7 \\ 144 & 128 & a & 96 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 4 & -a & 0 & -12 \\ 1 & 1 & 2 & 3 \\ 144-2a & 128 & a & 384 \end{vmatrix}$$

$$= \begin{vmatrix} -12 & 4 & -a \\ 3 & 1 & 1 \\ 384 & 144-2a & 128 \end{vmatrix} = 3\begin{vmatrix} -4 & 4 & -a \\ 1 & 1 & 1 \\ 128 & 144-2a & 128 \end{vmatrix}$$

$$= 3\begin{vmatrix} a-4 & 4 & -a \\ 0 & 1 & 1 \\ 0 & 144-2a & 128 \end{vmatrix} = 3(a-4)\begin{vmatrix} 1 & 144-2a \\ 1 & 128 \end{vmatrix}$$

$$= 6(a-4)(a-8).$$

Hence when $D=0$,

$$a = 4 \text{ or } 8.$$

It should be observed that one of the given equations is not independent, for it may be derived from the others when a has either of the values just determined. Thus, taking $a=8$, and multiplying the first and third equations by -31 and 16 respectively, and adding, the resulting equation is

$$18x + 16y + z = -12,$$

which is the fourth equation, after inserting the value of a and dividing throughout by 8. (See Ex. 10 (2), on p. 89.)

Exercises 2.

Evaluate each of the following determinants :

1. $\begin{vmatrix} 30 & 27 & 24 \\ 18 & 16 & 11 \\ 7 & 5 & 4 \end{vmatrix}$. (L.U.) **2.** $\begin{vmatrix} 1 & 4 & 27 \\ 2 & 9 & 64 \\ 3 & 16 & 125 \end{vmatrix}$. (L.U.) **3.** $\begin{vmatrix} 23 & 6 & 11 \\ 36 & 5 & 26 \\ 63 & 13 & 37 \end{vmatrix}$. (L.U.)

4. $\begin{vmatrix} 31 & 91 & 47 \\ 14 & 29 & 30 \\ 21 & 36 & 37 \end{vmatrix}$. (L.U.) **5.** $\begin{vmatrix} 37 & -3 & 11 \\ 16 & 2 & 3 \\ 5 & 3 & -2 \end{vmatrix}$. (L.U.) **6.** $\begin{vmatrix} 1 & 6 & 9 \\ 2 & 7 & 3 \\ 5 & 4 & 6 \end{vmatrix}$.

7. $\begin{vmatrix} 2 & -2 & 0 & 5 \\ -4 & 6 & -2 & 2 \\ -3 & -2 & 5 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix}$. **8.** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 16 & 9 & 4 & 1 \\ 64 & 27 & 8 & 1 \end{vmatrix}$.

9. $\begin{vmatrix} 13 & 7 & 19 & 3 \\ 22 & 4 & 15 & 16 \\ 14 & 5 & 14 & 12 \\ 36 & 6 & 25 & 20 \end{vmatrix}$. **10.** $\begin{vmatrix} 5 & -10 & 11 & 0 \\ -32 & -35 & 34 & 0 \\ 11 & 12 & -11 & 2 \\ 1 & 5 & 3 & 0 \end{vmatrix}$.

11. $\begin{vmatrix} 4 & -5 & 7 & 18 \\ -15 & 6 & 17 & 19 \\ 17 & 11 & 15 & -13 \\ 34 & -8 & -23 & -12 \end{vmatrix}$. **12.** $\begin{vmatrix} 5 & 6 & -7 & 4 \\ 7 & 8 & 0 & 9 \\ 9 & 10 & 5 & -44 \\ 8 & -15 & 21 & -41 \end{vmatrix}$.

Solve, by determinants, each of the following equations :

13. $7x + 5y - 13z + 4 = 0,$
$9x + 2y + 11z - 37 = 0,$
$3x - y - z - 2 = 0.$ (L.U.)

14. $7x - 3y + 5z = 21,$
$2x + 5y - z = 12,$
$x + 6y + 3z = 2.$ (L.U.)

15. $x + 2y + 3z = 0,$
$4x + 9y + 16z = 1,$
$27x + 64y + 125z = 2.$ (L.U.)

16. $x + y + z = 1,$
$11x + 12y - 7z = 11,$
$37x + 40y + 24z = 38.$ (L.U.)

17. $x + y + z = 9,$
$2x + 5y + 7z = 52,$
$2x + y - z = 0.$

18. $3x + 2y + 8z = 38,$
$x + 3y + 9z = 37,$
$2x + y + z = 15.$

19. $$\begin{aligned} 2x+5y+3z+4w-23&=0,\\ 3x+2y-5w+23&=0,\\ 5x-7z+6w-19&=0,\\ 9y-8z-7w+77&=0. \end{aligned}$$

20. $$\begin{aligned} 8x+9y&=35,\\ 10y+11z&=8,\\ 12z+13w&=67,\\ x+y+z+w&=9. \end{aligned}$$

21. If $xr_1-yr_2+vr_5=0$, $x-z-v=0$, $u-y-v=0$, $xr_1+zr_3=E$, and $yr_2+ur_4=E$, are five equations to determine x, y, z, u, v in terms of the other quantities, write down v as the quotient of two determinants of the fifth order, and, reducing the numerator to a determinant of the third order, shew that $v=0$, if $r_1r_4=r_2r_3$. (L.U.)

Solve the following equations :

22. $$\begin{vmatrix} x & 6 & 3 \\ 1 & x & 1 \\ -2 & 4 & x \end{vmatrix}=0.$$ (L.U.)

23. $$\begin{vmatrix} x & 2x+1 & 2x+1 \\ 2x+1 & 3x-1 & 4x \\ 3x-1 & 4x & 6x-1 \end{vmatrix}=0.$$ (L.U.)

24. $$\begin{vmatrix} 8+2x & 13 & 11 \\ 9+4x & 19 & 8 \\ 17-2x & 12 & 14 \end{vmatrix}=0.$$ (L.U.)

25. $$\begin{vmatrix} x^2 & x & 1 \\ -13 & -7 & 8 \\ -35 & 1 & 5 \end{vmatrix}=0.$$

26. $$\begin{vmatrix} x & c+x & b+x \\ c+x & x & a+x \\ b+x & a+x & x \end{vmatrix}=0.$$ (B.U., Sc.)

27. $$\begin{vmatrix} x^3 & x^2 & x & 1 \\ 5 & 3 & 65 & 147 \\ -3 & 5 & 3 & 65 \\ 0 & -3 & 5 & 3 \end{vmatrix}=0.$$

28. $$\begin{vmatrix} a-x & b & c & d \\ b & c-x & d & a \\ c & d & a-x & b \\ d & a & b & c-x \end{vmatrix}=0.$$

29. Shew that if the equations $ax^2+bx+c=0$, $px^2+qx+r=0$, have a common root, then
$$\begin{vmatrix} c & a \\ r & p \end{vmatrix}^2=\begin{vmatrix} b & c \\ q & r \end{vmatrix}\times\begin{vmatrix} a & b \\ p & q \end{vmatrix}.$$
Simplify this condition when $p=c$, $q=b$ and $r=a$.

30. Express, by means of determinants, the condition that the three cones
$$x^2+y^2-2z^2=0,\quad 2x^2-y^2+z^2=0,\quad ax^2+by^2+cz^2=0,$$
should have a generator in common, and find the value of a when $b=2$ and $c=-3$. (L.U.)

31. Prove that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. (L.U.)

32. Evaluate the determinant : $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$.

***33.** Prove that the order of a determinant may be depressed by unity by reducing to zero all the elements but one of any row or column. Hence verify the following rule for the formation of the equivalent determinant, first given by Chio in 1853 : In the given determinant D, select the smallest element p and delete its row (the rth) and its column (the cth) ; then each element in the new determinant = corresponding element in D diminished by (product of deleted elements in same row and column divided by p). The new determinant must then be multiplied by $(-p)^{r+c}$.

Apply this rule to shew that

$$\begin{vmatrix} 7 & 2 & -11 & 1 \\ 4 & -3 & -3 & -2 \\ 13 & 5 & -4 & 27 \\ 3 & -7 & -8 & 15 \end{vmatrix} = \begin{vmatrix} 25 & 3 & 35 \\ 41 & 13 & 17 \\ 59 & 9 & 73 \end{vmatrix},$$

and by similarly reducing the latter determinant, shew that its value is zero.

***34.** Establish the following results :

(a) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} = \begin{vmatrix} a_1c_1+a_2c_2 & a_1d_1+a_2d_2 \\ b_1c_1+b_2c_2 & b_1d_1+b_2d_2 \end{vmatrix}$;

(b) $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1p_1+a_2p_2+a_3p_3 & a_1q_1+a_2q_2+a_3q_3 & a_1r_1+a_2r_2+a_3r_3 \\ b_1p_1+b_2p_2+b_3p_3 & b_1q_1+b_2q_2+b_3q_3 & b_1r_1+b_2r_2+b_3r_3 \\ c_1p_1+c_2p_2+c_3p_3 & c_1q_1+c_2q_2+c_3q_3 & c_1r_1+c_2r_2+c_3r_3 \end{vmatrix}.$$

Hence, assuming the method to hold generally, deduce a practical rule for forming the product of two determinants as a determinant.

Evaluate each of the following products as a determinant :

***35.** $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$. ***36.** $\begin{vmatrix} 9 & 10 \\ 8 & 11 \end{vmatrix}^2$. ***37.** $\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} \cdot \begin{vmatrix} 8 & 3 \\ 9 & 2 \end{vmatrix}$.

***38.** $\begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & 2 \\ 3 & 3 & -1 \end{vmatrix} . \begin{vmatrix} 3 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 4 \end{vmatrix}.$ ***39.** $\begin{vmatrix} 1 & 11 & 8 \\ 2 & -4 & 3 \\ 3 & 1 & 7 \end{vmatrix}^2.$

***40.** If i denotes $\sqrt{-1}$, shew that

$$\begin{vmatrix} 2+3i & 4+5i \\ -4+5i & 2-3i \end{vmatrix} = 2^2+3^2+4^2+5^2,$$

and express $1^2+3^2+5^2+7^2$ as a determinant ; hence shew that

$$(2^2+3^2+4^2+5^2)(1^2+3^2+5^2+7^2)=8^2+12^2+22^2+62^2.$$

***41.** Establish the general result

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} = a^2+b^2+c^2+d^2;$$

hence prove Euler's theorem that the product of two sums, each of four squares, is itself the sum of four squares.

***42.** Shew that $\begin{vmatrix} bc+ca+ab & a^2+b^2+c^2 & bc+ca+ab \\ bc+ca+ab & bc+ca+ab & a^2+b^2+c^2 \\ a^2+b^2+c^2 & bc+ca+ab & bc+ca+ab \end{vmatrix}$

is a perfect square.

43. Prove that $\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ a & a & b & a \\ b & b & b & a \end{vmatrix} = -(a-b)^4.$

***44.** If P and Q denote respectively the determinants

$$\begin{vmatrix} a+x & b+y & c+z \\ b+x & c+y & a+z \\ c+x & a+y & b+z \end{vmatrix} \text{ and } \begin{vmatrix} a-x & b-y & c-z \\ b-x & c-y & a-z \\ c-x & a-y & b-z \end{vmatrix},$$

find the value of P^2-Q^2.

Find also the form assumed by P^2-Q^2 when $x=b-c$, $y=c-a$, and $z=a-b$.

***45.** Shew that if A, B, C are the angles of a plane triangle, then

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0.$$

Deduce that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

***46.** Prove that $\begin{vmatrix} \sin^2\alpha & \sin\alpha\cos\alpha & \cos^2\alpha \\ \sin^2\beta & \sin\beta\cos\beta & \cos^2\beta \\ \sin^2\gamma & \sin\gamma\cos\gamma & \cos^2\gamma \end{vmatrix}$

$$=\sin(\alpha-\beta)\sin(\alpha-\gamma)\sin(\beta-\gamma).$$

47. Shew that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$

***48.** Shew that the square of the determinant

$$\begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}$$

can be written in the form $\begin{vmatrix} 1 & \lambda & -\lambda \\ \lambda & 1 & \lambda \\ -\lambda & \lambda & 1 \end{vmatrix}$,

and determine the value of λ. (L.U., Sc.)

49. Evaluate $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$. (L.U., Sc.)

50. Find for what value, or values, of a the three equations

$$x^2+2y-x=11a,$$
$$3x^2+y+x-12a,$$
$$2x^2-y+x=0,$$

are compatible, and solve them when they are so. (L.U., Sc.)

CHAPTER III

TRIGONOMETRICAL FORMULAE. EXPONENTIAL SERIES. COMPLEX NUMBERS. DE MOIVRE'S THEOREM. HYPERBOLIC FUNCTIONS. SIMPLE SERIES

13. Trigonometrical Formulae. The following important formulae are here quoted, without proof, for future reference:

$$\left.\begin{aligned} &(a)\ \sin(A \pm B) = \sin A \cos B \pm \sin B \cos A,\\ &(b)\ \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,\\ &(c)\ \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(11)$$

By putting $A = B$ in the first of each of the above identities the following are easily deduced:

$$\left.\begin{aligned} &(a)\ \sin 2A = 2 \sin A \cos A,\\ &(b)\ \cos 2A = \cos^2 A - \sin^2 A\\ &\qquad\quad = 2\cos^2 A - 1 = 1 - 2\sin^2 A,\\ &(c)\ \tan 2A = 2\tan A/(1 - \tan^2 A). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(12)$$

By repeated application of (11) and (12), it may readily be shewn that:

$$\left.\begin{aligned} &(a)\ \sin 3A = 3\sin A - 4\sin^3 A,\\ &(b)\ \cos 3A = 4\cos^3 A - 3\cos A,\\ &(c)\ \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(13)$$

and finally, by adding and subtracting one to and from the other of each pair of (11), and writing P for $A + B$, and Q for $A - B$, the following identities are derived:

$$\left.\begin{aligned} &(a)\ \sin P + \sin Q = 2\sin\tfrac{1}{2}(P+Q) \cdot \cos\tfrac{1}{2}(P-Q),\\ &(b)\ \sin P - \sin Q = 2\cos\tfrac{1}{2}(P+Q) \cdot \sin\tfrac{1}{2}(P-Q),\\ &(c)\ \cos P + \cos Q = 2\cos\tfrac{1}{2}(P+Q) \cdot \cos\tfrac{1}{2}(P-Q),\\ &(d)\ \cos Q - \cos P = 2\sin\tfrac{1}{2}(P+Q) \cdot \sin\tfrac{1}{2}(P-Q). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(14)$$

14. General Solutions to Trigonometrical Equations. If $\sin A = a$, $\cos B = b$, $\tan C = c$, where a, b, c are positive, and α, β, γ are the three principal values of A, B, C respectively, *i.e.* the smallest positive angles satisfying the respective equations, then it can readily be shewn that :

$$\text{(i) } \mathbf{A = n\pi + (-1)^n \alpha}\ ; \quad \text{(ii) } \mathbf{B = 2n\pi + \beta}\ ; \quad \text{(iii) } \mathbf{C = n\pi + \gamma}, \quad \ldots\ldots\mathbf{(15)}$$

where n is a positive integer, including zero.

These formulae give all the positive angles satisfying the respective equations.

Ex. 1. (*a*) *Shew that*

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = (r - p)/(1 - q + s).$$

where $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, $\tan\theta_4$ *are the roots of the equation*

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

and deduce that when the roots of the equation are $\frac{3}{4}$, $\frac{3}{5}$, $-\frac{8}{19}$ *and* 0,

$$\theta_1 + \theta_2 - \theta_3 = \frac{\pi}{4}.$$

(*b*) *Solve the equation*

$$\sin 2x - \sin 3x - \sin 5x + \sin 6x + 2 \sin 4x = 0.$$

(*a*) Let $\tan\theta_r = a_r$, $(r = 1, 2, 3, 4)$:

then from (11*c*), if $A = \theta_1 + \theta_2$ and $B = \theta_3 + \theta_4$,

$$\tan A = (a_1 + a_2)/(1 - a_1 a_2), \quad \tan B = (a_3 + a_4)/(1 - a_3 a_4),$$

and since a_1, a_2, a_3, a_4, are roots of $x^4 + px^3 + qx^2 + rx + s = 0$,

$$\therefore\ p = -(a_1 + a_2 + a_3 + a_4). \quad q = a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4,$$

$$r = -(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4), \quad \text{and} \quad s = a_1a_2a_3a_4.$$

Hence $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$

$$= \tan(A + B) = (\tan A + \tan B)/(1 - \tan A \,.\, \tan B)$$

$$= \frac{a_1 + a_2 + a_3 + a_4 - (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)}{1 - (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4) + a_1a_2a_3a_4}$$

$$= \frac{\mathbf{r - p}}{\mathbf{1 - q + s}}.$$

Putting $\qquad a_1=\frac{3}{4},\quad a_2=\frac{3}{5},\quad a_3=-\frac{8}{19},\quad a_4=0,$

$$p=-\left(\tfrac{3}{4}+\tfrac{3}{5}-\tfrac{8}{19}\right)=-\tfrac{353}{380};\quad q=\tfrac{9}{20}-\tfrac{6}{19}-\tfrac{24}{95}=-\tfrac{45}{380},$$

$$r=\tfrac{18}{95}\quad\text{and}\quad s=0;$$

$$\therefore\ \tan(\theta_1+\theta_2-\theta_3)=\left(\tfrac{18}{95}+\tfrac{353}{380}\right)/\left(1+\tfrac{45}{380}\right)=1;$$

$$\therefore\ \theta_1+\theta_2-\theta_3=\frac{\pi}{4}.$$

(*b*) Observing that $2x+6x=3x+5x$, and rearranging terms accordingly,

$$(\sin 2x+\sin 6x)-(\sin 3x+\sin 5x)=-2\sin 4x.$$

$\therefore$ From (14*a*),

$$2\sin 4x\,.\cos 2x-2\sin 4x\,.\cos x=-2\sin 4x;$$

$$\therefore\ \sin 4x(\cos 2x-\cos x+1)=0;$$

or, from (12*b*),

$$\sin 4x(2\cos^2 x-1-\cos x+1)=0,$$

i.e.
$$\sin 4x\,.\cos x\,.(2\cos x-1)=0;$$

$$\therefore\ \sin 4x=0,\quad \cos x=0,\quad\text{or}\ \cos x=0\cdot5.$$

The principal values of x in these equations are given by

$$4x=0,\quad x=\frac{\pi}{2}\quad\text{or}\quad x=\frac{\pi}{3}.$$

$\therefore$ From (15),

$$\mathbf{x}=\tfrac{1}{4}\mathbf{n}\pi,\quad \tfrac{1}{2}(\mathbf{4n}\pm\mathbf{1})\pi\quad\text{or}\quad \tfrac{1}{3}(\mathbf{6n}\pm\mathbf{1})\pi.$$

15. The Exponential Series. By expanding $\left(1+\frac{1}{n}\right)^n$, when $n>1$, by the binomial theorem, and then increasing n indefinitely, it can be shewn that

$$\operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n=1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots+\frac{1}{\lfloor r}+\ldots\ \text{to infinity.}$$

This series is convergent and is denoted by e, the base of the Napierian system of logarithms.

In a similar manner, the more general theorem may be established :

$$\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots+\frac{x^r}{\underline{|r}}+\ldots.$$

But
$$\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^{nx}=\underset{n\to\infty}{\mathrm{Lt}}\left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x}=e^x,$$

so that
$$\mathbf{e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots+\frac{x^r}{\underline{|r}}+\ldots} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(16)}$$

This result is known as the **Exponential Series**, and it has already been shewn, in Ex. 5 of § 5, that this series is convergent for all values of x.

Ex. 2. *Calculate the value of e to four figures, and sum the series*

$$1+\frac{2^2}{\underline{|1}}+\frac{3^2}{\underline{|2}}+\frac{4^2}{\underline{|3}}+\ldots \textit{ to infinity.}$$

From § 15,

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots$$
$$=1+1+0\cdot5+0\cdot16667+0\cdot04167+0\cdot00833+0\cdot00139$$
$$+0\cdot00019+0\cdot00002+\ldots$$
$$=\mathbf{2\cdot718}\ldots \text{ to four figures.}$$

The rth term of the given series

$$=\frac{r^2}{\underline{|r-1}}=\frac{(r-1)^2+2r-1}{\underline{|r-1}}=\frac{r-1}{\underline{|r-2}}+\frac{2(r-1)+1}{\underline{|r-1}}$$

$$=\frac{r-2+1}{\underline{|r-2}}+\frac{2}{\underline{|r-2}}+\frac{1}{\underline{|r-1}}=\frac{1}{\underline{|r-3}}+\frac{3}{\underline{|r-2}}+\frac{1}{\underline{|r-1}}.$$

Hence, since the series is infinite, it may be written :

$$\text{Sum}=\sum_{r=1}^{\infty}\frac{1}{\underline{|r-1}}+3\sum_{r=2}^{\infty}\frac{1}{\underline{|r-2}}+\sum_{r=3}^{\infty}\frac{1}{\underline{|r-3}}$$
$$=e+3e+e=\mathbf{5e}.$$

16. De Moivre's Theorem. If a be any number, the roots of the equation $x^2+a^2=0$ are $x=\pm\sqrt{-a^2}$.

Since, however, a negative quantity cannot have a real square root, the quantity $\sqrt{-a^2}$ is called an imaginary quantity, and some meaning must be assigned to the symbol denoting it.

It is convenient to define $\sqrt{-a^2}$ by the relation

$$\sqrt{-a^2} \times \sqrt{-a^2} = -a^2;$$

hence $$\sqrt{-1} \times \sqrt{-1} = -1$$

and $$a\sqrt{-1} \times a\sqrt{-1} = -a^2;$$

$\therefore$ $\sqrt{-a^2}$ may be regarded as equivalent to $a\sqrt{-1}$.

The symbol $\sqrt{-1}$ is generally denoted by the letter i, so that $i^2 = -1$, $i^3 = i^2 \cdot i = -i$, $i^4 = (i^2)^2 = 1$, $i^5 = i^4 \cdot i = i$, $i^6 = (i^2)^3 = -1$, and so on. With this meaning of i, an important theorem, first given in 1730 by Abraham De Moivre, may now be established. The Theorem may be stated as follows:

For all values of n, $\cos n\theta + i \sin n\theta$ is one of the values of

$(\cos \theta + i \sin \theta)^n$. ....(17)

This very important theorem will now be proved.

Ex. 3. *Prove De Moivre's Theorem when n is any real number, integral or fractional.* (L.U., *for n positive.*)

Shew also that there are only q values of

$$(\cos\theta + i \sin\theta)^{\frac{1}{q}},$$

when q is a positive integer.

By multiplication:

$$\begin{aligned}(\cos \theta_1 + i \sin \theta_1)&(\cos \theta_2 + i \sin \theta_2)\\ &= \cos \theta_1 \cos \theta_2 + i(\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1)\\ &\qquad + i^2 \sin \theta_1 \sin \theta_2\\ &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i \sin(\theta_1 + \theta_2)\\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2), \quad \text{by (11)}.\end{aligned}$$

Again, $(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)(\cos \theta_3 + i \sin \theta_3)$

$$\begin{aligned}&= \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\}(\cos \theta_3 + i \sin \theta_3)\\ &= \cos(\theta_1 + \theta_2 + \theta_3) + i \sin(\theta_1 + \theta_2 + \theta_3), \text{ by the first result.}\end{aligned}$$

Similarly, by repeating the process,

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n),$$

where n is a positive integer.

Put $\theta_1 = \theta_2 = \ldots = \theta_n = \theta$, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Further, let $n = -m$, where m is a positive integer, then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = 1/(\cos\theta + i\sin\theta)^m$$
$$= \frac{1}{\cos m\theta + i\sin m\theta} \text{ by the previous result}$$
$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta - i^2\sin^2 m\theta} = \cos m\theta - i\sin m\theta$$
$$= \cos(-n\theta) - i\sin(-n\theta) = \cos n\theta + i\sin n\theta.$$

Finally, let $n = p/q$ where p and q are integers, q being positive but p unrestricted in sign; then, from the first result,

$$(\cos\phi + i\sin\phi)^q = \cos q\phi + i\sin q\phi.$$

Taking the qth root,

$$\cos\phi + i\sin\phi = (\cos q\phi + i\sin q\phi)^{\frac{1}{q}}.$$

Now put $\theta = q\phi$, and

$$\cos\frac{\theta}{q} + i\sin\frac{\theta}{q} = (\cos\theta + i\sin\theta)^{\frac{1}{q}}.$$

And from the previous results, whether p is positive or negative, since it is an integer, and

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = (\cos\theta + i\sin\theta)^{\frac{p}{q}},$$

it follows that $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta = (\cos\theta + i\sin\theta)^{\frac{p}{q}}$.

Hence the theorem is completely established for all real values of n.

Further, the expression $\cos\theta + i\sin\theta$ is unaltered if for θ is written $\theta + 2m\pi$, where m is an integer; hence

$$(\cos\theta + i\sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q} = \cos\frac{\theta+2m\pi}{q} + i\sin\frac{\theta+2m\pi}{q}.$$

By putting $m=0, 1, 2, \ldots$, different values are obtained until $m=q$, then

$$\cos\frac{\theta+2q\pi}{q}+i\sin\frac{\theta+2q\pi}{q}=\cos\left(\frac{\theta}{q}+2\pi\right)+i\sin\left(\frac{\theta}{q}+2\pi\right)$$
$$=\cos\frac{\theta}{q}+i\sin\frac{\theta}{q},$$

which is the value obtained when $m=0$.

Similarly, when $m=q+1$, the value at $m=1$ is repeated, and so on; hence:

There are q values only of $(\cos\theta+i\sin\theta)^{\frac{1}{q}}$, where q is a positive integer, and these are given by

$$\cos\frac{\theta+2m\pi}{q}+i\sin\frac{\theta+2m\pi}{q},\ \text{where } m=0, 1, 2, \ldots (q-1). \quad \ldots\ldots(18)$$

17. Complex Numbers. Any quantity of the form $a+ib$, where a and b are real numbers, is called a **complex number.** For two complex numbers to be equal, it is necessary and sufficient that the real parts shall be equal and the imaginary parts shall be equal; for let

$$a+ib=p+iq,$$

then
$$a-p+i(b-q)=0,$$

from which it is evident that $a=p$ and $b=q$.

From this fact it is always possible to put a complex number in the form of a De Moivre expression; thus, let

$$a+ib=r(\cos\theta+i\sin\theta),$$

then equating real and imaginary parts,

$$\left.\begin{array}{l} a=r\cos\theta, \quad b=r\sin\theta, \\ r^2=a^2+b^2 \quad \text{and} \quad \tan\theta=b/a. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(19)$$

from which

Since a, b are real, and a^2, b^2 are positive, r is always real: also a real value of θ can always be found to satisfy the equation $\tan\theta=b/a$; hence it is always possible to find real values of r and θ, so that

$$a+ib=r(\cos\theta+i\sin\theta).$$

Since $\tan\theta=\tan(\theta+2m\pi)$, where m is an integer, the **Principal Value of** θ, and therefore of $a+ib$, is defined as that which lies between π and $-\pi$.

Ex. 4. *Define the modulus and argument of a complex quantity.*
Prove that the modulus of the product of two complex quantities is equal to the product of the moduli of the quantities, and that the argument of the product is equal to the sum of the arguments of the quantities.

Writing $\dfrac{(Lip+R)/Cip}{Lip+R+1/Cip}$ *in the form* $L'ip+R'$, *find the values of* L' *and* R'. *Shew also that if* R *is small compared with* Lp, *and* LCp^2 *small compared with unity, then* $L'=L(1+p^2CL)$ *and* $R'=R(1+2p^2CL)$ *approximately.* (L.U.)

(i) Let $a+ib$ be any complex number; then it has just been shewn that $a+ib=r(\cos\theta+i\sin\theta)$, if $r^2=a^2+b^2$ and $\tan\theta=b/a$. The positive square root of a^2+b^2, which is always taken for the value of r, is called the **Modulus**, and θ, the **Argument** or **Amplitude** of the given complex number.

The modulus r of $a+ib$ is usually written $|a+ib|$, so that

$$|a+ib| \equiv +\sqrt{a^2+b^2}.$$

(ii) Let $a+ib$, $p+iq$ be two complex numbers, then

$$(a+ib)(p+iq)=ap-bq+i(aq+bp);$$

$$\begin{aligned}\therefore\quad |(a+ib)(p+iq)| &= \{(ap-bq)^2+(aq+bp)^2\}^{\frac{1}{2}}\\ &= (a^2p^2+b^2q^2+a^2q^2+b^2p^2)^{\frac{1}{2}}\\ &= \{(a^2+b^2)(p^2+q^2)\}^{\frac{1}{2}}\\ &= |a+ib|\,.\,|p+iq|;\end{aligned}$$

$\therefore$ The modulus of the product of the numbers $=$ the product of their moduli.

This may be shewn to be true of any number of complex numbers. Again, let

$$a+ib=r_1(\cos\theta_1+i\sin\theta_1),$$

and $$p+iq=r_2(\cos\theta_2+i\sin\theta_2)$$

then, on taking the product,

$$ap-bq+i(aq+bp)=r_1r_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\},$$

by Ex. 3, p. 40.

$\therefore$ Argument of product $=\theta_1+\theta_2$

$=$ sum of arguments of the numbers.

This is true for any number of complex numbers, by De Moivre's theorem.

(iii) $$L'ip + R' = \frac{(Lip + R)/Cip}{Lip + R + 1/Cip} = \frac{Lip + R}{1 - CLp^2 + iCRp}$$

$$= \frac{(Lip + R)(1 - CLp^2 - iCRp)}{(1 - CLp^2 + iCRp)(1 - CLp^2 - iCRp)}$$

$$= \frac{(L - CL^2p^2 - CR^2)ip + R}{(1 - CLp^2)^2 + C^2R^2p^2}.$$

Hence, equating real and imaginary parts :

$$L' = (\mathbf{L - CL^2p^2 - CR^2})/\{(\mathbf{1 - CLp^2})^2 + \mathbf{C^2R^2p^2}\}$$

and $$R' = \mathbf{R}/\{(\mathbf{1 - CLp^2})^2 + \mathbf{C^2R^2p^2}\}.$$

The numerator of $L' = L - C(L^2p^2 + R^2)$, and the common denominator to L' and $R' = 1 - 2CLp^2 + C^2p^2(L^2p^2 + R^2)$.

Hence, if R is small compared with Lp, it may be neglected as a first approximation ;

$$\therefore\ L' = (L - CL^2p^2)/(1 - CLp^2)^2 = L(1 - CLp^2)(1 - CLp^2)^{-2}$$
$$= L(1 - CLp^2)^{-1} ;$$

and since CLp^2 is small compared with unity, $(1 - CLp^2)^{-1}$ may be expanded by the binomial theorem, and the square and higher powers of CLp^2 neglected ;

$$\therefore\ L' = \mathbf{L(1 + CLp^2)} \text{ approximately.}$$

Similarly, $R' = R(1 - CLp^2)^{-2} = \mathbf{R(1 + 2CLp^2)}$, approximately.

Ex. 5. Express $75 - 100i$ in the form $R(\cos\theta + i\sin\theta)$, where R is positive, and write down expressions for its three cube roots. If $a + ib$ is such a cube root, find, for one root, the values of a and b correct to three places of decimals. (L.U.)

Let $$75 - 100i = R(\cos\theta + i\sin\theta),$$

then, by (18), $$R\cos\theta = 75,\quad R\sin\theta = -100$$

and $$\tan\theta = -100/75 = -1{\cdot}3333 ;$$

hence, $$R^2 = 75^2 + 100^2 = 25^2(3^2 + 4^2) = 25^2 . 5^2 ;$$

$\therefore$ $R = 125$, and from the tables, $\theta = -53^\circ\ 8'$ very nearly.

$$\therefore\ 75 - 100i = 125\ \{\cos(-53^\circ\ 8') + i\sin(-53^\circ\ 8')\}$$
$$= \mathbf{125\ \{\cos(-53^\circ\ 8') - i\sin 53^\circ\ 8'\}}.$$

Now the cube roots of $75 - 100i$ are given by

$$125^{\frac{1}{3}}(\cos(-53^\circ\,8') + i\sin(-53^\circ\,8'))^{\frac{1}{3}}$$

$$= 5\left\{\cos\left(-\frac{53^\circ\,8'}{3}\right) + i\sin\left(-\frac{53^\circ\,8'}{3}\right)\right\}, \text{ by (17)},$$

$$= 5\left\{\cos\frac{2m\pi - 53^\circ\,8'}{3} + i\sin\frac{2m\pi - 53^\circ\,8'}{3}\right\}, \text{ by (18)},$$

where $m = 0, 1, 2$.

Putting in these values of m, the cube roots become:

(i) $5\{\cos(-17^\circ\,42{\cdot}7') + i\sin(-17^\circ\,42{\cdot}7')\}$
$= 5(\cos 17^\circ\,42{\cdot}7' - i\sin 17^\circ\,42{\cdot}7')$
$= 5(0{\cdot}95263 - 0{\cdot}30412i) = 4{\cdot}76315 - 1{\cdot}52062\,.\,i,$

(ii) $5(\cos 102^\circ\,17{\cdot}3' + i\sin 102^\circ\,17{\cdot}3')$
$= 5(-\cos 77^\circ\,17{\cdot}3' + i\sin 77^\circ\,17{\cdot}3')$
$= 5(-0{\cdot}22010 + 0{\cdot}97519i) = -1{\cdot}00050 + 4{\cdot}87745\,.\,i,$

(iii) $5(\cos 222^\circ\,17{\cdot}3' + i\sin 222^\circ\,17{\cdot}3')$
$= 5(-\cos 42^\circ\,17{\cdot}3' - i\sin 42^\circ\,17{\cdot}3')$
$= -5(0{\cdot}74114 + 0{\cdot}67285i) = -(3{\cdot}70570 + 3{\cdot}36425\,.\,i).$

Since five-figure tables have been used, the values will be correct to four places of decimals. It should be observed that the question only required one value to be worked out, whereas the three have been calculated to exemplify the method.

The values of a and b, to three places, will therefore be

$a = -1{\cdot}001, \quad -3{\cdot}706, \quad 4{\cdot}763;$
$b = 4{\cdot}877, \quad -3{\cdot}364, \quad -1{\cdot}521.$

18. The Argand diagram. A complex number may be represented geometrically by means of a simple diagram, originally published in 1806 by J. B. Argand. Let P (Fig. 1) be any point (x, y), referred to rectangular axes OX, OY: then if $x + iy$ be denoted by z, the point P is said to correspond to z.

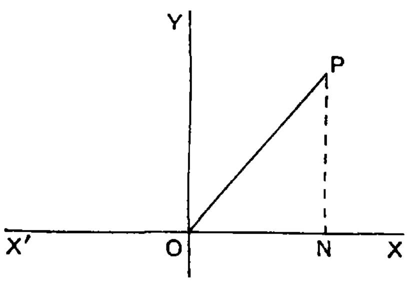

FIG. 1. The Argand diagram.

The modulus $|z| = |x+iy| = (x^2+y^2)^{\frac{1}{2}} = r = OP$,

and if $\angle XOP = \theta$,

argument of $z = \tan^{-1} y/x = \theta = 2m\pi + \theta$, where m is an integer.

It should be noticed that if OX' be equal to OX in length, then $OX' = -OX$; hence the operation -1 performed on OX turns it through two right angles.

But $-1 = \sqrt{-1} \times \sqrt{-1}$, so that the operation $\sqrt{-1}$ performed twice has the same effect as the operation -1; it is obvious, therefore, that the operation $\sqrt{-1}$ may be considered to have the effect of turning a length through one right angle in the positive direction.

Ex. 6. *Shew, by means of the Argand diagram, that the geometrical representation of the sum of two complex numbers follows the vector law; hence deduce that the modulus of the sum is less than the sum of the moduli of the numbers.*

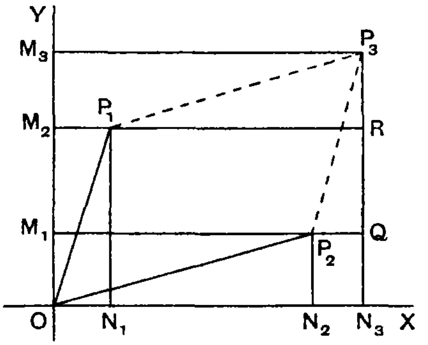

FIG. 2. Sum of two complex numbers.

Let $z_r = x_r + iy_r$ $(r=1, 2, 3)$, and let z_1, z_2, z_3 be represented by P_1, P_2, P_3 (Fig. 2) respectively, such that $z_3 = z_1 + z_2$. Join P_1P_3, P_2P_3, and draw parallels to the axes as indicated.

Now $$z_3 = z_1 + z_2;$$

$$\therefore\ x_3 + iy_3 = x_1 + x_2 + i(y_1 + y_2),$$

so that $$x_1 + x_2 = x_3 \quad \text{and} \quad y_1 + y_2 = y_3;$$

$$\therefore\ ON_1 + ON_2 = ON_3, \qquad OM_1 + OM_2 = OM_3;$$

hence $$ON_1 = N_2N_3 = P_2Q, \qquad OM_2 = M_1M_3 = P_3Q.$$

Let θ_1, θ_2, θ_3 be the arguments of z_1, z_2, z_3, so that

$$\angle XOP_1=\theta_1, \quad \angle XOP_2=\theta_2, \quad \angle XOP_3=\theta_3,$$

then $$\tan P_3P_2Q=P_3Q/P_2Q=M_1M_3/ON_1=OM_2/ON_1$$

$$=N_1P_1/ON_1=\tan\theta_1,$$

$$\tan P_3P_1R=P_3R/P_1R=N_3Q/N_1N_3=N_2P_2/ON_2=\tan\theta_2;$$

$$\therefore\ P_2P_3 \parallel OP_1 \quad \text{and} \quad P_1P_3 \parallel OP_2,$$

so that OP_3 is the diagonal of the parallelogram on OP_1, OP_2.

Hence the geometrical addition of complex numbers obeys the vector law.

Similarly it may be shewn that the vector law equally applies to the geometrical representation of the difference of two complex numbers.

Since $\quad OP_3<OP_1+P_1P_3, \quad$ *i.e.* $OP_3<OP_1+OP_2,$

it follows that the modulus of the sum of z_1 and z_2 is less than the sum of their moduli.

It is easy to shew also that the modulus of the sum is greater than the difference of the moduli of the numbers, hence

$$|z_1|\sim|z_2|<|z_3|<|z_1+z_2|.$$

19. Series and Exponential Values of sin θ and cos θ. When z is a complex number, some meaning must be assigned to e^z. It will be sufficient in this book to define e^z by what is called its **principal value,** viz., the series,

$$1+z+\frac{z^2}{\underline{|2}}+\frac{z^3}{\underline{|3}}+\ldots+\frac{z^r}{\underline{|r}}+\ldots,$$

where $z=x+iy$, and x, y are real numbers.

This series is convergent for all values of z, and with this meaning of e^z it will now be established that

$$\mathbf{\cos\theta+i\sin\theta=e^{i\theta}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(20)}$$

Ex. 7. *Establish the exponential value of* $\cos\theta+i\sin\theta$, *and from it deduce the series and exponential values of* $\sin\theta$, $\cos\theta$ *respectively; hence shew that* (1) *any complex number* $a+ib$ *may be expressed in the form* $e^{i\theta}\sqrt{a^2+b^2}$, *where* θ *is the principal value of* $\tan^{-1}b/a$, *and* (2) $16\cos^5\theta=\cos 5\theta+\cos 3\theta+10\cos\theta$.

In (17), let θ = one radian; then $\cos n + i \sin n$ is one of the values of $(\cos 1 + i \sin 1)^n$.

Now let $z = a + ib$, where a and b are real numbers, and suppose a and b can be determined so that the principal value of e^z is $\cos 1 + i \sin 1$, then

$$\cos\theta + i\sin\theta = (\cos 1 + i\sin 1)^\theta = e^{(a+ib)\theta} = e^{a\theta} \cdot e^{ib\theta}$$

$$= e^{a\theta}\left(1 + ib\theta + \frac{i^2b^2\theta^2}{\lfloor 2} + \frac{i^3b^3\theta^3}{\lfloor 3} + \dots\right)$$

$$= e^{a\theta}\left(1 - \frac{b^2\theta^2}{\lfloor 2} + \frac{b^4\theta^4}{\lfloor 4} - \frac{b^6\theta^6}{\lfloor 6} + \dots\right) + i\,e^{a\theta}\left(b\theta - \frac{b^3\theta^3}{\lfloor 3} + \frac{b^5\theta^5}{\lfloor 5} - \dots\right).$$

As a is real, $\qquad e^{a\theta}$ is real;
hence, equating real and imaginary parts,

$$\text{(i)}\ \cos\theta = e^{a\theta}\left(1 - \frac{b^2\theta^2}{\lfloor 2} + \frac{b^4\theta^4}{\lfloor 4} - \dots\right);$$

$$\text{(ii)}\ \sin\theta = e^{a\theta}\left(b\theta - \frac{b^3\theta^3}{\lfloor 3} + \frac{b^5\theta^5}{\lfloor 5} - \dots\right).$$

From (ii), $$\frac{\sin\theta}{\theta} = be^{a\theta}\left(1 - \frac{b^2\theta^2}{\lfloor 3} + \frac{b^4\theta^4}{\lfloor 5} - \dots\right).$$

Now $$\operatorname*{Lt}_{r\to\infty} \frac{(r+1)\text{th term of series}}{r\text{th term}} = -\operatorname*{Lt}_{r\to\infty} \frac{b^2\theta^2}{2r(2r+1)} = 0,$$
if b, θ are finite.

Hence the series converges to a finite limit, and

$$\operatorname*{Lt}_{\theta\to 0} \frac{\sin\theta}{\theta} = \operatorname*{Lt}_{\theta\to 0}\left\{be^{a\theta}\left(1 - \frac{b^2\theta^2}{\lfloor 3} + \frac{b^4\theta^4}{\lfloor 5} - \dots\right)\right\} = b.$$

But $\operatorname*{Lt}_{\theta\to 0} \sin\theta/\theta = 1$, by Elementary Trigonometry;

$$\therefore\ b = 1.$$

Again, the ratio test shews that the coefficient of $e^{a\theta}$ in (i) is convergent, so that, with $b = 1$,

$$\cos\theta = e^{a\theta}\left(1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \dots\right).$$

Write $-\theta$ for θ, then

$$\cos\theta = e^{-a\theta}\left(1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \dots\right),$$

since $\cos(-\theta) = \cos\theta$, and every power of θ in the series is even;

$\therefore\ e^{a\theta} = e^{-a\theta}$, or $e^{2a\theta} = 1$, from which $a = 0$, provided θ is not zero;

$$\therefore \cos\theta + i\sin\theta = e^{i\theta}, \quad \text{and} \quad \cos\theta - i\sin\theta = e^{-i\theta},$$

on writing $-\theta$ for θ.

Further, putting in the values of a and b in (i) and (ii),

$$\left.\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \dots + (-1)^{r-1}\frac{\theta^{2r-1}}{\lfloor 2r-1} + \dots, \\ \cos\theta &= 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \dots + (-1)^{r-1}\frac{\theta^{2r-2}}{\lfloor 2r-2} + \dots \end{aligned}\right\} \quad \dots\dots\dots (21)$$

These results are established by another method in Chap. VII.

Again, by addition and subtraction, of the expressions, $\cos\theta + i\sin\theta = e^{i\theta}$ and $\cos\theta - i\sin\theta = e^{-i\theta}$,

$$2i\sin\theta = e^{i\theta} - e^{-i\theta}, \quad 2\cos\theta = e^{i\theta} + e^{-i\theta}. \quad \dots\dots\dots (22)$$

(1) From § 17,

$$a + ib = r(\cos\theta + i\sin\theta),$$

where $r = \sqrt{a^2 + b^2}$, and θ = principal value of $\tan^{-1}b/a$.

Hence, from (20), $a + ib = \sqrt{a^2 + b^2}\,.\,e^{i\theta}$.

(2) $16\cos^5\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})^5$, from (22),

$$= \tfrac{1}{2}(e^{5i\theta} + 5e^{3i\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-3i\theta} + e^{-5i\theta}), \text{ by (1)},$$

$$= \tfrac{1}{2}(e^{5i\theta} + e^{-5i\theta}) + \tfrac{5}{2}(e^{3i\theta} + e^{-3i\theta}) + 5(e^{i\theta} + e^{-i\theta})$$

$$= \cos 5\theta + 5\cos 3\theta + 10\cos\theta.$$

20. Hyperbolic Functions. Just as the circular functions are related to the circle, so a corresponding group of functions is defined in relation to the rectangular hyperbola; hence the name, **Hyperbolic Functions**. These functions are denoted by adding h to the abbreviation for the corresponding circular function; thus, $\sinh u$, $\cosh u$, etc., denote the hyperbolic sine, cosine, etc., of u respectively. In analysis it is convenient to define the hyperbolic functions exponentially thus:

$$\left.\begin{aligned} 2\sinh u = e^u - e^{-u}, \quad 2\cosh u = e^u + e^{-u}, \\ \tanh u = \sinh u/\cosh u = (e^u - e^{-u})/(e^u + e^{-u}). \end{aligned}\right\} \quad \dots\dots\dots (23)$$

Ex. 8. *Prove that* $\cos ix = \cosh x$, *and* $\sin ix = i \sinh x$. *From these results, shew that* $\cosh (x+y) = \cosh x \,.\, \cosh y + \sinh x \sinh y$, *and obtain the formula for* $\tanh (x+y)$ *in terms of* $\tanh x$ *and* $\tanh y$. (L.U.)

From (22), $\quad 2 \cos \theta = e^{i\theta} + e^{-i\theta}$.

Let $\theta = ix$, then $\quad \cos ix = \frac{1}{2}(e^{-x} + e^{x}) = \mathbf{\cosh x}$, by (23).

Similarly, $2i \sin ix = e^{-x} - e^{x}$; multiply out by $-i$, then

$$\sin ix = \tfrac{1}{2} i (e^{x} - e^{-x}) = \mathbf{i \sinh x}.$$

From (11*b*), $\quad \cos (A+B) = \cos A \cos B - \sin A \sin B$;

put $\quad A = ix, \text{ and } B = iy,$

then $\quad \cos i(x+y) = \cos ix \cos iy - \sin ix \sin iy,$

or, $\cosh (x+y) = \cosh x \cosh y - i^2 \sinh x \sinh y$, from above results,

i.e., $\quad \mathbf{\cosh (x+y) = \cosh x \cosh y + \sinh x \sinh y.}$

Again, $\quad \tan ix = \sin ix / \cos ix = i \sinh x / \cosh x = \mathbf{i \tanh x}.$

But from (11*c*), writing ix, iy for A, B respectively,

$$i \tanh (x+y) = \tan i(x+y) = (\tan ix + \tan iy)/(1 - \tan ix \,.\, \tan iy)$$
$$= (i \tanh x + i \tanh y)/(1 - i^2 \tanh x \,.\, \tanh y);$$
$$\therefore \tanh (x+y) = \mathbf{(\tanh x + \tanh y)/(1 + \tanh x \tanh y)}.$$

Ex. 9. *If* $x = \log (\sec \theta + \tan \theta)$, *shew that* $\sec \theta = \cosh x$.
Find the values of $\sinh^{-1} x/a$ *and* $\cosh^{-1} x/a$ *in terms of* x *and* a.

(i) Since $\quad x = \log (\sec \theta + \tan \theta),$

$$\therefore \sec \theta + \tan \theta = e^{x},$$

or $\quad \tan^2 \theta = (e^{x} - \sec \theta)^2.$

$$\therefore \sec^2 \theta - 1 = e^{2x} - 2e^{x} \sec \theta + \sec^2 \theta;$$
$$\therefore \sec \theta = (e^{2x} + 1)/2e^{x} = \tfrac{1}{2}(e^{x} + e^{-x}) = \mathbf{\cosh x}.$$

(ii) Let $u = \sinh^{-1} x/a$, then

$$x = a \sinh u = \tfrac{1}{2} a (e^{u} - e^{-u});$$
$$\therefore ae^{2u} - 2xe^{u} + a = 0.$$

Solving this quadratic for e^{u},

$$e^{u} = \frac{x \pm \sqrt{x^2 + a^2}}{a}.$$

Now u is always defined by this equation, when the positive root of the radical is taken; hence, disregarding the negative root and taking logarithms,

$$u = \sinh^{-1} x/a = \log \frac{x + \sqrt{x^2 + a^2}}{a}.$$

In exactly the same way, by taking the exponential value of the hyperbolic cosine, it is readily shewn that

$$\cosh^{-1} x/a = \log \frac{x + \sqrt{x^2 - a^2}}{a}.$$

21. Formulae for Hyperbolic Functions. The formulae proved in the above examples are so important, that they are here collected for future reference.

$$(a)\quad \sin ix = i \sinh x, \quad \cos ix = \cosh x, \quad \tan ix = i \tanh x. \quad \ldots\ldots(24)$$

$$(b)\quad \left.\begin{aligned} &\sinh(x \pm y) = \sinh x \,.\, \cosh y \pm \cosh x \,.\, \sinh y, \\ &\cosh(x \pm y) = \cosh x \,.\, \cosh y \pm \sinh x \,.\, \sinh y, \\ &\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(25)$$

$$(c)\quad \left.\begin{aligned} &\sinh^{-1} \frac{x}{a} = \log \frac{x + \sqrt{x^2 + a^2}}{a}. \\ &\cosh^{-1} \frac{x}{a} = \log \frac{x + \sqrt{x^2 - a^2}}{a}. \\ &\tanh^{-1} \frac{x}{a} = \frac{1}{2} \log \frac{a + x}{a - x}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

22. Summation of Simple Series. The simple methods of finding the sum of a trigonometric series depend mainly upon a skilful application of the identities given in § 13, and sometimes upon the exponential values given in § 19.

Ex. 10. *Find the sum of the first n terms of the series:*

(i) $\cos \alpha + \cos(\alpha + \theta) + \cos(\alpha + 2\theta) + \ldots$;

(ii) $\cos \alpha \cos \beta + \cos(\alpha + \theta) \cos(\beta + \theta)$
$+ \cos(\alpha + 2\theta) \cos(\beta + 2\theta) + \ldots$. (L.U.)

(i) The rth term of this series $= \cos\{\alpha + (r-1)\theta\}$.

Now, from § 13,

$$2\cos\{\alpha + (r-1)\theta\} \,.\, \sin \tfrac{1}{2}\theta = \sin\left(\alpha + \frac{2r-1}{2} \,.\, \theta\right) - \sin\left(\alpha + \frac{2r-3}{2} \,.\, \theta\right);$$

$$\therefore\ 2\sin\tfrac{1}{2}\theta\,.\sum_{r=1}^{n}\cos\{a+(r-1)\theta\}$$

$$=\sum_{r=1}^{n}\left\{\sin\left(a+\frac{2r-1}{2}\,.\,\theta\right)-\sin\left(a+\frac{2r-3}{2}\,.\,\theta\right)\right\}$$

$$=\sin\left(a+\frac{\theta}{2}\right)-\sin\left(a-\frac{\theta}{2}\right)+\sin\left(a+\frac{3\theta}{2}\right)-\sin\left(a+\frac{\theta}{2}\right)+\ldots$$

$$+\sin\left(a+\frac{2n-1}{2}\,.\,\theta\right)-\sin\left(a+\frac{2n-3}{2}\,.\,\theta\right)$$

$$=\sin\left(a+\frac{2n-1}{2}\,.\,\theta\right)-\sin\left(a-\frac{\theta}{2}\right)$$

$$=2\cos\left(a+\frac{n-1}{2}\,.\,\theta\right).\sin\tfrac{1}{2}n\theta:$$

$$\therefore\ \sum_{r=1}^{n}\cos\{a+(r-1)\theta\}=\frac{\cos\left(a+\frac{n-1}{2}\,.\,\theta\right).\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}.$$

Or, let $C_n=\cos a+\cos(a+\theta)+\ldots+\cos\{a+(n-1)\theta\}$;

then, putting in the exponential values from (21), and writing a for e^{ia}, and u for $e^{i\theta}$,

$$2C_n=a+\frac{1}{a}+au+\frac{1}{au}+au^2+\frac{1}{au^2}+\ldots+au^{n-1}+\frac{1}{au^{n-1}}$$

$$=a(1+u+u^2+\ldots+u^{n-1})+\frac{1}{a}\left(1+\frac{1}{u}+\frac{1}{u^2}+\ldots+\frac{1}{u^{n-1}}\right)$$

$$=\frac{a(1-u^n)}{1-u}+\frac{1}{a}\,\frac{1-1/u^n}{1-1/u},\ \text{by (4)},$$

$$=\left\{a\left(u^{\frac{2n-1}{2}}-u^{-\frac{1}{2}}\right)+\frac{1}{a}\left(u^{\frac{1}{2}}-u^{-\frac{2n-1}{2}}\right)\right\}\Big/\left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)$$

$$=\left\{au^{\frac{n-1}{2}}\left(u^{\frac{n}{2}}-u^{-\frac{n}{2}}\right)+\frac{1}{au^{\frac{n-1}{2}}}\left(u^{\frac{n}{2}}-u^{-\frac{n}{2}}\right)\right\}\Big/\left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)$$

$$=\left(au^{\frac{n-1}{2}}+\frac{1}{au^{\frac{n-1}{2}}}\right)\left(u^{\frac{n}{2}}-u^{-\frac{n}{2}}\right)\Big/\left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)$$

$$=2\cos\left(a+\frac{n-1}{2}\,\theta\right)\sin\tfrac{1}{2}n\theta/\sin\tfrac{1}{2}\theta;$$

$$\therefore\ C_n=\cos\left(a+\frac{n-1}{2}\,\theta\right)\sin\tfrac{1}{2}n\theta/\sin\tfrac{1}{2}\theta,\ \text{as before.}$$

(ii) Let $C_n = \cos\alpha\cos\beta + \cos(\alpha+\theta)\cos(\beta+\theta) + \dots$,

the rth term $= \cos\{\alpha+(r-1)\theta\}\cos\{\beta+(r-1)\theta\}$

$= \frac{1}{2}\cos\{\alpha+\beta+2(r-1)\theta\} + \frac{1}{2}\cos(\alpha-\beta)$, by (13c);

$$\therefore\ C_n = \frac{1}{2}\sum_{r=1}^{n}[\cos\{\alpha+\beta+2(r-1)\theta\}+\cos(\alpha-\beta)].$$

Now the first sum is precisely the same as (1), $\alpha+\beta$ replacing α and 2θ replacing θ; the second is constant for every term; hence,

$$C_n = \tfrac{1}{2}\cos\{\alpha+\beta+(n-1)\theta\}\,.\,\sin n\theta/\sin\theta + \tfrac{1}{2}n\cos(\alpha-\beta)$$

$$= \frac{\cos\{\alpha+\beta+(n-1)\theta\}\,.\,\sin n\theta + n\sin\theta\cos(\alpha-\beta)}{2\sin\theta}.$$

Exercises 3.

1. Prove that

$$\sin A + \sin 3A + \sin 5A + \sin 7A = 4\cos A\cos 2A\sin 4A.$$

2. If $A\sin(x+\alpha) = 4\cos x + 3\sin x$ is identically true, find the values of A and α which are constants.

3. The space s described in time t by a moving particle is given by

$$s = (a^2-b^2)\sin\omega t + 2ab\cos\omega t.$$

Shew that this is the same as

$$s = (a^2+b^2)\sin(\omega t+\epsilon),$$

where $\tan\epsilon/2 = b/a$.

Hence find s when $a = 2{\cdot}3$, $b = 0{\cdot}92$, $\omega = 0{\cdot}5$ and $t = 0{\cdot}7008$.

4. Prove that if $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \dfrac{a+b}{a-b}$, then $a\tan\beta = b\tan\alpha$

5. Given that $(1+\sqrt{1+a})\tan\alpha = 1+\sqrt{1-a}$, prove that $\sin 4\alpha = a$.

6. If $\sin\theta + \sin 2\theta = a$, $\cos\theta + \cos 2\theta = b$, shew that

$$(a^2+b^2)(a^2+b^2-3) = 2b.$$

7. Given that $p = 1+\sin^2\theta$, and $q = 1+\cos^2\theta$, shew that

$$2(p^3+q^3) + 9q^2 = 27(1+\cos^4\theta).$$

8. Shew that the relation

$$X = a + b\cos x$$

may be transformed into the relation

$$X(1+t^2) = (a-b)(\mu^2+t^2),$$

where $t = \tan\dfrac{x}{2}$ and $\mu^2 = (a+b)/(a-b)$.

9. If $\sqrt{1-e}\,.\tan\frac{\theta}{2}=\sqrt{1+e}\,.\tan\frac{\phi}{2}$, prove that

$$(1+e\cos\theta)(1-e\cos\phi)=1-e^2.$$

10. If $a\sin^2 A+b\sin^2 B=c$, and $a\sin 2A=b\sin 2B$, shew that

$$\sin^2 A=\frac{c(b-c)}{a(a+b-2c)},$$

and find a similar expression for $\sin^2 B$.

11. Shew that $\tan^{-1}\alpha+\tan^{-1}\beta=\tan^{-1}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right)$; hence shew that

$$\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{10}+\tan^{-1}\frac{8}{21}=\frac{\pi}{4}.$$

12. Prove that $\sin^{-1}P+\sin^{-1}Q=\sin^{-1}R$, where $P=2ab/(a^2+b^2)$, $Q=2cd/(c^2+d^2)$ and $R=2(ac-bd)(ad+bc)/\{(ac-bd)^2+(ad+bc)^2\}$.

Hence shew that $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{56}{65}$.

Solve the following equations :

13. $\sin 3\theta+\cos 5\theta=\cos\theta$.

14. $3\,(\tan^2\theta+\sec 2\theta-7)+14=0$.

15. $\sin 7\theta+\sin 3\theta=13(\cos 7\theta+\cos 3\theta)$.

16. $\sin 2\theta\,.\sec 4\theta-\cos 2\theta+\cos 6\theta=0$.

17. $\sin 2\theta\,.(5+4\cos 2\theta)=3(\sin 3\theta+\sin\theta)$.

18. Calculate the value of x from the equations

$$3\sin\theta+5\cos\theta=5\,;\quad 5\sin\theta-3\cos\theta=x.$$

19. Shew that any complex number z may be written in the form $r(\cos\theta+i\sin\theta)$, where r is real and positive and θ is a real angle.

If, as usual, the symbol $|z|$ denotes r, shew either geometrically or analytically, that z_1 and z_2, being any two complex numbers, $|z_1+z_2|$, is not greater than $|z_1|+|z_2|$, and can only be equal to it when the ratio $z_1:z_2$ is real and positive. (L.U.)

20. Explain how the sum or difference of two complex quantities may be represented by means of an Argand diagram.

If z_1, z_2, z_3 be three complex quantities such that $z_3=z_1+z_2$, and if (r_1,θ_1), (r_2,θ_2), (r_3,θ_3) be the corresponding values of r and θ, prove that if $r_1>r_2$, then $r_1-r_2<r_3<r_1+r_2$, and

$$\theta_1-\sin^{-1}\frac{r_2}{r_1}<\theta_3<\theta_1+\sin^{-1}\frac{r_2}{r_1},$$

where $\sin^{-1}\frac{r_2}{r_1}$ denotes an angle between 0° and 90°. (L.U.)

21. O, A and B are the points which represent zero, unity, and $a+ib$ in the diagram of the complex variable; OC is the internal bisector of the angle BOA, and the parallel through B to OC meets AO in D. Shew that the point in which the circle through B, D and A meets OC represents one of the values of $(a+ib)^{\frac{1}{2}}$. Indicate the point which represents the second value. (L U.)

22. The centre of a regular hexagon is given by $2-i$, one of its vertices by $-1+i$, find the numbers giving the remaining vertices. (L.U.)

23. Define a complex number and prove that in general (i) the product of any number of complex numbers is a complex number, (ii) the quotient of two complex numbers is a complex number.

Express in the form $A+iB$,

$$\text{(i)}\ \frac{1}{(p+iq)^2}+\frac{1}{(p-iq)^2}; \quad \text{(ii)}\ \tan(x+iy). \qquad \text{(L.U.)}$$

24. Express $5+4i$ in the form $r(\cos\theta+i\sin\theta)$, hence extract its square root.

25. By expressing $-2{\cdot}35+1{\cdot}96i$ in the form $r(\cos\theta+i\sin\theta)$, find the three values of $(-2{\cdot}35+1{\cdot}96i)^{\frac{1}{3}}$.

26. Express $\log(a+ib)$ in the form $p+iq$, hence shew that

$$\text{(i)}\ \log\frac{a+ib}{a-ib}=2i\tan^{-1}b/a \quad \text{and} \quad \text{(ii)}\ 1{\cdot}7632+0{\cdot}5405i$$

is one of the logarithms of $5+3i$.

27. Express the complex products

$$\text{(i)}\ (7-8i)(8+7i), \quad \text{(ii)}\ (7-8i)^3(8+7i)^{\frac{1}{3}}$$

in the form $r(\cos\theta+i\sin\theta)$, tabulating corresponding values of r and θ. (L.U.)

28. If $P+iQ=\log(x-a+iy)/(x+a+iy)$, where P and Q are real, find the values of P and Q in terms of x and y. Shew that the family of curves $P=$ constant, $Q=$ constant, cut each other at right angles. (L.U.)

29. If $x+iy=\cosh(u+iv)$, shew that

$$\frac{x^2}{\cosh^2 u}+\frac{y^2}{\sinh^2 u}=1; \quad \frac{x^2}{\cos^2 v}-\frac{y^2}{\sin^2 v}=1. \qquad \text{(L.U.)}$$

30. If $\dfrac{1}{x+iy}+\dfrac{1}{u+iv}=1$, where x, y, u, v are real quantities, find the values of x and y in terms of u and v. (L.U., Sc.)

31. Shew that when a and b are both positive, $a+ib$ may be expressed in the form $\sqrt{a^2+b^2} \,.\, e^{i\theta}$, where $\tan\theta=b/a$.

Examine the cases when a and b are not both positive, illustrating by a diagram.

Given that
$$\frac{1}{\rho}=\frac{1}{Lpi}+Cpi+\frac{1}{R},$$
express ρ in the form $Ae^{i\theta}$, giving the values of A and θ. (L.U.)

32. Express $\cosh(1+i)$ in the two forms, $p+iq$ and $r(\cos\theta+i\sin\theta)$.

The current entering a telephone line is found to be given by the real part of
$$\frac{\cos\omega t+i\sin\omega t}{\cosh(p+ip)}.$$
Express the current in the form $a\sin(\omega t+b)$.

33. Shew that the modulus of the function
$$\frac{2+\cos\varphi+i\sin\varphi}{2+\cos\varphi+\overline{i\sin\varphi}}$$
is unity.

34. Express $(a+ib)^{p+iq}$ in the form $A+iB$, where a, b, p, q are all real; hence determine the condition that the given expression should be real.

35. Shew that if $x=\cos\theta+i\sin\theta$, $2\cos n\theta=x^n+x^{-n}$, and
$$2i\sin n\theta=x^n-x^{-n};$$
hence prove that
$$64\cos^7\theta=\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta.$$ (L.U., Sc.)

36. Prove that $\sin 7\theta/\sin\theta=1+2\cos 2\theta+2\cos 4\theta+2\cos 6\theta$.

37. Shew that

(i) $\cosh(n+1)x+\cosh(n-1)x=2\cosh nx \,.\, \cosh x$,

(ii) $\sinh(n+1)x-\sinh(n-1)x=2\cosh nx \,.\, \sinh x$.

38. If $\tan\frac{x}{2}=\tanh\frac{u}{2}$, prove that $\sinh u=\tan x$, $\cosh u=\sec x$, and $u=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$.

Calculate by the tables $\sinh(0{\cdot}5)$ and $\cosh(0{\cdot}5)$. (L.U.)

39. Define $\tanh x$ and shew that if $x=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, then
$$\tanh x=\sin\theta.$$

Prove that
$$\sinh^{-1}\frac{x}{a}=\log\frac{x+\sqrt{x^2+a^2}}{a}.$$ (L.U.)

40. By expressing $\sinh^{-1} \frac{5}{12}$ in logarithmic form, calculate its value to four places of decimals. (L.U.)

41. Prove that $\tanh^{-1} \dfrac{x^2-1}{x^2+1} = \log x.$ (M.U.)

42. Deduce from the exponential values of $\cos\theta$ and $\sin\theta$, their connection with the hyperbolic functions.

From the identity $\tan 2x = 2\tan x/(1-\tan^2 x)$, find the corresponding value of $\tanh 2x$.

Shew that by a proper choice of A and B, $Ae^{2i\theta} + Be^{-2i\theta}$ can be made equal to $5\cos 2\theta - 7\sin 2\theta$. (L.U.)

Solve the following equations :

43. $e^x - 9{\cdot}3e^{-x} = 8{\cdot}3.$

44. $8e^x + 15e^{-x} = 22.$

45. $\cosh x - 6\tan 23{\cdot}9^\circ = 0{\cdot}6.$

46. $\cosh x + 10\sinh x = 2{\cdot}007.$

47. $9\cosh x + 23\sinh x + 54 = 0.$

48. $x = \cosh^{-1}\left(\frac{58}{40}\right).$

49. $x = \sinh^{-1}(1{\cdot}05).$

50. $\tanh x = 0{\cdot}13 + \log_e 1{\cdot}6.$

51. $x = \tanh^{-1}(\log 1{\cdot}6).$

52. What is the sum of the series

$$1 + \tfrac{1}{2}ae^{i\theta} + \tfrac{1}{2}\cdot\tfrac{3}{4}a^2e^{2i\theta} + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}a^3e^{3i\theta} + \text{to } \infty$$

when $a^2 < 1$?

Deduce the sum of the series

$$1 + \tfrac{1}{2}a\cos\theta + \tfrac{1}{2}\cdot\tfrac{3}{4}a^2\cos 2\theta + \text{to } \infty$$

when θ is an angle in the first quadrant, and $a^2 < 1$. (M.U.)

Sum the following series :

53. $\sin\alpha + \sin\left(\alpha + \dfrac{\pi}{n}\right) + \sin\left(\alpha + \dfrac{2\pi}{n}\right) + \ldots$ to n terms. (L.U.)

54. $\cos x \,.\, \cos 2x + \cos 2x \,.\, \cos 3x + \cos 3x \,.\, \cos 4x + \ldots$ to n terms. (L.U.)

55. $1 + \dfrac{3}{\lfloor 2} + \dfrac{5}{\lfloor 3} + \dfrac{7}{\lfloor 4} + \ldots$ to infinity. (L.U.)

56. $2x - \dfrac{4}{\lfloor 3}x^3 + \dfrac{6}{\lfloor 5}x^5 - \dfrac{8}{\lfloor 7}x^7 + \ldots.$ (B.U., Sc.)

57. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots$ to n terms. (L.U.)

58. $\displaystyle\sum_{r=0}^{5} \cos\frac{2r+1}{13}\,.\,\pi.$

***59.** Prove that $\displaystyle\sum_{r=1}^{n} \sinh(2r-1)x \Big/ \sum_{r=1}^{n} \cosh(2r-1)x = \tanh nx.$

***60.** Shew that $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right) = \dfrac{1}{8}.$

***61.** Find the sum of the lengths of the lines joining a vertex of a regular polygon of n sides to each of the other vertices in terms of the length of a side of the polygon. (L.U.)

***62.** Find the sum to infinity of the series

$$x \cos \alpha - \tfrac{1}{2}x^2 \cos 2\alpha + \tfrac{1}{3}x^3 \cos 3\alpha - \dots . \qquad \text{(L.U., Sc.)}$$

***63.** Find the sum of the lengths of the lines joining any vertex of a regular polygon of n sides to each of the other vertices in terms of the radius of the circle circumscribing the polygon. By making n indefinitely great (or otherwise), shew that the average length of a chord of a circle of radius r drawn through a fixed point on the circumference is $4r/\pi$. (L.U.)

***64.** If C, S denote the infinite series

$$\cos \theta + x \cos 2\theta + x^2 \cos 3\theta + \dots .$$

$$\sin \theta + x \sin 2\theta + x^2 \sin 2\theta + \dots$$

respectively, shew that $C=(\cos \theta - x)/D$, and $S=\sin \theta/D$, where

$$D=1-2x \cos \theta + x^2,$$

it being assumed that x is less than unity.

CHAPTER IV

SPHERICAL TRIGONOMETRY

23. Spherical Triangles. The line of intersection of the surface of a sphere and any plane passing through the centre of the sphere is called a **great circle**.

The figure bounded by the arcs of three great circles is a **spherical triangle.**

Formulae for the Cosine and Sine Rules for spherical triangles will now be investigated.

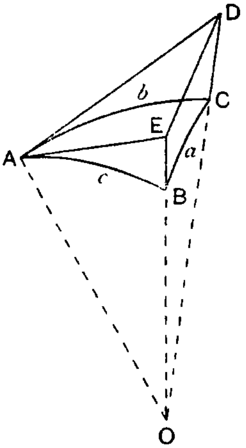

FIG. 3. Spherical triangle.

Ex. 1. *Establish the formula*

$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$,

for a spherical triangle ABC, and deduce from it the rule of sines.

Use the formulae to determine the angles of a triangle in which $a = 48° 18'$, $b = 65° 21'$, *and* $c = 83° 54'$.

Let ABC (Fig. 3) be a spherical triangle formed by the arcs AB, BC and CA of great circles on the sphere whose centre is O.

Suppose the tangent at A to AC meets OC produced in D, and the tangent at A to AB meets OB produced in E. Join DE, then $\angle DAE = A$.

From the triangle DEA, by the cosine rule for plane triangles,

$$DE^2 = EA^2 + AD^2 - 2 \cdot EA \cdot AD \cdot \cos A.$$

Similarly, for triangle DEO,

$$DE^2 = EO^2 + OD^2 - 2 \cdot EO \cdot OD \cdot \cos EOD.$$

Hence

$$EO^2+OD^2-2\,.\,EO\,.\,OD\,.\cos EOD$$
$$=EA^2+AD^2-2\,.\,EA\,.\,AD\,.\cos A.$$

But $\angle EOD=a$, any side being measured by the angle its arc subtends at O;
and since $\angle DAO=\angle EAO=90^\circ$,

$$\therefore\ EO^2=OA^2+AE^2,\quad\text{and}\quad OD^2=OA^2+AD^2,$$

and the above relation becomes

$$EO\,.\,OD\,.\cos a=OA^2+EA\,.\,AD\,.\cos A;$$

$$\therefore\ \cos a=\frac{OA}{OE}\cdot\frac{OA}{OD}+\frac{EA}{OE}\cdot\frac{AD}{OD}\cdot\cos A,$$

i.e. $$\cos a=\cos c\,.\cos b+\sin c\,.\sin b\,.\cos A,$$

which is the required formula.

Similar formulae may be proved for $\cos b$ and $\cos c$, and these may be written

$$\left.\begin{aligned}\cos A&=\frac{\cos a-\cos b\,.\cos c}{\sin b\,.\sin c},\\ \cos B&=\frac{\cos b-\cos c\,.\cos a}{\sin c\,.\sin a},\\ \cos C&=\frac{\cos c-\cos a\,.\cos b}{\sin a\,.\sin b}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

This is the **Cosine Rule** for spherical triangles.

Since $$\sin^2 A=1-\cos^2 A,$$

$\therefore$ by substitution from (27) for $\cos A$,

$$\sin^2 A=1-\left(\frac{\cos a-\cos b\,.\cos c}{\sin b\,.\sin c}\right)^2$$

$$=\frac{\sin^2 b\,.\sin^2 c-(\cos a-\cos b\cos c)^2}{\sin^2 b\,.\sin^2 c}$$

$$=\frac{(1-\cos^2 b)(1-\cos^2 c)-(\cos a-\cos b\,.\cos c)^2}{\sin^2 b\,.\sin^2 c}$$

$$=\frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\,.\cos b\,.\cos c}{\sin^2 b\,.\sin^2 c}.$$

The function forming the numerator of this fraction is usually denoted by $4n^2$, so that

$$\sin A = \frac{2n}{\sin b \, . \, \sin c};$$

$$\therefore \frac{\sin A}{\sin a} = \frac{2n}{\sin a \, . \, \sin b \, . \, \sin c}.$$

In a similar way it may be shewn that

$$\frac{\sin B}{\sin b} = \frac{2n}{\sin a \, . \, \sin b \, . \, \sin c} = \frac{\sin C}{\sin c}.$$

Hence the rule of sines :

$$\left.\begin{aligned} &\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \, . \, \sin b \, . \, \sin c}, \\ \text{where} \quad &4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \, . \, \cos b \, . \, \cos c. \end{aligned}\right\} \quad \ldots\ldots\ldots(28)$$

In the given triangle, substituting in (27),

$$\begin{aligned} \sin 65^\circ \, 24' \, . \, \sin 83^\circ \, 54' \, . \, \cos A &= \cos 48^\circ \, 18' - \cos 65^\circ \, 24' \, . \, \cos 83^\circ \, 54' \\ &= 0 \cdot 6652 - (0 \cdot 4163 \times 0 \cdot 1063) \\ &= 0 \cdot 6652 - 0 \cdot 0442 \\ &= 0 \cdot 6210 ; \end{aligned}$$

$$\begin{aligned} \therefore \log \cos A &= \log 0 \cdot 6210 - \log \sin 65^\circ \, 24' - \log \sin 83^\circ \, 54' \\ &= \bar{1} \cdot 7931 - \bar{1} \cdot 9587 + \bar{1} \cdot 9975 \\ &= \bar{1} \cdot 8369 = \log \cos 46^\circ \, 36' ; \end{aligned}$$

$$\therefore A = \mathbf{46^\circ \, 36'}.$$

Similarly for B,

$$\begin{aligned} \sin 83^\circ \, 54' \, . \, \sin 48^\circ \, 18' \, . \, \cos B &= \cos 65^\circ \, 24' - \cos 83^\circ \, 54' \, . \, \cos 48^\circ \, 18' \\ &= 0 \cdot 4163 - (0 \cdot 1063 \times 0 \cdot 6652) \\ &= 0 \cdot 4163 - 0 \cdot 0707 = 0 \cdot 3456 ; \end{aligned}$$

from which $\log \cos B = \bar{1} \cdot 6680 = \log \cos 62^\circ \, 15'$;

$$\therefore B = \mathbf{62^\circ \, 15'}.$$

Lastly, $\sin 48^\circ \, 18' \, . \, \sin 65^\circ \, 24' \, . \, \cos C$

$$\begin{aligned} &= \cos 83^\circ \, 54' - \cos 48^\circ \, 18' \, . \, \cos 65^\circ \, 24' \\ &= 0 \cdot 1063 - (0 \cdot 6652 \times 0 \cdot 4163) \\ &= 0 \cdot 1063 - 0 \cdot 2769 = -0 \cdot 1706. \end{aligned}$$

Since this value is negative, $\cos C$ must be replaced by

$$\cos(180^\circ - C),\quad \text{because}\quad \cos(180^\circ - C) = -\cos C\,;$$

$$\therefore\ \log\cos(180^\circ - C) = \log 0{\cdot}1706 - \log\sin 48^\circ\,18' - \log\sin 65^\circ\,21'$$

$$= 1{\cdot}2319 - 1{\cdot}8731 - 1{\cdot}9587 = 1{\cdot}4001$$

$$= \log\cos 75^\circ\,21'\,;$$

$$\therefore\ 180^\circ - C = 75^\circ\,21'\,;$$

$$\therefore\ C = \mathbf{104^\circ\,39'}.$$

Having found the angle A, the remaining angles could be found from the sine rule, thus, from (28), by taking logs :

$$\log\sin B = \log\sin A + \log\sin b - \log\sin a$$

$$= \log\sin 46^\circ\,36' + \log\sin 65^\circ\,24' - \log\sin 48^\circ\,18'$$

$$= 1{\cdot}8613 + 1{\cdot}9587 - 1{\cdot}8731 = 1{\cdot}9469$$

$$= \log\sin 62^\circ\,15'\,;$$

$$\therefore\ B = \mathbf{62^\circ\,15'},\ \text{as before.}$$

It must be remembered, however, that the sine rule always gives two possible solutions, since $\sin\theta = \sin(180^\circ - \theta)$, and **unless the data are sufficient to allow the proper value to be selected, it is better to avoid the sine rule.**

24. Spherical Excess. In the above triangle,

$$A + B + C = 46^\circ\,36' + 62^\circ\,15' + 104^\circ\,39'$$

$$= 213^\circ\,30'$$

$$= 180^\circ + 53^\circ\,30'\,;$$

thus **the sum of the three angles of the spherical triangle is greater than 180°.** This is true of every spherical triangle.

If E = spherical excess, *i.e.*

$$\mathbf{E = A + B + C - \pi},$$

the angles being expressed in radians, then

$$\textbf{Area of Spherical Triangle} = \mathbf{Er^2}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(29)}$$

where r = radius of sphere. (See Exercises 16, No. 20.)

Ex. 2. *Prove that in any spherical triangle ABC,*

$$\tan^2\frac{A}{2}=\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin s\,.\,\sin(s-a)},$$

where $2s=a+b+c$.

Use this formula to verify the angles found for the triangle given in Ex. 1, *viz. one in which* $a=48°\ 18'$, $b=65°\ 24'$, *and* $c=83°\ 54'$.

From Plane Trigonometry,

$$\cos A=\frac{1-\tan^2\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}};$$

$$\therefore\ \tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}=\frac{\sin b\,.\,\sin c-\cos a+\cos b\cos c}{\sin b\,.\,\sin c+\cos a-\cos b\cos c},\ \text{from (27)},$$

$$=\frac{\cos(b-c)-\cos a}{\cos a-\cos(b+c)}$$

$$=\frac{\sin\frac{1}{2}(a-b+c)\,.\,\sin\frac{1}{2}(a+b-c)}{\sin\frac{1}{2}(a+b+c)\,.\,\sin\frac{1}{2}(-a+b+c)}$$

$$=\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}.$$

Similar expressions can be likewise proved for $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$.

Hence

$$\left.\begin{aligned}\tan^2\frac{\mathbf{A}}{2}&=\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin s\,.\,\sin(s-a)};\\ \tan^2\frac{\mathbf{B}}{2}&=\frac{\sin(s-c)\,.\,\sin(s-a)}{\sin s\,.\,\sin(s-b)};\\ \tan^2\frac{\mathbf{C}}{2}&=\frac{\sin(s-a)\,.\,\sin(s-b)}{\sin s\,.\,\sin(s-c)},\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

where $\mathbf{2s=a+b+c}$.

In the given triangle,

$$2s=48°\ 18'+65°\ 24'+83°\ 54'=197°\ 36';$$

$\therefore\ s=98°\ 48'$, and $\log\sin s=\bar{1}\cdot 9949$;

$s-a=50°\ 30'$, $\log\sin(s-a)=\bar{1}\cdot 8874$;

$s-b=33°\ 24'$, $\log\sin(s-b)=\bar{1}\cdot 7407$;

$s-c=14°\ 54'$, $\log\sin(s-c)=\bar{1}\cdot 4102$.

$$\therefore \log \tan \frac{A}{2} = \tfrac{1}{2}(1\cdot7407 + \bar{1}\cdot4102 - \bar{1}\cdot9949 - \bar{1}\cdot8874) = \bar{1}\cdot6343$$
$$= \log \tan 23^\circ\ 18' ;$$
$$\therefore\ A = 46^\circ\ 36'.$$

$$\log \tan \frac{B}{2} = \tfrac{1}{2}(\bar{1}\cdot4102 + \bar{1}\cdot8874 - 1\cdot9949 - \bar{1}\cdot7407) = \bar{1}\cdot7810$$
$$= \log \tan 31^\circ\ 8' ;$$
$$\therefore\ B = 62^\circ\ 16'.$$

$$\log \tan \frac{C}{2} = \tfrac{1}{2}(\bar{1}\cdot8874 + 1\cdot7407 - 1\cdot9949 - 1\cdot4102) = 0\cdot1115$$
$$= \log \tan 52^\circ\ 17' ;$$
$$\therefore\ C = 104^\circ\ 34'.$$

These values agree quite closely with those previously found, and it will be seen that formulae (30) are much more convenient to apply than (27), as they are more readily adapted to logarithmic computation.

Ex. 3. *In any spherical triangle ABC, prove the formulae :*

$$(a)\ \tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cdot \cot \tfrac{1}{2}C,$$

$$(b)\ \tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cdot \cot \tfrac{1}{2}C,$$

and apply them to find the shortest distance between Southampton (Lat. 50° 54′ *N., Long.* 1° 24′ *W.) and New Orleans (Lat.* 30° *N., Long.* 90° *W.), regarding the earth as a sphere of radius* 3960 *miles.*

Let
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = k,$$

then
$$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{k \sin A + k \sin B}{k \sin A - k \sin B} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

Hence, from (14),

$$\frac{\sin \frac{1}{2}(a+b) \,.\, \cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b) \,.\, \sin \frac{1}{2}(a-b)} = \frac{\sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)};$$

$$\therefore \frac{\sin \frac{1}{2}(\mathbf{a}+\mathbf{b})}{\sin \frac{1}{2}(\mathbf{a}-\mathbf{b})} \cdot \frac{\cos \frac{1}{2}(\mathbf{a}-\mathbf{b})}{\cos \frac{1}{2}(\mathbf{a}+\mathbf{b})} = \frac{\tan \frac{1}{2}(\mathbf{A}+\mathbf{B})}{\tan \frac{1}{2}(\mathbf{A}-\mathbf{B})}. \quad \text{..............}(\mathbf{31})$$

Again,

$$\tan \tfrac{1}{2}(A+B) = \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)} = \frac{\sin \frac{1}{2}(A+B) \,.\, \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B) \,.\, \cos \frac{1}{2}(A-B)}$$

$$= \frac{\sin A + \sin B}{\cos A + \cos B}, \text{ from (14}a\text{) and (14}c\text{)},$$

$$= \frac{\dfrac{\sin a \,.\, \sin C}{\sin c} + \dfrac{\sin b \,.\, \sin C}{\sin c}}{\dfrac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c} + \dfrac{\cos b - \cos c \,.\, \cos a}{\sin c \,.\, \sin a}}, \text{ from (28) and (27)},$$

$$= \frac{\sin a \,.\, \sin b \,.\, \sin C(\sin a + \sin b)}{\sin a \,.\, \cos a - \sin a \,.\, \cos b \,.\, \cos c + \sin b \,.\, \cos b - \sin b \,.\, \cos c \,.\, \cos a}$$

$$= \frac{\sin a \,.\, \sin b \,.\, \sin C(\sin a + \sin b)}{\frac{1}{2}\sin 2a + \frac{1}{2}\sin 2b - \cos c(\sin a \,.\, \cos b + \sin b \,.\, \cos a)}$$

$$= \frac{\sin a \,.\, \sin b \,.\, \sin C(\sin a + \sin b)}{\sin(a+b)\cos(a-b) - \cos c \,.\, \sin(a+b)}, \text{ from (14)},$$

$$= \frac{\sin a \,.\, \sin b \,.\, \sin C\,(\sin a + \sin b)}{\sin(a+b)\{\cos(a-b) - \cos c\}}$$

$$= \frac{\sin a \,.\, \sin b \,.\, \sin C(\sin a + \sin b)}{\sin(a+b)\{\cos a \,.\, \cos b + \sin a \,.\, \sin b - \sin a \,.\, \sin b \,.\, \cos C - \cos a \,.\, \cos b\}}$$

from (27)

$$= \frac{\sin C \,.\, (\sin a + \sin b)}{\sin(a+b)(1-\cos C)} = \frac{2\sin\frac{C}{2} \,.\, \cos\frac{C}{2} \,.\, 2 \,.\, \sin\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{2\sin\frac{1}{2}(a+b) \,.\, \cos\frac{1}{2}(a+b) \,.\, 2 \,.\, \sin^2\frac{C}{2}}$$

from (12) and (14)

$$= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \,.\, \cot\frac{C}{2}, \text{ which is formula } (a) \text{ of the data.}$$

Divide this result by (31),

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} \,.\, \cot\frac{C}{2} = \tan\tfrac{1}{2}(A-B), \text{ which is formula } (b) \text{ of the data.}$$

Hence

$$\left.\begin{aligned} (a)\ \ \tan\tfrac{1}{2}(\mathbf{A}+\mathbf{B}) &= \frac{\cos\frac{1}{2}(\mathbf{a}-\mathbf{b})}{\cos\frac{1}{2}(\mathbf{a}+\mathbf{b})} \,.\, \cot\tfrac{1}{2}\mathbf{C}, \\ (b)\ \ \tan\tfrac{1}{2}(\mathbf{A}-\mathbf{B}) &= \frac{\sin\frac{1}{2}(\mathbf{a}-\mathbf{b})}{\sin\frac{1}{2}(\mathbf{a}+\mathbf{b})} \,.\, \cot\tfrac{1}{2}\mathbf{C}. \end{aligned}\right\} \quad \text{..................(32)}$$

These formulae are known as **Napier's Analogies.**

Let N in Fig. 4 be the North Pole, S, C, the respective positions of Southampton and New Orleans; draw the meridians through S and C, then the great circle arc SC is the shortest distance between the towns.

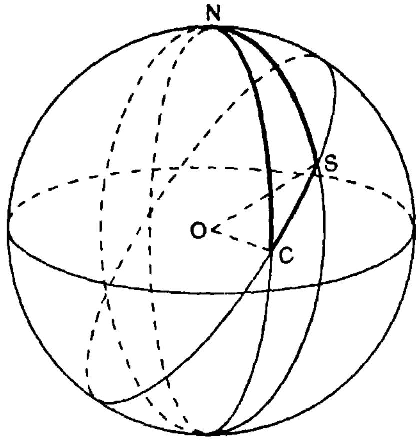

FIG 4. Distance between two places on the earth.

In the spherical triangle NSC,

NS = complement of latitude of $S = 90° - 50° 54' = 39° 6'$,

NC = ,, ,, ,, $C = 90° - 30° = 60°$,

$\angle SNC$ = difference of longitudes of S and C

$= 90° - 1° 24' = 88° 36'$.

Now, to find SC it is necessary to determine the angles NSC, SCN; denoting these by S and C, and applying (32),

$$\begin{aligned}
&\log\tan\tfrac{1}{2}(S+C)\\
&= \log\cos\tfrac{1}{2}(60° - 39°\,6') + \log\cot\frac{88°\,36'}{2} - \log\cos\tfrac{1}{2}(60° + 39°\,6')\\
&= \log\cos 10°\,27' + \log\cot 44°\,18' - \log\cos 49°\,33'\\
&= 1{\cdot}9927 + 0{\cdot}0106 - 1{\cdot}8121 = 0{\cdot}1912\\
&= \log\tan 57°\,13',
\end{aligned}$$

and $\log\tan\tfrac{1}{2}(S-C)$

$$\begin{aligned}
&= \log\sin 10°\,27' + \log\cot 44°\,18' - \log\sin 49°\,33'\\
&= \bar{1}{\cdot}2585 + 0{\cdot}0106 - \bar{1}{\cdot}8813 = \bar{1}{\cdot}3878\\
&= \log\tan 13°\,43'.
\end{aligned}$$

Hence
$$\tfrac{1}{2}(S+C)=57^\circ\ 13',$$
$$\tfrac{1}{2}(S-C)=13^\circ\ 43';$$
$$\therefore\ S=\mathbf{70^\circ\ 56'}\ ;\quad C=\mathbf{43^\circ{\cdot}30'}.$$

Finally, from the sine rule (28),

$$\sin SC=\frac{\sin CNS\ .\ \sin NC}{\sin S}=\frac{\sin 88^\circ\ 36'\ .\ \sin 60^\circ}{\sin 70^\circ\ 56'};$$

$$\therefore\ \log\sin SC=\log\sin 88^\circ\ 36'+\log\sin 60^\circ-\log\sin 70^\circ\ 56'$$
$$=1{\cdot}9999+1{\cdot}9375-\bar{1}{\cdot}9755=1{\cdot}9619$$
$$=\log\sin 66^\circ\ 21';$$
$$\therefore\ SC=66^\circ\ 21'=1{\cdot}158\text{ radians.}$$

This is the angle SOC, and since $OS=OC=3960$ miles,

$$\therefore\ \text{length of arc } SC=1{\cdot}158\times 3960\text{ miles}$$
$$=4585\text{ miles.}$$

$\therefore$ Distance between Southampton and New Orleans is approximately **4585 miles.**

The side SC might have been found from formula (14), as follows:

$$\cos SC=\cos NS\ .\ \cos NC+\sin NS\ .\ \sin NC\ .\ \cos CNS$$
$$=\cos 39^\circ\ 6'\ .\ \cos 60^\circ+\sin 39^\circ\ 6'\ .\ \sin 60^\circ\ .\ \cos 88^\circ\ 36'$$
$$=0{\cdot}3881+0{\cdot}0134=0{\cdot}4015$$
$$=\cos 66^\circ\ 20';$$

$$\therefore\ SC=60^\circ\ 20'=1{\cdot}158\text{ radians};$$

$$\therefore\ \text{length of arc } SC=1{\cdot}158\times 3960\text{ miles}$$
$$=\mathbf{4585\ miles}$$

as before.

It should be observed, in general, that if the positions of S and C are given by lat. α°, long. θ°; lat. β°, long. ϕ°, respectively, then $\angle SNC=\theta\sim\phi$, if longitudes are either both E. or both W., but $\angle SNC=\theta+\phi$, when one is E. and one is W.

Also $\angle NSC=90^\circ\mp\alpha^\circ$, according as the latitude is N. or S., and similarly for $\angle NCS$.

If both latitudes are S., the south pole should be taken as a vertex of the triangle.

EXERCISES 4.

1. Prove that in a spherical triangle ABC, having a right angle at C,

(*a*) $\cos c = \cos a \cdot \cos b$,

(*b*) $\tan c = \tan a \cdot \sec B$,

(*c*) $\sin a = \sin c \cdot \sin A$.

Hence, solve the triangle in which $a = 45° 42'$, and $b = 51° 6'$.

2. Shew that in any triangle ABC, right-angled at C,

$$\cos A = \cos a \cdot \sin B.$$

Hence, solve the triangle in which $A = 68° 18'$, and $B = 43° 54'$.

3. From the results proved in Exs. 1 and 2, shew that the relationships between the sides and angles of a right-angled spherical triangle are given by the two following rules of Napier :

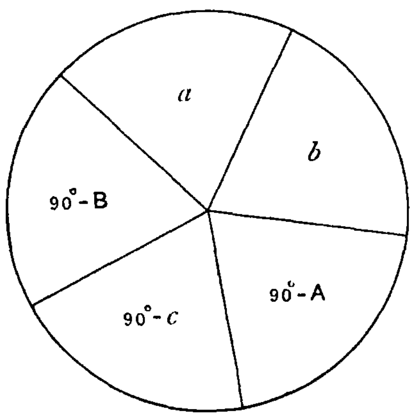

FIG. 5.

Taking C as the right angle, the sides a, b, together with the complements of the remaining parts, are called the **Circular Parts** of the triangle.

If a circle be divided into five sectors, and the circular parts be written in these sectors respectively in the order in which they stand in the triangle, then, after selecting any one part, called the **Middle Part,** the two contiguous parts are called the **adjacent parts,** whilst the two remaining parts are called the **opposite parts.**

Napier's Rules are then :

(i) Sine of Middle Part = Product of tangents of adjacent parts ;

(ii) Sine of Middle Part = Product of cosines of opposite parts.

Solve the following right-angled triangles :

4. $a=38° 30'$, $B=90°$, $c=52° 48'$.

5. $a=71°$, $b=52°$, $C=90°$.

6. $A=71° 18'$, $B=90°$, $C=58° 42'$.

7. $a=50°$, $c=75°$, $C=90°$.

8. $A=37° 18'$, $c=79° 42'$, $C=90°$.

9. $a=42° 36'$, $B=58° 12'$, $C=90°$.

10. $A=a$, $C=90°$.

11. $a=81°$, $B=51° 42'$, $C=90°$.

12. $A=73° 3'$, $B=42° 26'$, $C=90°$.

***13.** If E = spherical excess of a triangle in which $C=90°$, shew that

$$\tan \tfrac{1}{2}E = \tan \tfrac{1}{2}a \,.\, \tan \tfrac{1}{2}b\,;$$

hence, solve the triangle in which $a=82°$, and $b=64°$.

***14.** In a spherical triangle ABC, a perpendicular is drawn from A to BC meeting it in D; shew that

$$\cos CD : \cos DB = \cos B : \cos c,$$

and deduce that $\qquad \cot BD = \tan c \,.\, \cos B.$

Solve the following spherical triangles :

15. $a=62° 15'$, $\quad b=43° 38'$, $\quad c=51° 43'$.

16. $a=48° 51'$, $\quad b=39° 54'$, $\quad c=61° 53'$.

17. $a=42° 15'$, $\quad b=53° 54'$, $\quad c=37° 27'$.

18. $a=63° 41'$, $\quad b=58° 38'$, $\quad c=84° 27'$.

19. $a=41° 16'$, $\quad b=43° 18'$, $\quad c=71° 36'$.

20. $a=48° 42'$, $\quad b=46° 54'$, $\quad C=76° 1'$.

21. $a=68° 18'$, $\quad B=107° 36'$, $\quad c=86° 24'$.

22. $A=55° 10'$, $\quad b=72° 18'$, $\quad c=88° 48'$.

23. $a=82° 18'$, $\quad B=28° 49'$, $\quad c=64° 42'$.

24. $a=44° 24'$, $\quad b=38° 42'$, $\quad C=38° 25'$.

25. In an equilateral triangle, prove that

$$2 \cos \frac{a}{2} = \operatorname{cosec} \frac{A}{2}.$$

***26.** Shew from the cosine and sine rules that

$$\cot a \,.\, \sin b = \cot A \sin C + \cos b \,.\, \cos C\,;$$

hence find A when $a=45°$, $b=2C=60°$.

***27.** Prove that

$$\sin(s-b) \,.\, \sin(s-c) + \sin s \,.\, \sin(s-a) = \sin b \sin c,$$

where $2s = a+b+c$; hence shew from (30) that

$$\cos^2 \frac{A}{2} = \frac{\sin s \,.\, \sin(s-a)}{\sin b \,.\, \sin c},$$

and deduce that $$\sin^2 \frac{A}{2} = \frac{\sin(s-b) \,.\, \sin(s-c)}{\sin b \,.\, \sin c}.$$

***28.** Assuming the expressions for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$, from Ex. 27, prove that $$\sin A = \frac{2\sqrt{\sin s \,.\, \sin(s-a) \,.\, \sin(s-b) \,.\, \sin(s-c)}}{\sin b \,.\, \sin c};$$
hence shew that

$$4 \sin s \,.\, \sin(s-a) \,.\, \sin(s-b) \,.\, \sin(s-c)$$
$$= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \,.\, \cos b \,.\, \cos c.$$

***29.** Shew from the expressions for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ in Ex. 27, that in any spherical triangle:

$$\sin \tfrac{1}{2}(A+B) \,.\, \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a-b) \,.\, \cos \tfrac{1}{2}C.$$

Hence deduce that

$$\cos \tfrac{1}{2}(A+B) \,.\, \cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a+b) \,.\, \sin \tfrac{1}{2}C.$$

If $C = 90°$, prove that

$$\sin(A+B) = \frac{\cos a + \cos b}{1 + \cos a \,.\, \cos b}.$$

***30.** The area of an equilateral triangle is $\frac{1}{16}$th of the whole surface of the sphere upon which it is traced; find the angles of the triangle, and if a is one of the equal sides, shew without tables that

$$\sec a = \sqrt{6} + \sqrt{2} - 1,$$

having given that $$4 \sin 15° = \sqrt{6} - \sqrt{2}.$$

Hence find a from the tables, taking $\sqrt{6} = 2{\cdot}4495$ and $\sqrt{2} = 1{\cdot}4142$.

***31.** Without using tables, and taking the value for sin 15° given in Ex. 30, shew from (32), that, if in a spherical triangle,

$$a+b = \tfrac{2}{3}\pi, \quad a-b = \tfrac{1}{6}\pi, \quad \text{and} \quad C = \tfrac{2}{3}\pi;$$

then $$\tfrac{1}{2}(A+B) = \tan^{-1} \frac{3\sqrt{2}+\sqrt{6}}{6}, \quad \tfrac{1}{2}(A-B) = \tan^{-1} \frac{\sqrt{6}-\sqrt{2}}{6}.$$

Hence, from the formula of Ex. 11, page 54, shew that

$$\tan A = \frac{5\sqrt{6}+6\sqrt{2}}{13}, \quad \tan B = \frac{9\sqrt{2}-\sqrt{6}}{13}.$$

***32.** The angular distance between two places, B, C, measured along the arc of a great circle on the earth's surface, is $a°$. B is situated in latitude $\theta°$ N., longitude $\mu°$ W., whilst C is situated in latitude $\phi°$ N., longitude $\psi°$ W., μ being greater than ψ. Prove that

$$\cos a = \cos\theta . \cos\phi . \cos(\mu - \psi) + \sin\theta . \sin\phi,$$

assuming the earth to be a perfect sphere.

Hence shew that if C were on the equator and the longitude of B were $(\psi + 60°)$ W., then

$$2\cos a = \cos\theta.$$

***33.** The sum of the sides of a spherical triangle is $180°$; prove that

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1.$$

Find the angles of the triangle in which

$$a = 58°, \quad b = 52°, \quad c = 70°,$$

and verify the above formula.

Assuming the Earth to be a sphere of mean radius 3960 miles, calculate the shortest distance, measured along the arc of a great circle, between each of the following pairs of towns, using the appended table of approximate latitudes and longitudes.

Town.	Latitude	Longitude
Adelaide	34° 57′ S.	138° 38′ E.
Bombay	18° 55′ N.	72° 54′ E.
Cape Town	33° 40′ S.	18° 30′ E.
Gibraltar	36° 10′ N.	5° 20′ W.
Havre	49° 30′ N.	0° 10′ E.
Honolulu	21° 18′ N.	157° 51′ W.
Liverpool	53° 25′ N.	2° 59′ W.
London	51° 30′ N.	0° 5′ W.
Melbourne	37° 50′ S.	144° 59′ E.
New York	40° 45′ N.	74° 0′ W.
Plymouth	50° 22′ N.	4° 9′ W.
Port Nelson	56° 30′ N.	92° 59′ W.
Quebec	46° 48′ N.	71° 13′ W.
Yokohama	35° 30′ N.	139° 35′ E.

34. Liverpool to Port Nelson.

35. Liverpool to Quebec.

36. Liverpool to New York.

37. London to Gibraltar.

38. Havre to New York.

39. Plymouth to Cape Town.

40. Cape Town to Adelaide.

41. Bombay to Melbourne.

42. Yokohama to Honolulu.

43. From A, a point on the Earth's surface in latitude 51°, a distance AB of length 20,000 feet is measured along a great circle perpendicular to the meridian through A. Calculate the angle which the great circle through B perpendicular to BA makes with the meridian through B. (Consider the Earth as a sphere of radius 3960 miles.) (L.U.)

***44.** Two ports are in the same latitude l, their difference of longitude being 2λ. Shew that the distance saved in sailing from one port to the other along a great circle instead of due East or West is

$$2r\{\lambda \cos l - \sin^{-1}(\sin \lambda \cos l)\},$$

where r is the radius of the Earth.

Calculate the distance thus saved if the latitude is 60° and the difference in longitude 90°, taking the radius of the Earth as 3960 miles. (L.U.)

45. Calculate the great circle distance between Singapore (1° 24′ N., 103° 51′ E.) and Yokohama. (Lc.U.)

46. For a spherical triangle ABC, prove that

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \qquad \text{(M.U., Sc.)}$$

CHAPTER V

DIFFERENTIATION

25. Total Differential Coefficient. Let P (x, y) be a point on a curve $y=f(x)$, and let $P'(x+\Delta x, y+\Delta y)$ be another point on the curve close to P, then from Fig. 6, $\tan\theta = RP'/PR = \Delta y/\Delta x$.

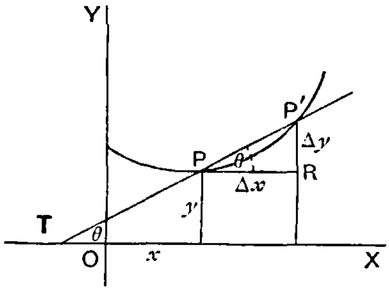

FIG. 6. Differential coefficient.

Now let P' approach P so that ultimately the secant TPP' becomes the tangent at P; then

$$\operatorname*{Lt}_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \tan\theta,$$

where θ is the angle made by the tangent at P with the x-axis. This angle is called the **slope** and $\tan\theta$ the **gradient** of the curve at P, and $\operatorname*{Lt}_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ is denoted by $\dfrac{dy}{dx}$; hence

$$\left.\begin{array}{l} \dfrac{\mathbf{dy}}{\mathbf{dx}} = \tan\theta, \\ \textbf{and the gradient of the curve } \mathbf{y=f(x)} \textbf{ at the point } \mathbf{(x, y)} \textbf{ is } \dfrac{\mathbf{dy}}{\mathbf{dx}}. \end{array}\right\} \quad \textbf{(33)}$$

$\frac{dy}{dx}$ is the **total differential coefficient of y** with respect to x, and expresses the rate at which y changes with x.

In general, if Δx, Δy are corresponding increments in x and y, then since $y=f(x)$, $y+\Delta y=f(x+\Delta x)$, so that

$$\Delta y=f(x+\Delta x)-f(x);$$

$$\therefore \frac{dy}{dx}=\underset{\Delta x\to 0}{\mathrm{Lt}}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

It is evident from this result that the differential coefficient of $f(x)$ is also a function of x, so that it has a gradient at every finite point. This may be written $\frac{d}{dx}\cdot\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$, and is called the **second derivative** or differential coefficient of $f(x)$. Similarly the process may be repeated n times, the nth derivative being written $\frac{d^ny}{dx^n}$.

Ex. 1. *Differentiate* $\sin\frac{\theta}{2}$ *with regard to* θ *from first principles.* (L.U.)

Let $y=\sin\frac{\theta}{2}$, and $\Delta\theta$, Δy be corresponding increments in θ and y, then

$$y+\Delta y=\sin\frac{\theta+\Delta\theta}{2};$$

$$\therefore \Delta y=\sin\frac{\theta+\Delta\theta}{2}-\sin\frac{\theta}{2}=2\cos\left(\frac{\theta}{2}+\frac{\Delta\theta}{4}\right)\sin\frac{\Delta\theta}{4},\ \text{by (11}a\text{)};$$

$$\therefore \frac{\Delta y}{\Delta\theta}=\frac{2\cos\left(\frac{\theta}{2}+\frac{\Delta\theta}{4}\right)\cdot\sin\frac{\Delta\theta}{4}}{\Delta\theta}=\frac{1}{2}\cdot\cos\left(\frac{\theta}{2}+\frac{\Delta\theta}{4}\right)\frac{\sin\frac{\Delta\theta}{4}}{\frac{\Delta\theta}{4}};$$

$$\therefore \frac{dy}{d\theta}=\frac{1}{2}\underset{\Delta\theta\to 0}{\mathrm{Lt}}\cos\left(\frac{\theta}{2}+\frac{\Delta\theta}{4}\right)\frac{\sin\frac{\Delta\theta}{4}}{\frac{\Delta\theta}{4}}$$

$$=\frac{1}{2}\cos\frac{\theta}{2},\ \text{since}\ \underset{\Delta\theta\to 0}{\mathrm{Lt}}\frac{\sin\frac{\Delta\theta}{4}}{\frac{\Delta\theta}{4}}=1.$$

26. Fundamental Standard Forms. The determination of the differential coefficient of any function of a single variable depends upon the following elementary standard forms :

$$\left.\begin{array}{lll} \text{(i)} & y = ax^n, & y_1 = nax^{n-1}, \\ \text{(ii)} & y = ae^{bx}, & y_1 = abe^{bx}, \\ \text{(iii)} & y = \sin ax, & y_1 = a\cos ax. \\ \text{(iv)} & y = \cos ax, & y_1 = -a\sin ax, \end{array}\right\} \qquad \text{..................(34)}$$

where
$$y_1 \equiv \frac{dy}{dx},$$

together with the important theorems established in § 27.

Ex. 2. Find the differential coefficients of the following functions : a^x, $\log_a x$, $\sin^{-1}x$; and deduce the derivatives of $\log_e x$ and $\cos^{-1}x$.

(i) Let $y = a^x$, and suppose that $a = e^c$, so that $\log_e a = c$, then $y = e^{cx}$; hence by (34) $y_1 = ce^{cx} = a^x \log_e a$.

(ii) Let $y = \log_a x$, then $x = a^y$;

$$\therefore \text{ by (i),} \qquad \frac{dx}{dy} = a^y \,.\, \log_e a = x \log_e a.$$

Now let $k = \log_e a$, then $a = e^k$, so that $e = a^{\frac{1}{k}}$;

$$\therefore \log_a e = \frac{1}{k} = \frac{1}{\log_e a};$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \,.\, \log_a e.$$

(iii) Let $y = \sin^{-1}x$, then $x = \sin y$, and $\dfrac{dx}{dy} = \cos y = \sqrt{1 - x^2}$;

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

In (ii) let $a = e$, then $y = \log_e x$, and

$$\frac{dy}{dx} = \frac{1}{x}.$$

Also
$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x;$$

$$\therefore \frac{d}{dx}(\cos^{-1}x) = -\frac{d}{dx}(\sin^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, \text{ by (iii).}$$

27. Function of a Function, Products and Quotients of Functions. Let $y=f(u)$, $u=F(v)$, $v=\phi(x)$, all the functions being continuous; and let Δv, Δu, Δy be increments corresponding to a small increment Δx in x, then

$$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta v}\cdot\frac{\Delta v}{\Delta x};$$

hence, on proceeding to the limit,

$$\frac{dy}{dx}=\underset{\Delta x=0}{\mathrm{Lt}}\frac{\Delta y}{\Delta x}=\underset{\Delta u=0}{\mathrm{Lt}}\frac{\Delta y}{\Delta u}\cdot\underset{\Delta v=0}{\mathrm{Lt}}\frac{\Delta u}{\Delta v}\cdot\underset{\Delta x=0}{\mathrm{Lt}}\frac{\Delta v}{\Delta x},$$

since Δy, Δu, Δv each tend to zero with Δx;

hence $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}. \qquad \text{(35a)}$$

Again, let $y=uv$, where $u=F(x)$ and $v=\phi(x)$, then

$$\log y=\log u+\log v.$$

Hence, by the results of Ex. 2 and the preceding paragraph,

$$\left.\begin{aligned}\frac{1}{y}\cdot\frac{dy}{dx}&=\frac{1}{u}\cdot\frac{du}{dx}+\frac{1}{v}\cdot\frac{dv}{dx},\\ \text{or}\quad \frac{dy}{dx}&=v\cdot\frac{du}{dx}+u\cdot\frac{dv}{dx}.\end{aligned}\right\} \qquad \text{(35b)}$$

Similarly, if $y=u/v$,

$$\left.\begin{aligned}\frac{1}{y}\cdot\frac{dy}{dx}&=\frac{1}{u}\cdot\frac{du}{dx}-\frac{1}{v}\cdot\frac{dv}{dx}.\\ \text{or}\quad \frac{dy}{dx}&=\left(v\cdot\frac{du}{dx}-u\frac{dv}{dx}\right)\Big/v^2.\end{aligned}\right\} \qquad \text{(35c)}$$

Ex. 3. *Differentiate* $\tan x$, $\tan^{-1}x$, *and* $\sec x$ *with respect to* x.

(i) Let $$y=\tan x=\sin x/\cos x,$$

then $$\log y=\log\sin x-\log\cos x;$$

$$\therefore\ \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{\sin x}\cdot\frac{d}{dx}\cdot(\sin x)-\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)$$

$$=\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x},\ \text{by (34)};$$

$$\therefore\ \frac{dy}{dx}=1+\tan^2 x=\sec^2 x.$$

(ii) Let $y = \tan^{-1}x$, then $x = \tan y$, so that $\dfrac{dx}{dy} = \sec^2 y = 1 + x^2$;

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}.$$

(iii) Let $y = \sec x$, and suppose $u = \cos x$, then $y = u^{-1}$;

$$\therefore \frac{du}{dx} = -\sin x \quad \text{and} \quad \frac{dy}{du} = -\frac{1}{u^2};$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{\sin x}{\cos^2 x} = \sec^2 x \,.\, \sin x.$$

28. Summary of Standard Forms. The results of Examples 2 and 3 are so important that they are here collected for reference.

(i) $y = a^x$, $\quad y_1 = a^x \log_e a$.	(v) $y = \sin^{-1} x$, $\quad y_1 = (1-x^2)^{-\frac{1}{2}}$.	
(ii) $y = \log_a x$, $\quad y_1 = \dfrac{1}{x} . \log_a e$.	(vi) $y = \cos^{-1} x$, $\quad y_1 = -(1-x^2)^{-\frac{1}{2}}$.	(36)
(iii) $y = \log_e x$, $\quad y_1 = \dfrac{1}{x}$.	(vii) $y = \tan^{-1} x$, $\quad y_1 = (1+x^2)^{-1}$.	
(iv) $y = \tan x$, $\quad y_1 = \sec^2 x$.	(viii) $y = \sec x$, $\quad y_1 = \sin x \,.\, \sec^2 x$.	

Ex. 4. *Differentiate the functions :*

(*a*) $y = x^5 \log a + 5a^x + 2\tan^{-1} bx - \log(1-bx) + \log(1+bx)$,

(*b*) $y = \log \sin \dfrac{3x-4}{4x-3}$.

(*a*) From the standard forms,

$$\frac{dy}{dx} = 5x^4 \log a + 5a^x \log a + \frac{2b}{1+b^2x^2} + \frac{b}{1-bx} + \frac{b}{1+bx}$$

$$= 5(x^4 + a^x) \log a + \frac{4b}{1-b^4x^4}.$$

(*b*) Let $u = \dfrac{3x-4}{4x-3}$, then, by (35*c*),

$$\frac{du}{dx} = \frac{3(4x-3) - 4(3x-4)}{(4x-3)^2} = 7/(4x-3)^2.$$

Let $\quad v = \sin u$, then $\dfrac{dv}{du} = \cos u = \cos \dfrac{3x-4}{4x-3}$,

and $\quad y = \log v$, so that $\dfrac{dy}{dv} = \dfrac{1}{v} = \operatorname{cosec} \dfrac{3x-4}{4x-3}$;

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \frac{7}{(4x-3)^2} \cot \frac{3x-4}{4x-3}.$$

Ex. 5. *Find the gradient at the point* (x, y) *of each of the following curves :*

$$(a)\ 3x^2+12xy-12y^2+18x-6y+5=0,$$

$$(b)\ y^2=\frac{3x^2-5}{5x^2-3}.$$

(*a*) Since $\quad 3x^2+12xy-12y^2+18x-6y+5=0,$

$\therefore$ by differentiation,

$$6x+12(xy_1+y)-24yy_1+18-6y_1=0,$$

from which
$$y_1 \equiv \frac{dy}{dx}=\frac{3+x+2y}{1-2x+4y}.$$

(*b*) Taking logarithms,

$$2 \log y=\log(3x^2-5)-\log(5x^2-3);$$

$$\therefore \frac{2}{y}\cdot\frac{dy}{dx}=\frac{6x}{3x^2-5}-\frac{10x}{5x^2-3}=\frac{32x}{(3x^2-5)(5x^2-3)};$$

$$\therefore \frac{dy}{dx}=\frac{16x}{(3x^2-5)(5x^2-3)}\cdot\sqrt{\frac{3x^2-5}{5x^2-3}}=\frac{16x}{\sqrt{(3x^2-5)(5x^2-3)^3}}.$$

Ex. 6. *Find the nth differential coefficients of* $e^{ax}\sin bx$ *and* $1/(x^2-a^2)$. *If* $y=(ax+b)/(cx+d)$, *prove that*

$$2\cdot\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}=3\left(\frac{d^2y}{dx^2}\right)^2. \qquad \text{(L.U.)}$$

(i) Let
$$y=e^{ax}\sin bx,$$

then
$$y_1=e^{ax}(a\sin bx-b\cos bx)$$
$$=(a^2+b^2)^{\frac{1}{2}}e^{ax}\sin(bx+\phi),$$

where
$$\tan^{-1}\frac{b}{a}=\phi.$$

Repeating the process,

$$y_2=(a^2+b^2)e^{ax}\sin(bx+2\phi).$$

Hence
$$y_n=(a^2+b^2)^{\frac{n}{2}}e^{ax}\sin\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

(ii) Let
$$y=\frac{1}{x^2-a^2}=\frac{1}{2a}\left\{\frac{1}{x-a}-\frac{1}{x+a}\right\};$$

$$\therefore y_1=-\frac{1}{2a}\left\{\frac{1}{(x-a)^2}-\frac{1}{(x+a)^2}\right\}.$$

Hence, on repeating the process,

$$y_n = (-1)^n \frac{\lfloor n}{2a}\left\{\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right\}.$$

(iii) The numerator $= ax + b = \frac{a}{c}\left(cx + \frac{bc}{a}\right) = \frac{a}{c}\left(cx + d + \frac{bc}{a} - d\right)$;

hence $$y = \frac{a}{c} + \frac{k}{cx+d}, \text{ where } k = b - \frac{ad}{c};$$

$$\therefore\ y_1 = -\frac{kc}{(cx+d)^2},\quad y_2 = \frac{2kc^2}{(cx+d)^3},\quad y_3 = -\frac{6kc^3}{(cx+d)^4};$$

$$\therefore\ 2y_1y_3 = \frac{12k^2c^4}{(cx+d)^6} = 3\left\{\frac{2kc^2}{(cx+d)^3}\right\}^2 = 3y_2^2.$$

Exercises 5a.

Find the differential coefficient with respect to x of each of the following functions :

1. $(2x-3)(5x-1)$. **2.** $(3x+4)(x-1)(2x+7)$. **3.** $\frac{1}{3}x^{-2}$.

4. $\frac{1}{2\sqrt{x}} + \frac{2}{x^3}$. **5.** $\frac{2}{3x+2}$. **6.** $\frac{7}{(2x+3)^2}$.

7. 10^x. **8.** ae^{cx}. **9.** a^{ax+b}.

10. $\log(4x-9)$. **11.** $\tan 4x$. **12.** $a\sin(5x+3)$.

13. $\sin^{-1}\frac{x}{2a}$. **14.** $\sin^{-1}\frac{3x-1}{5}$. **15.** $\tan^{-1}\frac{3x}{a}$.

16. $\tan^{-1}(2x-1)$. **17.** $\sinh(2x+3)$. **18.** $\cosh^{-1}\frac{4x}{5}$.

19. $x^5 + \log\frac{x}{5}$. **20.** $\tan x - x$.

Find the gradient at the point (x, y) of each of the following curves :

21. $y = (6x^2 - 2x + 7)^5$. **22.** $y = (a^2 + x^2)^{\frac{3}{2}}$. **23.** $y = (b^2 - x^2)^{\frac{2}{3}}$.

24. $y = \frac{1}{\sqrt{4x^2+3x+1}}$. **25.** $y = \log(x + \sqrt{x^2-1})$.

26. $y = \log\tan\left(2x + \frac{\pi}{4}\right)$. **27.** $y = \cos^3 2x$.

28. $y = a\cos^4\frac{x}{a}$. **29.** $y = \log\sin x$. **30.** $y = \tan^3 3x$.

31. $y = x\sqrt{x^2+2x+4}$. **32.** $y = x^2 \log 4x$. **33.** $y = (x-a)^2(x-b)^3$.

34. $y = x^2 \log (x+3)$. **35.** $y = x \log \cot x$. **36.** $y = 4x \sin x$.

37. $y = (4x-3) \sin 3x$. **38.** $y = \tan (x+5) \log (x+9)$.

39. $y = 2x^2 \sin (3x+2)$. **40.** $y = a \sin^4(bx+c)$.

41. $y = x \sin (\log x)$. **42.** $y = e^x \sin x$. **43.** $y = e^{\sin 2x}$.

44. $y = e^{-2x} \sin 3x$. **45.** $y = \dfrac{2x}{1-x^2}$. **46.** $y = \dfrac{1-2x}{3+x^2}$.

47. $y = \dfrac{3x}{2x-1}$. **48.** $y = \dfrac{3x+7}{4x-9}$. **49.** $y\sqrt{a^2+x^2} = x$.

50. $y(a-bx)^2 = (a+bx)^2$. **51.** $y(1-x) = 2+x^2$.

52. $y(x^2-x+1) = 3-5x$. **53.** $y\sqrt{1+x^2} = 1-x^2$.

54. $y\sqrt{a^2+x^2} = 2-x$. **55.** $y = \dfrac{2x^2-3x+5}{3x-5}$. **56.** $y = \dfrac{7x^2-4x+2}{5-3x}$.

57. $y = \dfrac{4x^2-3x-7}{5x^2+6x-9}$. **58.** $y^2 = \dfrac{a-2x}{a+3x}$. **59.** $y = \dfrac{x^2(5+x)^3}{(3+x)^5}$.

60. $y = \log \dfrac{4x^2+3}{2x^2+5}$. **61.** $y = \log \dfrac{x^2+x+1}{x^2-x+1}$. **62.** $y = x^2 \log \dfrac{5x-9}{6x+11}$.

63. $y = \log \dfrac{1+x \sin x}{1-x \sin x}$. **64.** $y = \dfrac{\log x}{1+x \log x}$.

65. $y = a^{2 \sin x} \log \dfrac{\sqrt{a^2-x^2}}{a+2x}$. **66.** $y = (b^2-x^2)^{2c}$.

67. $y = e^{ax} \cos (bx+c)$. **68.** $y(1+\cos x) = \sin x$.

69. $y(a+b \sin x) = \cos x$. **70.** $y = x^2 \tan^{-1} \dfrac{2x-3}{3x-2}$.

71. $y = \tan^{-1} \dfrac{x(3-x^2)}{1-3x^2}$. **72.** $y = \tan^{-1} (\sec x + \tan x)$.

73. $y = \dfrac{4(1-x)}{3(2-x)} . \sin^{-1}(2x-1)$. **74.** $y = \sin \dfrac{e^x}{a} . \log \dfrac{1}{x}$.

***75.** $y = (\sin x)^x$. **76.** $y = e^x \sin^2 x$. ***77.** $(\tan x)^y = (\tan y)^x$.

78. $ax^2 + 2hxy + by^2 = 1$. ***79.** $y = x^y$. ***80.** $y = e^{x^x}$.

***81.** $y = x^{x^x}$. ***82.** $y = \sin^{-1}\left(\dfrac{a+b \cos x}{b+a \cos x}\right)$.

***83.** $y = x^x + x^{\frac{1}{x}}$. ***84.** $e^y = \log x$. ***85.** $x^2 = e^{xy}$.

***86.** $y \log \cos x = \log \sin x$. ***87.** $x \sin y = y \tan x$.

***88.** $x = be^{x+y}$. ***89.** $x^2(x^2+y^2) = a^2(y^2-x^2)$.

90. $y=\tan^{-1}x+\frac{1}{2}\log\frac{1+x}{1-x}$.

91. If $y=\sec\theta$, shew that $y\cdot\frac{d^2y}{d\theta^2}+\left(\frac{dy}{d\theta}\right)^2=y^2(3y^2-2)$.

92. Evaluate $\frac{d}{dx}\tan^{-1}\frac{x+b}{a}$. (L.U.)

93. Find $\frac{dy}{dx}$,
(i) when $y=10^{-5x}$,
(ii) when $y=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$,
(iii) when $y=\sin^{-1}(1-3x^2)$. (L.U.)

94. If $xy=ax^3+b$, prove that $x^2\frac{d^2y}{dx^2}=2y$.

95. Given that $y=k\cos(\log x)+l\sin(\log x)$, shew that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

96. Evaluate $\frac{d}{dx}\log(\sin x^2)$. (L.U.)

97. If $xy=ax^2+\frac{b}{x}$, prove that $x^2\frac{d^2y}{dx^2}+2\left(x\frac{dy}{dx}-y\right)=0$.

98. Differentiate $\operatorname{cosec} x$ and $\tan^{-1}\left(\frac{b}{a}\tan x\right)$.

Find $\left(\frac{d}{dx}\right)^n\frac{1}{(x-1)(2x+1)}$; and shew that if $y=(\sin^{-1}x)^2$,

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0. \qquad \text{(L.U.)}$$

99. If $y=x\sin x$, find the value of $\frac{d^5y}{dx^5}$; hence, shew that

$$\frac{d^5y}{dx^5}-\frac{dy}{dx}=4\frac{y}{x}.$$

***100.** Shew that, if the independent variable x in the differential coefficient dy/dx can be expressed in terms of t, where x is a given function of t, then

$$\frac{dy}{dx}\text{ becomes }\frac{dy}{dt}\bigg/\frac{dx}{dt},$$

and
$$\frac{d^2y}{dx^2}\text{ becomes }\left(\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right)\bigg/\left(\frac{dx}{dt}\right)^3.$$

Prove that if $x=e^t$, the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+p^2y=0\text{ becomes }\frac{d^2y}{dt^2}+p^2y=0. \qquad \text{(L.U.)}$$

***101.** A ladder AB, l feet long, hinged at its lower end A, is raised to a vertical position by a rope attached to it at a point C distant s ft. from A. If C moves with a uniform horizontal velocity u ft. per sec., shew that when B has risen y ft. vertically, its vertical velocity is $lu\sqrt{l^2-y^2}/sy$ ft. per sec., and that its actual velocity on the circle described is l^2u/sy ft. per sec.

Calculate the value of y when $l=26$, $s=11$, $u=4{\cdot}5$ and B's vertical velocity is 4 ft. per sec.

***102.** A heavy uniform rod of weight W is suspended symmetrically and in a horizontal position by two vertical strings of length l distant $2a$ apart; a couple of moment M is applied twisting the bar about the vertical axis of symmetry through an angle θ; shew by virtual work, or otherwise, that

$$M=\frac{Wa^2\sin\theta}{\sqrt{l^2-4a^2\sin^2\frac{\theta}{2}}}. \qquad \text{(L.U.)}$$

PARTIAL DIFFERENTIATION

29. Partial Derivative. Let $u=f(x, y)$ be a continuous function of two variables, and suppose any constant value be given to y, so that u becomes a function of x only, then the differential coefficient of u with respect to x is called the **partial differential coefficient** of $f(x, y)$ with respect to x, and is written $\frac{\partial u}{\partial x}$ to distinguish it from the total derivative $\frac{du}{dx}$.

Similarly, if a constant value be assigned to x, then $\frac{\partial u}{\partial y}$ is the partial derivative of $f(x, y)$ with respect to y. In the same way partial derivatives of higher orders are defined, and, in general, for a function of n variables, the partial derivatives with respect to any one variable are formed according to the ordinary rules of differentiation, after the remaining $(n-1)$ variables are considered to have constant values.

Ex. 7. *For the following curves :*

$$(a)\ ax^2+2hxy+by^2=0, \quad (b)\ x^3+ax^2y+bxy^2+cy^3=0,$$

find $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ *and* $\dfrac{dy}{dx}$ *; hence exhibit the relation between these three derivatives, and shew that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu$,
where u stands for the whole expression equated to zero, and n is the degree of each expression.

(*a*) Let $u=ax^2+2hxy+by^2$, then, considering y to be constant,

$$\frac{\partial u}{\partial x}=2ax+2hy,$$

and considering x to be constant, $\dfrac{\partial u}{\partial y}=2hx+2by.$

Further, since $\quad ax^2+2hxy+by^2=0,$

$\therefore$ by differentiation, when both x and y vary,

$$2ax+2h\left(y+x\frac{dy}{dx}\right)+2by\frac{dy}{dx}=0;$$

$$\therefore\ 2(hx+by)\frac{dy}{dx}=-2(ax+hy),$$

or
$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=-\frac{\partial u}{\partial x},$$

or
$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial u}{\partial x}=0.$$

Since each term involving the variables is of the 2nd degree,

$$\therefore\ n=2.$$

But
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2ax^2+2hxy+2hxy+2by^2$$
$$=2(ax^2+2hxy+by^2)=2u.$$

(*b*) Let $u=x^3+ax^2y+bxy^2+cy^3$, then, with y constant,

$$\frac{\partial u}{\partial x}=3x^2+2axy+by^2,$$

and with x constant, $\dfrac{\partial u}{\partial y}=ax^2+2bxy+3cy^2.$

Finally, when both x and y vary,

$$3x^2+a\left(2xy+x^2\frac{dy}{dx}\right)+b\left(y^2+2xy\frac{dy}{dx}\right)+3cy^2\frac{dy}{dx}=0\;;$$

$$\therefore\;(ax^2+2bxy+3cy^2)\frac{dy}{dx}=-(3x^2+2axy+by^2)\;;$$

$$\therefore\;\frac{dy}{dx}=-\frac{3x^2+2axy+by^2}{ax^2+2bxy+3cy^2}.$$

The numerator and denominator of this fraction are equal to $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ respectively, so that, again,

$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial u}{\partial x}=0.$$

The degree of the expression is obviously 3 :

$$\therefore\;x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3x^3+2ax^2y+bxy^2+ax^2y+2bxy^2+3cy^3$$
$$=3(x^3+ax^2y+bxy^2+cy^3)=3u.$$

30. General Theorems. The formula

$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial u}{\partial x}=0,\quad\text{or}\quad\frac{dy}{dx}=-\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}\quad\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

is generally true, and is very useful for finding the total derivative of any implicit function, $f(x, y)=0$, as the above example shews.

The formula $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ is illustrative of **Euler's Theorem on Homogeneous Functions,** which may be stated generally as follows.

If $u=f(x_1, x_2, \ldots x_n)$ be a homogeneous function in n variables and of the mth degree, then

$$x_1\frac{\partial u}{\partial x_1}+x_2\frac{\partial u}{\partial x_2}+x_3\frac{\partial u}{\partial x_3}+\ldots+x_n\frac{\partial u}{\partial x_n}=mu.\quad\ldots\ldots\ldots\ldots\ldots(38)$$

Ex. 8. *A rectangular plate is x inches long and y inches wide ; shew by considering an infinitesimal expansion of the plate, that if a be its area, then*

$$da=\frac{\partial a}{\partial x}\cdot dx+\frac{\partial a}{\partial y}\cdot dy.$$

Here $\qquad a=xy.$

Let the side x increase to $x+\Delta x$, whilst y remains constant, the corresponding increase in area being Δa ;

then $\qquad a+\Delta a=(x+\Delta x)y$;

and since $a=xy$, $\qquad \therefore\ \Delta a=y\,.\,\Delta x$,

so that $$\frac{\partial a}{\partial x}=\underset{\Delta x\to 0}{\mathrm{Lt}}\frac{\Delta a}{\Delta x}=y.$$

This gives the rate at which the area increases as x increases.

Similarly, $$\frac{\partial a}{\partial y}=x.$$

If, however, both x and y increase together, then

$$\frac{da}{dx}=\frac{d}{dx}(xy)=y+x\cdot\frac{dy}{dx}=\frac{\partial a}{\partial x}+\frac{\partial a}{\partial y}\cdot\frac{dy}{dx},$$

from which $$da=\frac{\partial a}{\partial \mathbf{x}}\cdot \mathbf{dx}+\frac{\partial a}{\partial \mathbf{y}}\cdot \mathbf{dy},$$

or $$\frac{da}{dy}=\frac{d}{dy}(xy)=y\cdot\frac{dx}{dy}+x=\frac{\partial a}{\partial x}\cdot\frac{dx}{dy}+\frac{\partial a}{\partial y};$$

$$\therefore\ da=\frac{\partial a}{\partial \mathbf{x}}\cdot \mathbf{dx}+\frac{\partial a}{\partial \mathbf{y}}\cdot \mathbf{dy}.$$

31. Total Differential. The result shewn in (37) on p. 84 gives the total differential in terms of partial derivatives, and the formula may be shewn to be true generally for functions whose partial derivatives are continuous. Hence:

If $\mathbf{u=f(x_1, x_2, \dots x_n)}$ be a continuous function in n variables, whose partial derivatives are also continuous, then the total differential of u is given by

$$\mathbf{du=\frac{\partial u}{\partial x_1}\cdot dx_1+\frac{\partial u}{\partial x_2}\cdot dx_2+\dots+\frac{\partial u}{\partial x_n}\cdot dx_n.} \qquad \dots\dots\dots\dots\dots(\mathbf{39})$$

32. Partial Derivative of a Composite Function. Let $\omega=f(x, y)$ where $x=A(u, v)$ and $y=B(u, v)$; then by substitution

$$\omega=f\{A(u, v),\ B(u, v)\},$$

and, assuming v is constant,

$$\left.\begin{aligned}&\frac{\partial\omega}{\partial \mathbf{u}}=\frac{\partial \mathbf{f}}{\partial \mathbf{u}}=\frac{\partial \mathbf{f}}{\partial \mathbf{A}}\frac{\partial \mathbf{A}}{\partial \mathbf{u}}+\frac{\partial \mathbf{f}}{\partial \mathbf{B}}\frac{\partial \mathbf{B}}{\partial \mathbf{u}}=\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{u}}+\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{u}}.\\ &\text{Similarly, if } u \text{ is constant,}\\ &\frac{\partial \mathbf{f}}{\partial \mathbf{v}}=\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{v}}+\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{v}}.\end{aligned}\right\} \qquad \dots\dots\dots(\mathbf{40})$$

These results are easily extended for any number of variables.

Multiply the above equations by du, dv respectively, and add; then

$$\frac{\partial f}{\partial u}\cdot du+\frac{\partial f}{\partial v}\cdot dv=\frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}\cdot du+\frac{\partial x}{\partial v}\cdot dv\right)+\frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}\cdot du+\frac{\partial y}{\partial v}\cdot dv\right),$$

i.e. $$df=\frac{\partial f}{\partial x}\cdot dx+\frac{\partial f}{\partial y}\cdot dy, \text{ by (39).}$$

Hence the equation $$df=\frac{\partial f}{\partial x}\cdot dx+\frac{\partial f}{\partial y}\cdot dy$$

is equivalent to the two equations,

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u};\quad \frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}, \quad\ldots\ldots\ldots(41)$$

when f is a function of x, y each of which is a function of u, v.

Ex. 9. *If* $\omega=\phi(x, y)$ *where* $x=F(u, v)$ *and* $y=f(u, v)$, *find the partial derivatives* $\frac{\partial\omega}{\partial x}, \frac{\partial\omega}{\partial y}$ *in terms of u and v; hence determine these derivatives when* $x=r\cos\theta$ *and* $y=r\sin\theta$. *Find also, in this case, the values of* $\frac{\partial^2\omega}{\partial x^2}, \frac{\partial^2\omega}{\partial y^2}$, *and shew that*

$$\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}=\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\omega}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\omega}{\partial\theta^2}.$$

Suppose the values of x, y are substituted in the given function, so that it becomes $\omega=\psi(u, v)$, then, from (40),

$$\frac{\partial\omega}{\partial u}=\frac{\partial\omega}{\partial x}\cdot\frac{\partial F}{\partial u}+\frac{\partial\omega}{\partial y}\cdot\frac{\partial f}{\partial u},$$

$$\frac{\partial\omega}{\partial v}=\frac{\partial\omega}{\partial x}\cdot\frac{\partial F}{\partial v}+\frac{\partial\omega}{\partial y}\cdot\frac{\partial f}{\partial v}.$$

Solving these equations for $\frac{\partial\omega}{\partial x}, \frac{\partial\omega}{\partial y}$,

$$\frac{\dfrac{\partial\omega}{\partial x}}{\begin{vmatrix}\dfrac{\partial f}{\partial u} & -\dfrac{\partial\omega}{\partial u}\\[2ex] \dfrac{\partial f}{\partial v} & -\dfrac{\partial\omega}{\partial v}\end{vmatrix}}=\frac{\dfrac{\partial\omega}{\partial y}}{\begin{vmatrix}-\dfrac{\partial\omega}{\partial u} & \dfrac{\partial F}{\partial u}\\[2ex] -\dfrac{\partial\omega}{\partial v} & \dfrac{\partial F}{\partial v}\end{vmatrix}}=\frac{1}{\begin{vmatrix}\dfrac{\partial F}{\partial u} & \dfrac{\partial f}{\partial u}\\[2ex] \dfrac{\partial F}{\partial v} & \dfrac{\partial f}{\partial v}\end{vmatrix}};$$

$$\therefore\ \frac{\partial\omega}{\partial x}=\left(\frac{\partial\omega}{\partial u}\cdot\frac{\partial f}{\partial v}-\frac{\partial\omega}{\partial v}\cdot\frac{\partial f}{\partial u}\right)\Big/J,\qquad \frac{\partial\omega}{\partial y}=\left(\frac{\partial\omega}{\partial v}\cdot\frac{\partial F}{\partial u}-\frac{\partial\omega}{\partial u}\cdot\frac{\partial F}{\partial v}\right)\Big/J,$$

where
$$J=\frac{\partial F}{\partial u}\cdot\frac{\partial f}{\partial v}-\frac{\partial F}{\partial v}\cdot\frac{\partial f}{\partial u}.$$

J is called the **Jacobian** or **Functional Determinant** of F, f, with respect to u, v, and is sometimes written $\dfrac{\partial(F,f)}{\partial(u,v)}$. Similarly, each of the numerators above is a Jacobian, and the transformations may be written

$$\frac{\partial\omega}{\partial x}=\frac{\partial(\omega,f)}{\partial(u,v)}\Big/\frac{\partial(F,f)}{\partial(u,v)},\quad \frac{\partial\omega}{\partial y}=\frac{\partial(F,\omega)}{\partial(u,v)}\Big/\frac{\partial(F,f)}{\partial(u,v)}.$$

When $x=r\cos\theta$ and $y=r\sin\theta$, r,θ replace u, v, and

$$\frac{\partial F}{\partial r}=\frac{\partial x}{\partial r}=\cos\theta,\quad \frac{\partial f}{\partial r}=\frac{\partial y}{\partial r}=\sin\theta,\quad \frac{\partial F}{\partial\theta}=-r\sin\theta,\quad \frac{\partial f}{\partial\theta}=r\cos\theta.$$

Putting in these values,

$$\frac{\partial\omega}{\partial x}=\frac{\partial\omega}{\partial r}\cdot\cos\theta-\frac{\sin\theta}{r}\cdot\frac{\partial\omega}{\partial\theta},\quad \frac{\partial\omega}{\partial y}=\frac{\partial\omega}{\partial r}\cdot\sin\theta+\frac{\cos\theta}{r}\cdot\frac{\partial\omega}{\partial\theta}.$$

To find $\dfrac{\partial^2\omega}{\partial x^2}$ and $\dfrac{\partial^2\omega}{\partial y^2}$, let $\xi=\dfrac{\partial\omega}{\partial x}=\cos\theta\cdot\dfrac{\partial\omega}{\partial r}-\dfrac{\sin\theta}{r}\cdot\dfrac{\partial\omega}{\partial\theta}$, and

$$\eta=\frac{\partial\omega}{\partial y}=\frac{\partial\omega}{\partial r}\sin\theta+\frac{\cos\theta}{r}\cdot\frac{\partial\omega}{\partial\theta};$$

then $\dfrac{\partial^2\omega}{\partial x^2}=\dfrac{\partial}{\partial x}\cdot\left(\dfrac{\partial\omega}{\partial x}\right)=\dfrac{\partial\xi}{\partial x}=\cos\theta\cdot\dfrac{\partial\xi}{\partial r}-\dfrac{\sin\theta}{r}\cdot\dfrac{\partial\xi}{\partial\theta}$, on writing ξ for ω in the above value for $\dfrac{\partial\omega}{\partial r}$, *i.e.*

$$\frac{\partial^2\omega}{\partial x^2}=\cos\theta\left(\cos\theta\frac{\partial^2\omega}{\partial r^2}-\frac{\sin\theta}{r}\cdot\frac{\partial^2\omega}{\partial r\,\partial\theta}+\frac{\sin\theta}{r^2}\cdot\frac{\partial\omega}{\partial\theta}\right)$$
$$-\frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2\omega}{\partial r\,\partial\theta}-\sin\theta\cdot\frac{\partial\omega}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial^2\omega}{\partial\theta^2}-\frac{\cos\theta}{r}\cdot\frac{\partial\omega}{\partial\theta}\right)$$
$$=\cos^2\theta\cdot\frac{\partial^2\omega}{\partial r^2}+\frac{\sin^2\theta}{r}\cdot\frac{\partial\omega}{\partial r}-\frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2\omega}{\partial r\,\partial\theta}+\frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial\omega}{\partial\theta}$$
$$+\frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2\omega}{\partial\theta^2}.$$

Similarly

$$\frac{\partial^2\omega}{\partial y^2}=\frac{\partial\eta}{\partial y}=\sin\theta\cdot\frac{\partial\eta}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial\eta}{\partial\theta}$$

$$=\sin^2\theta\cdot\frac{\partial^2\omega}{\partial r^2}+\frac{\cos^2\theta}{r}\cdot\frac{\partial\omega}{\partial r}+\frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2\omega}{\partial r\,\partial\theta}-\frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial\omega}{\partial\theta}$$
$$+\frac{\cos^2\theta}{r^2}\cdot\frac{\partial^2\omega}{\partial\theta^2}.$$

Hence, on addition,

$$\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}=\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\omega}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\omega}{\partial\theta^2}.$$

33. Independence Test for a System of Equations. A test for the independence of a system of homogeneous linear equations was given in § 12. A more general test will now be given, which is applicable to any system.

Let $u_r(r=1, 2, \dots n)$ be n functions of n independent variables x_r, then the functions will not be independent if the Jacobian J vanishes; *i.e.* if

$$J=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n}\end{vmatrix}=0. \quad \dots\dots(42)$$

Ex. 10. *Shew that the following systems of equations are consistent, i.e. the equations of each system are not all independent.*

(1) $u_1=3x^2+yz,$
$u_2=x^4+x^2yz,$
$u_3=y^2z^2-3x^2yz.$

(2) $2x+z-1=0,$
$x-y-5=0,$
$5x+y+2z-7=0,$
$36x+32y+z+21=0.$

(1) $\dfrac{\partial u_1}{\partial x}=6x,\quad \dfrac{\partial u_1}{\partial y}=z,\quad \dfrac{\partial u_1}{\partial z}=y;$

$\dfrac{\partial u_2}{\partial x}=4x^3+2xyz,\quad \dfrac{\partial u_2}{\partial y}=x^2z,\quad \dfrac{\partial u_2}{\partial z}=x^2y;$

$\dfrac{\partial u_3}{\partial x}=-6xyz,\quad \dfrac{\partial u_3}{\partial y}=2yz^2-3x^2z,\quad \dfrac{\partial u_3}{\partial z}=2y^2z-3x^2y;$

$$\therefore\ J = \begin{vmatrix} 6x & z & y \\ 2x(2x^2+yz) & x^2z & x^2y \\ -6xyz & z(2yz-3x^2) & y(2yz-3x^2) \end{vmatrix}$$

$$= 2xyz \begin{vmatrix} 3 & 1 & 1 \\ 2x^2+yz & x^2 & x^2 \\ -3yz & 2yz-3x^2 & 2yz-3x^2 \end{vmatrix} = 0, \text{ from § 10 (iii) (p. 23).}$$

It is easy to shew otherwise that the given equations are not all independent, for

$$u_1{}^2 - 9u_2 = -3x^2yz + y^2z^2 = u_3.$$

(2) In this system there are four equations and only three unknowns: in order therefore to apply the general rule, another symbol, w, may be introduced, which will be put equal to unity after the partial derivatives have been found. The system may thus be written:

$$u_1 = 2x + z - 4w = 0,$$
$$u_2 = x - y - 5w = 0,$$
$$u_3 = 5x + y + 2z - 7w = 0,$$
$$u_4 = 36x + 32y + z + 24w = 0.$$

Since the system is linear, the partial derivatives will be equal respectively to the corresponding coefficients, and the Jacobian will become precisely the determinant of the coefficients, the vanishing of which implies that the given equations are not all independent, as proved in § 12: hence the condition expressed in (10) is a particular case of the general condition (42).

$$\text{Now}\quad J = \begin{vmatrix} 2 & 0 & 1 & -4 \\ 1 & -1 & 0 & -5 \\ 5 & 1 & 2 & -7 \\ 36 & 32 & 1 & 24 \end{vmatrix} = \begin{vmatrix} 2-0 & 0+4 & 1 & -4 \\ 1+1 & -1+5 & 0 & -5 \\ 5-1 & 1+7 & 2 & -7 \\ 36-32 & 32-24 & 1 & 24 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 2 & 1 & -4 \\ 2 & 2 & 0 & -5 \\ 4 & 4 & 2 & -7 \\ 4 & 4 & 1 & 24 \end{vmatrix} = 0.$$

Thus the equations are not all independent, as is known from Ex. 8, p. 30, from which the given system is derived by taking $w=1$.

Exercises 5B.

Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ in each of the following cases:

1. $x^3+y^3+3axy=0$.
2. $ax^2+2hxy+by^2+2gx+2fy+c=0$.
3. $x^2/a^2-y^2/b^2=1$.
4. $u=2\log(x^2+y^2+a^2)$.
5. $u=\tan^{-1}\dfrac{x^2+y^2}{x+y}$.
6. $y^2+axy+bx^2+x^3-cy^3=0$.

7. The tangent plane at the point (u, v, w) on a surface whose equation is $f(x, y, z)=0$, is given by

$$(x-u)\frac{\partial f}{\partial u}+(y-v)\frac{\partial f}{\partial v}+(z-w)\frac{\partial f}{\partial w}=0.$$

Find the equations of the tangent planes to the surfaces:

(1) $xy=az$, at the point $(4, 6, 8)$;

(2) $x^2/a^2+y^2/b^2+z^2/c^2=1$, at the point $(4, 3, 3)$.

8. Define the partial differential coefficients of a function of two variables with respect to one of them, and shew that if $f(x, y)=0$, then

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}.$$

Prove that the conic $3x^2+2xy-y^2+2x+4y-1=0$ is parallel to Ox at the points where it intersects the line $3x+y+1=0$. (L.U.)

9. Denoting by suffixes those variables which are taken as independent in each case, shew that if x, y, z are connected by a relation $f(x, y, z)=0$, it is not, in general, true that

$$\left(\frac{\partial x}{\partial z}\right)_{x,y}=1\Big/\left(\frac{\partial z}{\partial x}\right)_{x,y}.$$

The pressure p, volume v, and absolute temperature T of a gas are connected by the equation $pv=RT$, where R is a constant. Prove that if Q is a function of the state of the gas such that

$$\left(\frac{\partial Q}{\partial v}\right)_{v,T}=p,$$

then

$$\left(\frac{\partial Q}{\partial T}\right)_{p,T}=\left(\frac{\partial Q}{\partial T}\right)_{v,T}+R,$$

and find the value of $\left(\dfrac{\partial Q}{\partial p}\right)_{p,T}$. (L.U.)

10. If $v=f(x, y)$, what do $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ represent?

If $v=\dfrac{A}{t^{\frac{1}{2}}}\cdot e^{-\frac{x^2}{4a^2t}}$, prove that $\dfrac{\partial v}{\partial t}=a^2\dfrac{\partial^2 v}{\partial x^2}$. (L.U.)

11. Verify Euler's theorem for the equations :

$$(a)\quad u = axy + byz + czx,$$

$$(b)\quad u = x^5yz^2 + 4x^3y^2z^3 + 7x^3y^4z - 8z^4x^4.$$

12. From Euler's theorem on homogeneous functions :

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu\ ;$$

shew that
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

13. Illustrate the theorem that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}:$$

(a) when $u = ax^2 + 2hxy + by^2$,

(b) when $u = \log\tan(y/x)$.

***14.** If $F(x, y) = a$, $f(x, z) = b$, where a and b are constants, prove that

$$\frac{dy}{dz} = \frac{\partial F}{\partial x}\cdot\frac{\partial f}{\partial z}\Big/\frac{\partial F}{\partial y}\cdot\frac{\partial f}{\partial x}.$$

Shew also that if $v = \dfrac{c}{t}e^{-\mu}$, where $\mu = \dfrac{1}{4t}\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)$, then

$$\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2} + b^2\frac{\partial^2 v}{\partial y^2}.$$

***15.** The equation $\dfrac{\partial u}{\partial t} = \mu\dfrac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation ; shew that if

$$u = Ae^{-gx}\sin(nt - gx),$$

where A, g, n are positive constants, then

$$g = \sqrt{\frac{n}{2\mu}}.$$

16. Verify that if $u = a\cosh b(y + c)\,.\,\cos(bx + d)$, where a, b, c, d are constants, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

***17.** Shew that if $u = a\cosh b(y + c)\,.\,\cos(bx - pt)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \mu\cdot\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2},$$

a, b, c, p, μ **being constants, then** $\mu = (b^2 - p^2)/b^2$, **provided neither** b **nor** u **are zero.**

18. What is meant by partial as distinguished from total differentiation ?

Find the partial differential coefficients of x^2y with respect to x and y, and its total differential coefficient with respect to x when x and y are connected by the relation $x^2 + xy + y^2 = 1$. (L.U.)

***19.** The equation $f(x, y) = 0$, where $f(x, y)$ denotes the function

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$

represents a conic. Shew that by changing the origin to the point (u, v), $f(x + u, y + v) = 0$ becomes

$$ax^2 + 2hxy + by^2 + x\frac{\partial s}{\partial u} + y\frac{\partial s}{\partial v} + s = 0.$$

Shew, further, that the equations $\frac{\partial s}{\partial u} = 0$, $\frac{\partial s}{\partial v} = 0$, can only give finite solutions for u and v, provided

$$\begin{vmatrix} a & h \\ h & b \end{vmatrix} \text{ is not zero.}$$

Find $\frac{\partial s}{\partial u}$, $\frac{\partial s}{\partial v}$ when $a = 4$, $b = 1$, $h = 3$, $f = 1$, $g = -2$, $c = -5$, and assuming these derivatives both vanish, shew that $f(x + u, y + v) = 0$ becomes

$$4x^2 + 6xy + y^2 = 1.$$

***20.** If $F(x, y, z)$ denotes the function $f(x, y, z) + 2lx + 2my + 2nz + d$, where $f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, shew that

$$F(x + u, y + v, z + w)$$

may be written in the form

$$f(x, y, z) + x\,.\frac{\partial s}{\partial u} + y\,.\frac{\partial s}{\partial v} + z\,.\frac{\partial s}{\partial w} + s.$$

Shew also that the equations $\frac{\partial s}{\partial u} = 0$, $\frac{\partial s}{\partial v} = 0$, $\frac{\partial s}{\partial w} = 0$,

can only give finite solutions for u, v, w, when

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{ is not zero.}$$

Shew further that $s = \frac{1}{2}\left(u\frac{\partial s}{\partial u} + v\frac{\partial s}{\partial v} + w\frac{\partial s}{\partial w}\right) + lu + mv + nw + d,$

and that when the derivatives vanish, subject to the above condition, then

$$s = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix} \div \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

21. If $z=\log(\tan x+\tan y)$, shew that

$$\sin 2x\,.\,\frac{\partial z}{\partial x}+\sin 2y\,.\,\frac{\partial z}{\partial y}=2.$$

22. Given z as a function of two independent variables x, y, define the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and illustrate their meanings geometrically.

If $f(z)=\phi(u)$, where z and u are each functions of x and y, prove that

$$\frac{\partial z}{\partial x}\,.\,\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}\,.\,\frac{\partial z}{\partial y}. \qquad \text{(L.U.)}$$

23. If $u=x^2-y^2$ and $v=2xy$, find in terms of x and y the partial differential coefficient $\frac{\partial u}{\partial x}$:

(i) when u and v are regarded as functions of x and y,

(ii) when u and y are regarded as functions of x and v. (L.U.)

24. If $\tan u=\cos x/\sinh y$, $\tanh v=\sin x/\cosh y$, shew that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad\text{and}\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

***25.** If $x=e^u\cos v$, $y=e^u\sin v$, prove that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=e^{-2u}\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right). \qquad \text{(L.U.)}$$

***26.** Verify that $z=A\sin(x-at)+B\cos(x-at)$ satisfies the differential equation

$$\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}. \qquad \text{(D.U.)}$$

***27.** If $x=\cosh u\,.\,\cosh v$, $y=\sinh u\,.\,\sinh v$, prove that

$$\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}=(\sinh^2 u-\sinh^2 v)\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right). \qquad \text{(L.U., Sc.)}$$

***28.** If $x=r\cos\theta$, $y=r\sin\theta$, prove that

$$\frac{\partial^2\theta}{\partial x\,\partial y}=\frac{\partial^2}{\partial x^2}(\log r)=-\frac{\partial^2}{\partial y^2}(\log r)=-\frac{1}{r^2}\cos 2\theta. \qquad \text{(L.U., Sc.)}$$

***29.** If $u=x+y$, $v=xy$, shew that

$$\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\,\partial y}+\frac{\partial^2\phi}{\partial y^2}=(u^2-4v^2)\frac{\partial^2\phi}{\partial v^2}-2\frac{\partial\phi}{\partial v}. \qquad \text{(L.U., Sc.)}$$

***30.** If $x=e^u+e^{-v}$, $y=e^v+e^{-u}$, prove that

(i) $\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}=x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}$,

(ii) $\dfrac{\partial^2 z}{\partial u^2}-2\dfrac{\partial^2 z}{\partial u\,\partial v}+\dfrac{\partial^2 z}{\partial v^2}=x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}+x^2\dfrac{\partial^2 z}{\partial x^2}-2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}.$

(L.U., Sc.)

***31.** If $u=f(y/x)+a\sqrt{x^2+y^2}$, shew that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=a\sqrt{x^2+y^2}.$$

***32.** If $x=r\cos\theta$, $y=r\sin\theta$, prove that

$$(x^2-y^2)\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)+4xy\frac{\partial^2 u}{\partial x\partial y}=r^2\frac{\partial^2 u}{\partial r^2}-r\frac{\partial u}{\partial r}-\frac{\partial^2 u}{\partial\theta^2}.\quad\text{(L.U., Sc.)}$$

***33.** If $x=cuv$, $y=c\sqrt{(1+u^2)(1-v^2)}$, prove that

$$\frac{1}{y}\left(y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}\right)=\left(v\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial v}\right)\Big/\{c(u^2+v^2)\}.\quad\text{(L.U., Sc.)}$$

34. Shew that the equations, $8x^3-27y^3+z^3+18xyz=0$,

$$4x^2+9y^2+z^2+6xy-2zx+3yz=0,\quad 2x-3y+z=0$$

are consistent.

35. Prove that the equations:

$$u=x^3+x^2y+x^2z-xz^2-yz^2-z^3,$$
$$v=x+z,$$
$$w=x^2-z^2+xy-zy$$

are not all independent.

MAXIMA AND MINIMA

34. Practical Rules. If $y=f(x)$ be a continuous curve, $\frac{dy}{dx}$, the gradient of the curve measures the rate at which y changes with x. Hence, when $\frac{dy}{dx}=0$, y passes through either (*a*) a maximum value, (*b*) a minimum value, or (*c*) a point of inflexion.

If in passing through the zero value $\frac{dy}{dx}$ changes

(*a*) from a positive to a negative value, *i.e.* $\frac{d^2y}{dx^2}$ is negative, y passes through a **maximum** value;

(*b*) from a negative to a positive value, *i.e.* $\frac{d^2y}{dx^2}$ is positive, y passes through a **minimum** value;

(*c*) from a positive to a positive, or from a negative to a negative value, the curve passes through **a point of inflexion.**

Hence, the following **practical rules** for finding maximum and minimum values :

1. Find $\frac{dy}{dx}$ and equate to zero.
2. Solve this equation for x ; let $x=a$ be a root.
3. Find $\frac{d^2y}{dx^2}$ and substitute $x=a$ in it ; if the result is :
   (i) **Positive,** y has a **minimum value** at $x=a$.
   (ii) **Negative,** y has a **maximum value** at $x=a$.
   (iii) **Zero,** a further differential coefficient must be found, and the first one of **even order** which does not vanish at $x=a$ must be used in place of $\frac{d^2y}{dx^2}$ for tests (i) and (ii). If $\frac{d^3y}{dx^3}$ does not vanish at $x=a$, there is no maximum nor minimum value at this point.

Ex. 11. *Find the values of x which make y either a maximum or minimum in the following functions :*

$$(a)\quad y=4x^3+3x^2-90x+144,$$

$$(b)\quad y=\frac{3x^2+2x+11\cdot1}{4x^2+6x+19\cdot3},$$

$$(c)\quad y=x^5-5x^4+6,$$

nd calculate the corresponding values of y.

(a) Since $\quad y=4x^3+3x^2-90x+144,$

$$\therefore\quad \frac{dy}{dx}=12x^2+6x-90=6(2x^2+x-15)=6(2x-5)(x+3),$$

nd $$\frac{d^2y}{dx^2}=6(4x+1).$$

Hence, when $\frac{dy}{dx}=0$, $x=2\cdot5$ or -3.

When $x=2\cdot5$, $\frac{d^2y}{dx^2}=66$, and $y=0\cdot25$, and when $x=-3$,

$$\frac{d^2y}{dx^2}=-66 \quad\text{and}\quad y=333.$$

$\therefore$ y is a maximum at $x=$ **-3**, its value there being **333** ; **and** is a minimum at $x=$ **2·5**, its value there being **0·25**.

(*b*) Since

$$y=\frac{3x^2+2x+11\cdot1}{4x^2+6x+19\cdot3},$$

$$\therefore\ \frac{dy}{dx}=\frac{(4x^2+6x+19\cdot3)(6x+2)-(3x^2+2x+11\cdot1)(8x+6)}{(4x^2+6x+19\cdot3)^2}$$

$$=\frac{10x^2+27x-28}{(4x^2+6x+19\cdot3)^2},$$

and $$\frac{d^2y}{dx^2}=\frac{-80x^3-324x^2+672x+857\cdot1}{(4x^2+6x+19\cdot3)^3}.$$

When $\frac{dy}{dx}=0$, $\quad 10x^2+27x-28=0,$

or $\quad (5x-4)(2x+7)=0,$

giving $\quad x=0\cdot8 \text{ or } -3\cdot5.$

When $x=0\cdot8$, $\frac{d^2y}{dx^2}=\frac{1146\cdot38}{(26\cdot66)^3}$, which is positive, and

$$y=\frac{14\cdot62}{26\cdot66}=0\cdot5109.$$

When $x=-3\cdot5$, $\frac{d^2y}{dx^2}=\frac{-2033\cdot9}{(47\cdot3)^3}$, which is negative, and

$$y=\frac{10\cdot85}{47\cdot3}=0\cdot8636.$$

$\therefore$ y is a maximum at $x=-3\cdot5$ its value there being **0·8636**; and y is a minimum at $x=0\cdot8$, its value there being **0·5109**.

(*c*) Since $y=x^5-5x^4+6$,

$$\frac{dy}{dx}=5x^4-20x^3=5x^3(x-4),$$

$$\frac{d^2y}{dx^2}=20x^2(x-3).$$

When $\frac{dy}{dx}=0$, $\quad x=0$ or 4.

When $x=4$, $\frac{d^2y}{dx^2}=320$ and $y=-250$.

When $x=0$, $\frac{d^2y}{dx^2}=0$, and further differential coefficients must be found.

$$\frac{d^3y}{dx^3}=60x(x-2), \text{ which also vanishes at } x=0,$$

$$\frac{d^4y}{dx^4}=120(x-1), \text{ which becomes } -120 \text{ at } x=0 \text{ and } y=6.$$

$\therefore$ y is a maximum at $x=\mathbf{0}$, its value there being **6**; and y is a minimum at $x=\mathbf{4}$, its value there being $\mathbf{-250}$.

Ex. 12. *A cylindrical tank closed at each end, of capacity V cubic feet, is to be constructed from sheets of metal having a total area of S square feet; shew that when S is a minimum:*

(*a*) *the length and diameter are equal,*

(*b*) $V^2 : S^3 = 1 : 54\pi$.

If the capacity is to be 20096 *gallons, find, without using tables, the radius of the tank when S is a minimum. Take* $\pi=3{\cdot}14$, *and* 6·25 *gallons to a cubic foot. Calculate also the total area of plate required.*

Let $r=$ the radius and $l=$ the length of the tank in feet; then, since it is closed at both ends,

$$S=2\pi r^2+2\pi rl=2\pi r(r+l).$$

Also $V=\pi r^2 l$, so that $l=\frac{V}{\pi r^2}$;

$$\therefore\ S=2\pi r\left(r+\frac{V}{\pi r^2}\right)=2\pi r^2+\frac{2V}{r};$$

$$\therefore\ \frac{dS}{dr}=4\pi r-\frac{2V}{r^2}=\frac{2}{r^2}(2\pi r^3-V).$$

For a maximum or minimum, $\frac{dS}{dr}=0$;

$$\therefore\ r^3=V/2\pi \quad \text{or} \quad \infty.$$

The former obviously gives a minimum, and the latter a maximum.

Now, from above, $\frac{l}{r}=\frac{V}{\pi r^3}=\frac{V}{\pi}\cdot\frac{2\pi}{V}=2$;

$$\therefore\ l=2r;$$

$\therefore$ **Length = Diameter.**

Again, since $l = 2r$ and $S = 2\pi r(r + l)$,

$$\therefore \ S = 2\pi r(r + 2r) = 6\pi r^2 ;$$

$$\therefore \ \frac{S^3}{V^2} = \frac{216\pi^3 r^6}{V^2} = \frac{216\pi^3}{V^2} \cdot \frac{V^2}{4\pi^2} = 54\pi ;$$

$$\therefore \ \mathbf{S^3 : V^2 = 54\pi : 1.}$$

In the particular case,

$$20096 \text{ gallons} = 20096 \div 6{\cdot}25 = 3215{\cdot}36 \text{ cu. ft.},$$

and from the above analysis,

$$r^3 = \frac{V}{2\pi} = \frac{3215{\cdot}36}{6{\cdot}28} = 512 \text{ cu. ft.};$$

$$\therefore \ r = 8 \textbf{ feet.}$$

Also, it has already been proved that

$$S = 6\pi r^2 ;$$

$$\therefore \ S = 6 \times 3{\cdot}14 \times 64 = 1205{\cdot}76 \textbf{ sq. ft.}$$

Ex 13. *The shape of a hole bored by a drill is a cone surmounting a cylinder. If the cylinder be of height h and radius r, and the semi-vertical angle of the cone be α where tan α = h/r, show that, for a total fixed depth H of the hole, the volume removed is a maximum if* $h = \frac{H}{6}(\sqrt{7} - 1)$. (L.U.)

Let l = height of cone and V = volume removed, then

$$H = h + l = h + r\cot\alpha = h + r^2/h ,$$

$$\therefore \ l = H - h \quad \text{and} \quad r^2 = h(H - h),$$

and $$V = \pi r^2 h + \tfrac{1}{3}\pi r^2 l = \tfrac{1}{3}\pi r^2(3h + l) = \tfrac{1}{3}\pi h(H - h)(2h + H)$$

$$= \tfrac{1}{3}\pi(H^2 h + h^2 H - 2h^3) ;$$

$$\therefore \ \frac{dV}{dh} = \tfrac{1}{3}\pi(H^2 + 2Hh - 6h^2) = 0 \text{ for max. or min.},$$

i.e. $$-6h^2 + 2Hh + H^2 = 0,$$

giving $$h = \tfrac{1}{6}H(-1 \pm \sqrt{7}).$$

It is obvious, therefore, for a maximum value of V,

$$h = \tfrac{1}{6}\mathbf{H}(\sqrt{7} - 1).$$

35. Concavity, Convexity and Points of Inflexion. It will readily be seen from the results of § 34, that a curve $y=f(x)$ is concave upwards, or convex upwards in the immediate neighbourhood of a point $x=a$, according as $\dfrac{d^2y}{dx^2}$, not being zero, is positive or negative.

Further, at a point of inflexion, $\dfrac{dy}{dx}$ does not change sign, but it does pass through a turning value ; hence, as long as both derivatives are continuous at the point considered, a necessary condition for a point of inflexion is that $\dfrac{d^2y}{dx^2}$ should vanish.

Ex. 14. *Find the maximum and minimum ordinates and the points of inflexion on the curve*

$$y(x^2+x+1)=2x^2+1. \qquad \text{(L.U.)}$$

Since
$$y=\frac{2x^2+1}{x^2+x+1}.$$
$$\therefore \frac{dy}{dx}=\frac{2x^2+2x-1}{(x^2+x+1)^2},$$
and
$$\frac{d^2y}{dx^2}=-\frac{2(2x+1)(x^2+x-2)}{(x^2+x+1)^3}.$$

For maximum or minimum values, $\dfrac{dy}{dx}=0$;

$$\therefore\ 2x^2+2x-1=0,$$
or
$$x^2+x-\tfrac{1}{2}=0,$$
giving
$$x=\tfrac{1}{2}(-1\pm\sqrt{3}).$$

Substitute in $\dfrac{d^2y}{dx^2}$,

$$\frac{d^2y}{dx^2}=-\frac{2(2x+1)(x^2+x-\frac{1}{2}-\frac{3}{2})}{(x^2+x-\frac{1}{2}+\frac{3}{2})^3}=-\frac{2(-1\pm\sqrt{3}+1)(-\frac{3}{2})}{\frac{27}{8}}$$
$$=\pm\tfrac{8}{9}\sqrt{3}.$$

Hence y is a maximum when $x=-\frac{1}{2}(1+\sqrt{3})$, and a minimum when $x=-\frac{1}{2}(1-\sqrt{3})$, and the values of y are

$$\frac{2x^2+1}{x^2+x+1}=\frac{(2x^2+2x-1)+2-2x}{x^2+x-\frac{1}{2}+\frac{3}{2}}=\frac{2+1\mp\sqrt{3}}{\frac{3}{2}}=\tfrac{2}{3}(3\mp\sqrt{3}).$$

$\therefore$ **y is a maximum at $x = -\frac{1}{2}(1+\sqrt{3})$**, its value there being $\frac{2}{3}(3+\sqrt{3})$; and y is a minimum at $x = -\frac{1}{2}(1-\sqrt{3})$, its value there being $\frac{2}{3}(3-\sqrt{3})$.

Again, for a point of inflexion, $\dfrac{d^2y}{dx^2}=0$, *i.e.* $(2x+1)(x^2+x-2)=0$, or $(2x+1)(x-1)(x+2)=0$, giving $x=-\frac{1}{2}$, 1, or -2.

To test these, denote $(2x+1)(x-1)(x+2)$ by $f(x)$, and let h be a small positive quantity less than unity.

Then considering signs only,

$$f(-\tfrac{1}{2}-h) = -2h(-\tfrac{3}{2}-h)(\tfrac{3}{2}-h) = (-)(-)(+) = +.$$

$$f(-\tfrac{1}{2}+h) = \quad 2h(-\tfrac{3}{2}+h)(\tfrac{3}{2}+h) = (+)(-)(+) = -.$$

Hence, at $x=-\frac{1}{2}$, $\dfrac{dy}{dx}$ passes through a turning value.

Similarly it may be shewn that $\dfrac{dy}{dx}$ passes through turning values at each of the other points.

Hence points of inflexion occur at

$$x = -\tfrac{1}{2},\ 1 \text{ and } -2.$$

Exercises 5c.

Determine at what points the ordinates of the following curves have maximum and minimum values, and find also any points of inflexion :

1. $y=x^3-6x^2-15x-6$.

2. $y=4x^3-33x^2+90x+6{\cdot}75$.

3. $y=4x^3-15x^2+18x+7$.

4. $y=\dfrac{x^2}{3}+2+\dfrac{27}{x^2}$.

5. $y=x^2(3x^2-40x+150)-4$.

6. $y=12x^5-45x^4+40x^3+7$.

7. $y=\dfrac{x^2+1}{4x-3}$.

8. $y=\dfrac{(x-3)(2x-9)}{x-5}$.

9. $y=(x-1)(x-2)^2$.

10. $y=x^6-3x^4+8$.

11. $y=x^{\frac{2}{a}}-x^{1+\frac{1}{a}}$, where $a=1{\cdot}4$.

12. $y=a\sin x+b\cos x$.

13. Find the magnitudes of the maximum and the minimum ordinates of the curve $y=(x+2)^2(x-3)$, and the position of the point of inflexion. (L.U.)

14. A man has n hurdles of equal length and he wishes to enclose a rectangular portion of ground which shall have maximum area ; find the number of hurdles he must place on each side of the plot.

15. A hollow cylindrical vessel is to be made from a sheet of metal twenty inches square. Find its base radius and its length when its volume is a maximum, allowing for both ends and 18·49 square inches for joins and waste. Take $\pi = 3{\cdot}14$.

16. A length of wire 71·4 ft. long is cut into two portions, which are bent into the shapes of a circle and square respectively. Find the radius of the circle and the side of the square when the sum of the areas is a maximum.

17. A cylindrical tank is to be made, closed at each end, to hold 39,250 gallons of water. Find its length and radius if the quantity of iron plate required is to be a minimum, taking $\pi = 3{\cdot}14$ and 6·25 gallons to a cubic foot.

18. Find the maximum value of xy when $x+y=20$.

19. If in testing for a maximum or a minimum value of $y=f(x)$, it is found that $\frac{d^2y}{dx^2}=0$, what further tests must be tried ?

A wire 3 feet long has to be bent into the form of a rectangle with an external circular loop at one corner and the rectangle is to have one side double the other. Find the dimensions of the circle and rectangle so that the total area enclosed is a minimum. (L.U.)

20. Calculate the side of a square prism of maximum volume which can be cut from a right circular cone 27 ft. high and 9 ft. in base radius. Find also the volume of the prism, and shew that its ratio to the volume of the cone is $8 : 9\pi$.

21. From a given circular sheet of metal it is required to cut out a sector so that the remainder can be bent into a conical vessel of maximum capacity ; find the angle of the sector to be removed, and if r is the radius of the metal, find the volume of the vessel.

22. A conical tent is to have a given volume V. Find what is the ratio of its height to its base radius when the least possible amount of canvas is used.

If, in this case, the canvas is spread out on the ground, what fraction of a complete circle is it ?

23. *The stiffness of a beam of rectangular section is proportional to* the product of the breadth and the cube of the depth. Find the ratio of the sides of the stiffest beam of rectangular section with given perimeter. (L.U.)

24. A beam rectangular in section is cut from a cylindrical tree trunk one foot in diameter. The sides of the section are x and z inches long respectively, and the stiffness, y, of the beam is given by the formula

$$y = Axz^3,$$

where A is constant. Find the values of x and z which make y a maximum.

25. Find the radius and length of a right cylinder of maximum volume which can be cut from a right circular cone of height h and base radius r, and shew that the volume of this cylinder is $\frac{4}{9}$ths that of the cone.

26. Find the dimensions of the cylinder of maximum volume which can be cut from a sphere of 10 in. radius. (L.U.)

27. Calculate the radius and length of a right cylinder of maximum volume which can be cut from a sphere of radius r. Prove that the volumes of the cylinder and the sphere are as $1 : \sqrt{3}$.

28. A rectangular window has a semicircular top. The perimeter of the whole window is 35·7 ft. Find the radius of the semicircle and the height of the rectangle when the area of the window is a maximum.

29. The range R, of a projectile on a horizontal plane, is given by the formula

$$Rg = 2V^2 \sin\theta \,.\, \cos\theta,$$

where V and g are constants; find the value of θ when R is a maximum.

30. The sum of three sides of a sector of a circle is two feet. Find the radius of the circle and the length of the arc of the sector when its area is a maximum.

31. The length and girth of a right circular cylinder are together equal to a given constant c; shew that the whole surface is a maximum when the diameter of a transverse section equals $c/(2\pi - 1)$.

32. A firm is satisfied from its past experience that its expenditure per week in pounds is

$$120 + 3{\cdot}5x + \frac{A}{x+5} + 0{\cdot}01A,$$

where x is the number of horses employed and A the usual turnover. If $A = £2744$, find the number of horses which will cause the expenditure to be a minimum, and calculate the cost in this case.

33. A body is projected from the foot of a plane inclined at an angle ϕ to the horizontal, with a velocity V whose direction makes an angle θ with the horizontal. Its range R is then given by the formula

$$R = \frac{2V^2}{g} \,.\, \cos\theta \,.\, \sec^2\phi \,.\, \sin(\theta - \phi).$$

Shew that if g, V and ϕ are constant, R is a maximum when the direction of projection bisects the angle between the inclined plane and the vertical.

34. Find the maximum and minimum values of $ae^{-\alpha x}\sin\beta x$, and prove that the ratio of consecutive maximum values is constant. (L.U.)

35. A box without a lid is to be made from a rectangular piece of tin, 32 in. by 20 in., by cutting out squares from the four corners and turning up the projecting pieces to make the sides of the box. Determine the dimensions of the box that it may contain the greatest possible volume. (L.U.)

36. Find the relative dimensions of a cone which has the maximum volume for a given area of surface, including the area of base. Verify that for such a cone, the area of the curved surface is three times the area of base. (L.U.)

37. The sum of the perimeters of n equal squares and a circle is $2a$ feet; prove that when the sum of the areas is greatest, the side of each square is equal to the diameter of the circle, and that the area A of each square is given by

$$(\pi+4n)^2 \,.\, A=4a^2.$$

Calculate a side and the radius of the circle when n is 2, a is 78, and $7\pi=22$.

38. The sum of the perimeters of n equal circles and a square is $2b$ feet; find the relation between the diameter of a circle and the side of the square when the sum of the areas is a maximum, and calculate the diameter and side when n is 7, b is 52, and $7\pi=22$.

39. A tall telegraph pole is to be strained to a vertical position by a sloping wire from the pole to the ground. The wire has to pass over a wall 7 feet high and 5 feet from the pole. What is the least length of wire that can be employed between pole and the ground? (L.U.)

***40.** Find the maximum values of the expression

$$\frac{M^2\omega^2}{\left(L\omega-\dfrac{1}{C\omega}\right)^2+R^2}$$

first, when C is regarded as the only variable, and secondly, when ω is regarded as the only variable. Shew that the function is, in fact, a maximum on each occasion. (L.U.)

***41.** ABC is any triangle and AD is drawn to meet BC such that $AD=AB$. A point G is taken in AC, and GF drawn parallel to AD meeting BC in F; if GB intersects AD in K, $BC=a$, $AB=b$, $BD=c$ and $BF=x$, shew that

$$(a)\quad GF=\frac{b(a-x)}{a-c},$$

$$(b)\quad AK=\frac{ab(x-c)}{x(a-c)}.$$

Hence prove that the product $GF\,.\,AK$ is a maximum when $x^2=ac$.

42. Sketch the curve $y(x^2-4)=x^2+x+1$, and find the maximum and minimum values of y. (B.U.)

43. Sketch the curve $y=\dfrac{x}{(a+x)(b+x)}$, where a and b are positive. Shew that it has a maximum at $x=\sqrt{(ab)}$ and a minimum at $x=-\sqrt{(ab)}$. (M.U.)

***44.** Given the total surface, that is, the sum of the curved surface and the base, of a cone, prove that the volume of the cone will be greatest when half the vertical angle is equal to $\tan^{-1} 1/2\sqrt{2}$. (S.U., Sc.)

***45** A uniform thin rod of length l and mass m is suspended by one end so that it can oscillate as a pendulum; a particle of mass $\frac{9}{4}m$ is attached to the rod at a distance x from the point of support. Determine x so that the period for small oscillations may be a minimum, and find the period. (L.U.)

CHAPTER VI

INTEGRATION

36. The Problem of Integration. If $y = f(x)$, and $\frac{dy}{dx} = \phi(x)$, where both $f(x)$ and $\phi(x)$ are continuous functions, then the problem of integration consists in finding the function $f(x)$ when only $\phi(x)$ is given. Since $f(x) + A$, where A is an arbitrary constant, has the same derivative as $f(x)$, it follows that the problem of finding $f(x)$ will have an indefinite number of solutions, differing only by constants. Each of these solutions is known as the **indefinite integral** of $\phi(x)$, and in finding such an integral the arbitrary constant should always be added.

The integration of simple functions depends upon the following **fundamental standard forms** :

$$\left.\begin{aligned}
&(a)\ \int \mathbf{x^n\,dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}} + \mathbf{A}, \text{ for all values of } n \text{ except } -1,\\
&(b)\ \int \frac{\mathbf{dx}}{\mathbf{x}} = \mathbf{\log_e x + A}, \quad \ldots\ldots\ldots\ldots\ldots\ldots ,\\
&(c)\ \int \mathbf{e^{ax}\,dx} = \frac{\mathbf{e^{ax}}}{\mathbf{a}} + \mathbf{A}, \qquad a \text{ being a constant},\\
&(d)\ \int \mathbf{\sin ax\,.\,dx} = -\frac{\mathbf{1}}{\mathbf{a}}\mathbf{\cos ax + A}, \quad \ldots\ldots\ldots\ldots ,\\
&(e)\ \int \mathbf{\cos ax\,.\,dx} = \frac{\mathbf{1}}{\mathbf{a}}\mathbf{\sin ax + A}, \quad \ldots\ldots\ldots\ldots ,\\
&(f)\ \int \mathbf{\sec^2 ax\,.\,dx} = \frac{\mathbf{1}}{\mathbf{a}}\mathbf{\tan ax + A}, \quad \ldots\ldots\ldots\ldots ,\\
&(g)\ \int \mathbf{cosec^2 ax\,.\,dx} = -\frac{\mathbf{1}}{\mathbf{a}}\mathbf{\cot ax + A}, \ldots\ldots\ldots\ldots .
\end{aligned}\right\}\ldots\ldots\ldots\mathbf{(43)}$$

Ex. 1. *Evaluate the integral* $\int(1-2x)^n dx$,

(*a*) *when* $n=3$, (*b*) *when* $n=-4$, (*c*) *when* $n=\frac{3}{5}$, (*d*) *when* $n=-1$.

(*a*) When $n=3$,
$$\int(1-2x)^n \,.\, dx = \int(1-2x)^3 dx$$
$$= \int(1-6x+12x^2-8x^3)dx$$
$$= x - 6\,.\,\frac{x^2}{2} + 12\,.\,\frac{x^3}{3} - 8\frac{x^4}{4} + A, \text{ by } (43a),$$
$$= \mathbf{x - 3x^2 + 4x^3 - 2x^4 + A.}$$

The expansion of the function is only used here because n has a small positive integral value; this, however, is only a particular case, and, in general, the following method of substitution must be employed:

Put $v=1-2x$, then $dv=-2\,.\,dx$, and $dx=-\frac{1}{2}dv$:
$$\therefore \int(1-2x)^3 dx = -\tfrac{1}{2}\int v^3\,.\,dv = -\tfrac{1}{8}v^4 + B = -\tfrac{1}{8}(\mathbf{1-2x})^4 + \mathbf{B}.$$

This differs only by a constant from the value found above, for
$$-\tfrac{1}{8}(1-2x)^4 + B = -\tfrac{1}{8}(1-8x+24x^2-32x^3+16x^4) + B$$
$$= \mathbf{x - 3x^2 + 4x^3 - 2x^4 + A}, \text{ where } A = B - \tfrac{1}{8}.$$

(*b*) When $n=-4$, the integral becomes
$$\int(1-2x)^{-4}dx = -\tfrac{1}{2}\int v^{-4}\,.\,dv, \text{ from the above substitution,}$$
$$= -\tfrac{1}{2}\cdot\frac{v^{-4+1}}{-4+1} + A = \tfrac{1}{6}\,.\,v^{-3} + A$$
$$= \tfrac{1}{6}(\mathbf{1-2x})^{-3} + \mathbf{A}.$$

(*c*) With the same substitution, when $n=\frac{3}{5}$,
$$\int(1-2x)^{\frac{3}{5}}\,.\,dx = -\tfrac{1}{2}\int v^{\frac{3}{5}}\,.\,dv = -\tfrac{1}{2}\cdot\frac{v^{\frac{3}{5}+1}}{\frac{3}{5}+1} + A = -\tfrac{5}{16}v^{\frac{8}{5}} + A$$
$$= -\tfrac{5}{16}(\mathbf{1-2x})^{\frac{8}{5}} + \mathbf{A}.$$

(*d*) With the same substitution, when $n=-1$,
$$\int(1-2x)^{-1}dx = -\tfrac{1}{2}\int v^{-1}dv = -\tfrac{1}{2}\int\frac{dv}{v} = -\tfrac{1}{2}\log v + A, \text{ by } (43b),$$
$$= -\tfrac{1}{2}\log(1-2x) + A = \tfrac{1}{2}\,\mathbf{\log\frac{1}{1-2x} + A}.$$

37. Definite Integrals. When the integral of a function $\phi(x)$ is required between two specified values of x, the integral at once becomes **definite,** and the given values of x are called the **limits of integration.** Thus suppose the integral of $\phi(x)$ is required between $x=a$ and $x=b$, then this is written $\int_a^b \phi(x) \, . \, dx$, and $(b-a)$ is called the range of integration.

Let
$$\int \phi(x) \, . \, dx = f(x) + A.$$

then
$$\int_a^b \phi(x) \, . \, dx = \Big[f(x) + A \Big]_a^b = f(b) + A - f(a) - A = f(b) - f(a),$$

so that there is no arbitrary constant needed in evaluating a definite integral, since this cancels out.

Further, it should be carefully noted that when a substitution is used to evaluate an integral, the limits should be changed to correspond with the new variable. No later transformation back to the old variable is then needed.

Ex. 2. *Evaluate* $\int_0^1 \frac{x}{\sqrt[3]{9-7x}} \, . \, dx.$ (L.U.)

Let $u=9-7x$, then $x=\frac{1}{7}(9-u)$, and $dx=-\frac{1}{7} \, . \, du$.
Now, when $x=1$, $u=9-7=2$, and when $x=0$, $u=9$;

$$\therefore \int_0^1 \frac{x}{\sqrt[3]{9-7x}} \, . \, dx = -\frac{1}{49}\int_9^2 \frac{9-u}{u^{\frac{1}{3}}} \, . \, du = -\frac{1}{49}\int_9^2 (9u^{-\frac{1}{3}} - u^{\frac{2}{3}}) du$$

$$= -\frac{1}{49}\left[\frac{27}{2} \, . \, u^{\frac{2}{3}} - \frac{3}{5} u^{\frac{5}{3}}\right]_9^2 = \left\{\frac{u^{\frac{2}{3}}}{490}[6u-135]\right\}_9^2$$

$$= \frac{\sqrt[3]{4}}{490}(12-135) - \frac{\sqrt[3]{81}}{490}(54-135)$$

$$= \frac{3}{490}(81\sqrt[3]{3} - 41\sqrt[3]{4})$$

$$= 0{\cdot}3167,$$

on taking $\sqrt[3]{3} = 1{\cdot}4422$, and $\sqrt[3]{4} = 1{\cdot}5874$.

Ex. 3. *If* $(2x-3)y=24x^3$, *find the value of* $\int_2^6 y\,.\,dx.$

Let $z=2x-3$, then $x=\frac{1}{2}(z+3)$, and $dx=\frac{1}{2}\,.\,dz$; also when $x=6$, $z=9$, and when $x=2$, $z=1$;

$$\therefore \int_2^6 \frac{24x^3}{2x-3}\,.\,dx$$

$$=\frac{1}{2}\int_1^9 \frac{3z^3+27z^2+81z+81}{z}\,.\,dz$$

$$=\frac{1}{2}\int_1^9 \left(3z^2+27z+81+\frac{81}{z}\right)dz$$

$$=\frac{1}{2}\Big[z^3+\tfrac{27}{2}z^2+81z+81\log z\Big]_1^9$$

$$=\tfrac{1}{2}\{729+\tfrac{27}{2}\,.\,81+729+81\log 9-1-\tfrac{27}{2}-81-81\log 1\}$$

$$=1228+81\log 3=1317 \text{ nearly.}$$

Ex. 4. *Evaluate the integrals*:

$$(a)\ \int\cos^2\theta\,.\,d\theta,\quad (b)\ \int\sin^3\theta\,.\,d\theta,\quad (c)\ \int\tan\theta\,.\,d\theta$$

Shew, by putting $\tan\theta$ *for* $2t$, *that*

$$\int_0^{0\cdot 5}\frac{8t}{(1+4t^2)^{\frac{3}{2}}}\,dt=2-\sqrt{2}$$

The indefinite integrals of the second and third powers of a sine or cosine may be readily found by means of the identities given by (12) and (13), p. 36.

Thus from (12), (a) becomes:

$$\int\cos^2\theta\,.\,d\theta=\tfrac{1}{2}\int(1+\cos 2\theta)\,d\theta=\tfrac{1}{2}(\theta+\tfrac{1}{2}\sin 2\theta)+A$$

$$=\tfrac{1}{4}(2\theta+\sin 2\theta)+A.$$

(b) From (13),

$$\int\sin^3\theta\,.\,d\theta=\tfrac{1}{4}\int(3\sin\theta-\sin 3\theta)\,.\,d\theta=\tfrac{1}{4}(-3\cos\theta+\tfrac{1}{3}\cos 3\theta)+A$$

$$=\tfrac{1}{12}(\cos 3\theta-9\cos\theta)+A.$$

In (c), put $x=\cos\theta$, then $dx=-\sin\theta\,.\,d\theta$;

$$\therefore \int\tan\theta\,.\,d\theta=\int\frac{\sin\theta}{\cos\theta}\,.\,d\theta=-\int\frac{dx}{x}=-\log x+A=-\log\cos\theta+A$$

$$=\log\sec\theta+A.$$

Putting $2t=\tan\theta$, gives $2\,.\,dt=\sec^2\theta\,.\,d\theta$, and

$$1+4t^2=1+\tan^2\theta=\sec^2\theta\,;$$

$$\therefore\ (1+4t^2)^{\frac{3}{2}}=\sec^3\theta.$$

When $\qquad t=0{\cdot}5,\quad \tan\theta=1,\quad \therefore\ \theta=\frac{\pi}{4}\,;$

when $\qquad t=0,\quad \tan\theta=0,\quad \therefore\ \theta=0.$

$$\therefore\ \int_0^{0\cdot5}\frac{8t}{(1+4t^2)^{\frac{3}{2}}}\,.\,dt=\int_0^{\frac{\pi}{4}}\frac{4\tan\theta\,.\,\sec^2\theta\,.\,d\theta}{2\sec^3\theta}=2\int_0^{\frac{\pi}{4}}\sin\theta\,.\,d\theta$$

$$=2\Big[-\cos\theta\Big]_0^{\frac{\pi}{4}}=2\left(-\frac{1}{\sqrt{2}}+1\right)=2-\sqrt{2}.$$

Ex. 5. *If* $2\int v\,.\,dx=v-\log(1+v)+A$, *where* v *is a function which is zero when* $x=0$, *prove that* $v=2e^x\sinh x$.

Since $\qquad 2\int v\,.\,dx=v-\log(1+v)+A,$

$\therefore$ by differentiating each side,

$$2v\,.\,dx=\left(1-\frac{1}{1+v}\right)dv=\frac{v}{1+v}\,.\,dv\,;$$

$$\therefore\ 2\,.\,dx=\frac{dv}{1+v}.$$

Integrating each side, $\quad 2x=\log(1+v)+B\,;$

but $v=0$, when $x=0$, $\therefore\ B=0$; so that $1+v=e^{2x}$;

$$\therefore\ v=e^{2x}-1=e^x(e^x-e^{-x})$$

$$=\mathbf{2e^x\sinh x}.$$

Exercises 6A.

Integrate each of the following functions with respect to x:

1. $3x^{\frac{8}{5}}$. **2.** $(2x-5)^8$. **3.** $(4-x)^{-6}$.

4. $x^{\frac{2}{3}}+x^{-\frac{3}{2}}$. **5.** $\dfrac{1}{3x+7}$. **6.** $x(3x+4)^2$.

7. $(x+1)(x-2)$. **8.** $(1+2x^2)\left(1-\dfrac{3}{x^2}\right)$. **9.** $\dfrac{4}{5x+3}$.

10. $(3x-7)^8$. **11.** $\dfrac{5x}{5x+2}$. **12.** $\dfrac{x^3-5}{x+2}$.

13. $\dfrac{x^2-5}{2\sqrt{x}}$. **14.** $\sqrt{2a+x}$. **15.** $\dfrac{x^3+a^3}{2x}$.

16. $(3-2x)^{\frac{2}{3}}$. **17.** $\dfrac{3x^2+1}{2x+1}$. **18.** $\dfrac{5x^3}{1-x}$.

19. $\sin^2 x$. **20.** $\cos^3 x$. **21.** $\sin 5x$.

22. $\tan(2x+3)$. **23.** $\sin x \,.\, \cos x$. **24.** $\cot x$.

Evaluate the following definite integrals :

25. $\displaystyle\int_4^8 \frac{x+7}{x-3}\,.\,dx$. **26.** $\displaystyle 2\int_1^2 \frac{9x-7}{3x-2}\,.\,dx$. **27.** $\displaystyle\int_2^5 \frac{6x-9}{3x-4}\,.\,dx$.

28. $\displaystyle\int_{-2}^2 \frac{x^3+3}{x+3}\,.\,dx$. **29.** $\displaystyle 8\int_1^7 \frac{dx}{4x+7}$. **30.** $\displaystyle\int_2^3 (3x-5)^2 dx$.

31. $\displaystyle\int_1^1 \frac{4x}{4x+5}\,.\,dx$. **32.** $\displaystyle\int_0^{0.1\pi} \sin 5x \,.\, dx$. **33.** $\displaystyle\int_0^{\pi} \sin^2 x \,.\, dx$.

34. $\displaystyle 4\int_0^{\frac{\pi}{4}} \cos x \,.\, \sin x \,.\, dx$. **35.** $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x \,.\, dx$. **36.** $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \,.\, dx$.

37. By putting $u=4x^3+12x+7$, evaluate the integral

$$12\int_0^2 \frac{x^2+1}{4x^3+12x+7}\,.\,dx.$$

38. Evaluate $\displaystyle\int_3^6 \frac{x+1}{\sqrt{x-2}}\,.\,dx$, by means of the substitution $u^2=x-2$.

39. Prove that $\displaystyle\int_{-a}^{a} \frac{10x-2+5a}{a+2x}\,.\,dx=10a-\log 3$.

40. Find $\displaystyle\int \frac{x^2}{x+2}\,.\,dx$. (L.U.)

41. If X denotes the function $x/(a^2+x^2)$, evaluate

$$\int X\,.\,dx \quad \text{and} \quad \int_0^a X\,.\,dx.$$

By putting $x=a\tan\theta$ in the indefinite integral and its value, shew that

$$\int \tan\theta \,.\, d\theta = \log\sec\theta + A.$$

42. Shew, by putting $x=\tan\theta$, that $\displaystyle 8\int_0^1 \frac{dx}{(1+x^2)^2}=\pi+2$.

***43.** Evaluate $\int \frac{v^3}{1+v^2} \,.\, dv$ by means of the substitution $v^2 = z^2 - 1$; hence, by putting $v = \tan\theta$ in both the integral and its value, shew that $\int \tan^3\theta d\theta = \frac{1}{2}\sec^2\theta + \log\cos\theta$.

***44.** By means of the substitution $x = \alpha\sin^2\theta + \beta\cos^2\theta$, evaluate

$$\int_{\frac{1}{2}(\alpha+\beta)}^{\alpha} \frac{dx}{\sqrt{(\alpha - x)(x-\beta)}}. \qquad \text{(Li. U.)}$$

***45.** Find a function y of x, which is zero when $x = 1$, and which satisfies the equation

$$3\int y \,.\, dx = 2y^3 + A.$$

38. Integration by means of Partial Fractions. Many algebraic functions of the form $\phi(x)/\psi(x)$, where $\psi(x)$ is resolvable into factors, can be readily integrated by first splitting the function into partial fractions according to the methods of § 6, pp. 7-10. The following will illustrate the process.

Ex. 6. *If* $y(4x^2 - 16x + 15) = 4x^2 - 12x + 13$, *evaluate*

$$(a) \int y \,.\, dx, \quad (b) \int_3^6 y \,.\, dx.$$

(*a*) From Ex. 6, p. 8,

$$\frac{4x^2 - 12x + 13}{4x^2 - 16x + 15} = 1 + \frac{4}{2x-5} - \frac{2}{2x-3};$$

$$\therefore \int y \,.\, dx = \int dx + 4\int \frac{dx}{2x-5} - 2\int \frac{dx}{2x-3}$$

$$= x + 2\log(2x-5) - \log(2x-3) + A'$$

$$= \log e^x + \log(2x-5)^2 - \log(2x-3) + \log A,$$

where $A' = \log A$,

$$= \log\left\{\frac{Ae^x(2x-5)^2}{2x-3}\right\}.$$

$$(b) \int_3^6 y \,.\, dx = \Big[x + 2\log(2x-5) - \log(2x-3)\Big]_3^6$$

$$= 6 + 2\log 7 - \log 9 - 3 - 2\log 1 + \log 3$$

$$= 3 + 2\log 7 - \log 3$$

$$= 3 + 3{\cdot}8918 - 1{\cdot}0986 = \mathbf{5{\cdot}7932}.$$

Ex. 7. *Find the values of* (*a*) $\int y \, . \, dx$, *and* (*b*) $\int_1^3 y \, . \, dx$, *where*

$$y = 3(4x^3 + x^2 + 5)/\{x(2x^3 + 1)(x^2 + 3)\}.$$

(*a*) From Ex. 6, p. 9,

$$\frac{3(4x^3 + x^2 + 5)}{x(2x^3 + 1)(x^2 + 3)} = \frac{5}{x} - \frac{2x}{x^2 + 3} - \frac{6x^2}{2x^3 + 1};$$

$$\therefore \int \frac{3(4x^3 + x^2 + 5)}{x(2x^3 + 1)(x^2 + 3)} \, . \, dx = 5\int \frac{dx}{x} - \int \frac{2x \, dx}{x^2 + 3} - \int \frac{6x^2 \, . \, dx}{2x^3 + 1}$$

$$= 5\int \frac{dx}{x} - \int \frac{du}{u} - \int \frac{dv}{v},$$

where $u = x^2 + 3$, and $v = 2x^3 + 1$

$$= 5 \log x - \log u - \log v + \text{constant}$$

$$= \log \frac{x^5}{(x^2 + 3)(2x^3 + 1)} + \text{constant.}$$

$$(b) \int_1^3 \frac{3(4x^3 + x^2 + 5)}{x(2x^3 + 1)(x^2 + 3)} . \, dx = \left[\log \frac{x^5}{(x^2 + 3)(2x^3 + 1)} \right]_1^3$$

$$= 5 \log 3 - \log 55$$

$$= 5(1 \cdot 0986) - (2 \cdot 3026 + 1 \cdot 7047) = \mathbf{1 \cdot 4857}$$

Ex. 8. *Establish the results* :

$$(a) \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} + A, \text{ when } x > a.$$

$$(b) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a + x}{a - x} + A, \text{ when } x < a,$$

$$(c) \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a},$$

and use the results to prove that

$$\int_3^5 \frac{41 \, dx}{x(41 - x^2)\sqrt{25 - x^2}} = 0 \cdot 416.$$

(*a*) By the methods of § 6,

$$\frac{1}{x^2 - a^2} = \frac{1}{2a} \left\{ \frac{1}{x - a} - \frac{1}{x + a} \right\};$$

$$\therefore \int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a}\{\log(x-a) - \log(x+a)\} + A,$$

or $$\int \frac{\mathbf{dx}}{\mathbf{x^2-a^2}} = \frac{\mathbf{1}}{\mathbf{2a}} \cdot \mathbf{log}\, \frac{\mathbf{x-a}}{\mathbf{x+a}} + \mathbf{A}. \qquad \textbf{(44a)}$$

(b) In a similar way,

$$\frac{1}{a^2-x^2} = \frac{1}{2a}\left\{\frac{1}{a+x} + \frac{1}{a-x}\right\};$$

$$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\{\log(a+x) - \log(a-x)\} + A,$$

or $$\int \frac{\mathbf{dx}}{\mathbf{a^2-x^2}} = \frac{\mathbf{1}}{\mathbf{2a}}\, \mathbf{log}\, \frac{\mathbf{a+x}}{\mathbf{a-x}} + \mathbf{A}. \qquad \textbf{(44b)}$$

(c) This integral may be readily found by putting $x = a \tan\theta$, then $dx = a \sec^2\theta \, . \, d\theta$:

$$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a \sec^2\theta}{a^2 \sec^2\theta} \cdot d\theta = \frac{1}{a}\int d\theta = \frac{1}{a} \cdot \theta = \frac{\mathbf{1}}{\mathbf{a}}\, \mathbf{tan^{-1}}\, \frac{\mathbf{x}}{\mathbf{a}}.$$

No constant of integration is needed because both sides vanish at $x = 0$.

Hence $$\int \frac{\mathbf{dx}}{\mathbf{a^2+x^2}} = \frac{\mathbf{1}}{\mathbf{a}}\, \mathbf{tan^{-1}}\, \frac{\mathbf{x}}{\mathbf{a}}. \qquad \textbf{(44c)}$$

In the given integral put $25 - x^2 = u^2$, then $-x\,dx = u\,du$, $41 - x^2 = 16 + u^2$, and the new limits become 0 and 4;

$$\therefore \int_3^5 \frac{41\,dx}{x(41-x^2)\sqrt{25-x^2}} = -\int_4^0 \frac{41\,du}{(16+u^2)(25-u^2)}$$

$$= \int_0^4 \left(\frac{1}{25-u^2} + \frac{1}{16+u^2}\right) du$$

$$= \left[\frac{1}{10} \log \frac{5+u}{5-u} + \frac{1}{4} \tan^{-1} \frac{u}{4}\right]_0^4, \text{ by (44}b\text{) and (44}c\text{)},$$

$$= \frac{1}{10} \log 9 + \frac{\pi}{16} = 0{\cdot}2197 + 0{\cdot}1964$$

$$= \mathbf{0{\cdot}416}.$$

EXERCISES 6B.

In each of the following examples resolve the expression for y into partial fractions and evaluate the indefinite integral $\int y \,.\, dx$:

1. $y(x^2+5x+6)=1$. **2.** $y(x^2-5x+6)=2x-5$.
3. $y(2x^2+x-6)=5x-11$. **4.** $y(14x^2-17x-6)=25$.
5. $y(12x^2-25x+12)=7$. **6.** $y(x^2+x-12)=8x+11$.
7. $y(x^2+4x-21)=10(x+4)$. **8.** $y(12-17x+6x^2)=14-11x$.
9. $y(56-29x+3x^2)=38-11x$. **10.** $y(2+5x-12x^2)=22$.
11. $y(x-1)(x-2)(x-3)=2(3x^2-11x+9)$.
12. $y(x-2)(2x-3)(3x-1)=16-13x$.
13. $yx(x+3)(x^2+4)=12+8x-3x^2$.
14. $y(1-x^4)=4x$. **15.** $y(4-x^2)=4$.
***16.** $y(x-3)(x^2+2)(x^2+3)=2(3x^3+4x^2+3x+6)$.
***17.** $y(x^6-1)=6x^2$. ***18.** $y(8x^3-1)=3(2x+1)$.
***19.** $y(a^4-x^4)(b^2-x^2)=a^2(a^2+2b^2)-2a^2x^2-x^4$.
***20.** $y(x^4+x^2+1)=2(x^2-1)$.

Evaluate the following definite integrals:

21. $\int_3^7 \frac{3\,dx}{x^2-x-2}$. **22.** $\int_{-1}^{2} \frac{3\,dx}{3x-x^2}$. **23.** $\int_0^4 \frac{dx}{2x^2+5x+3}$.

24. $\int_2^5 \frac{31}{12x^2-x-20}\,.\,dx$. **25.** $\int_0^6 \frac{5}{12x^2+11x+2}\,.\,dx$.

26. $\int_4^8 \frac{-x^2+6x+11}{(x+1)(x+2)(x-3)}\,.\,dx$. **27.** $\int_2^5 \frac{22x\,.\,dx}{3x^4-7x^2-6}$.

28. $\int_{3\cdot5}^{9\cdot5} \frac{10x-21}{(2x-5)(2x-3)}\,.\,dx$. **29.** $\int_3^7 \frac{2(2x-1)}{x(x-1)(x-2)}\,.\,dx$.

***30.** $\int_1^4 \frac{4(4+6x^2-x^3)}{x(x^2+2)(x^3+8)}\,.\,dx$. **31.** $\int_3^5 \frac{6x^2\,.\,dx}{(x-1)(x+1)(x-2)}$.

32. $\int_{1\cdot5}^{2\cdot5} \frac{dx}{4x^2-12x+13}$. ***33.** $\int_0^1 \frac{1-3x}{(x^2+1)(x+1)}\,.\,dx$.

***34.** Find the numerical values of the constants, A, B, C, D, such that for all values of x the following relation is identically true:

$$\frac{16x}{16-x^4}=\frac{Ax+B}{4+x^2}+\frac{C}{2+x}+\frac{D}{2-x};$$

hence, prove that $$16\int_0^a \frac{x\,.\,dx}{16-x^4}=\log\frac{4+a^2}{4-a^2}.$$

By the substitution, $x=2\sin\theta$, evaluate the integral $4\int_0^{\frac{\pi}{6}} \frac{\tan\theta}{1+\sin^2\theta}\,.\,d\theta$.

35. Evaluate the integral $\int_{3\cdot 8}^{8} \frac{12x^2 - 38x + 31}{6x^2 - 19x + 10} \, . \, dx$.

***36.** Transform the integral $\int \frac{dx}{\sin x(3 \cos x - \sin x - 1)}$, by the substitution $t = \tan \frac{x}{2}$. Hence evaluate it.

37. Evaluate the definite integral $\int_0^{0\cdot 75} \frac{x^3 \, . \, dx}{x^4 + 3x^2 + 2}$.

38. Evaluate $\int_1^2 \frac{17x^2 + 8x - 12}{x^2(4 + x^2)(3 + x)} \, . \, dx$.

39. Prove that $\int_0^2 \frac{5x}{(x+1)(x^2+4)} \, . \, dx = 0\cdot 819$ nearly. (L.U.)

***40.** Evaluate $\int_0^2 \frac{x^2 + 1}{(9 - x^2)(x^2 + x + 4)} \, . \, dx$. (L.U.)

***41.** By making the substitution $x = 2t/(1 + t^2)$, prove that

$$\int_0^1 \frac{6 \, dx}{(4 + 5x)\sqrt{1 - x^2}} = 2 \log 2.$$

39. Integration of Rational Quadratic Functions. If

$$X \equiv ax^2 + 2bx + c,$$

then the evaluation of $\int dx/X$ falls into two groups, **according as X is, or is not resolvable into real rational factors.**

In the former case, it is best to split $1/X$ into partial fractions, thus

$$\int \frac{dx}{3x^2 - 5x + 2} = \int \frac{dx}{(x-1)(3x-2)} = \int \left(\frac{1}{x-1} - \frac{3}{3x-2} \right) dx$$

$$= \log (x-1) - \log (3x-2) + A$$

$$= A + \log \frac{x-1}{3x-2},$$

where A is an arbitrary constant.

When, however, X is not resolvable into real rational factors, the function must be expressed in one of the forms $x^2 \pm a^2$, or $a^2 - x^2$, and the corresponding standard form established in equation (44) (p. 113) applied.

Ex. 9. $$\int\frac{dx}{2x^2-5x+1}=\frac{1}{2}\int\frac{dx}{x^2-\frac{5}{2}x+\frac{1}{2}}=\frac{1}{2}\int\frac{dx}{(x-\frac{5}{4})^2-\frac{17}{16}}$$
$$=\frac{1}{2}\int\frac{du}{u^2-a^2},\text{ where } u=x-\tfrac{5}{4},\ a=\tfrac{1}{4}\sqrt{17},$$
$$=\frac{1}{4a}\log\frac{u-a}{u+a}+\text{constant, by (44}a\text{)},$$
$$=\frac{1}{\sqrt{17}}\cdot\log\frac{4x-5-\sqrt{17}}{4x-5+\sqrt{17}}+\text{constant.}$$

Ex. 10. $$\int\frac{dx}{2x^2-5x+6}=\frac{1}{2}\int\frac{dx}{(x-\frac{5}{4})^2+\frac{23}{16}}$$
$$=\frac{1}{2}\int\frac{du}{u^2+a^2},\text{ where } u=x-\tfrac{5}{4},\ a=\tfrac{1}{4}\sqrt{23},$$
$$=\frac{1}{2a}\tan^{-1}\frac{u}{a}+\text{constant, by (44}c\text{)},$$
$$=\frac{2}{\sqrt{23}}\cdot\tan^{-1}\frac{4x-5}{\sqrt{23}}+\text{constant.}$$

Ex. 11. $$\int\frac{dx}{3+3x-2x^2}=\frac{1}{2}\int\frac{dx}{\frac{33}{16}-(x-\frac{3}{4})^2}$$
$$=\frac{1}{2}\int\frac{du}{a^2-u^2},\text{ where } u=x-\tfrac{3}{4},\ a=\tfrac{1}{4}\sqrt{33},$$
$$=\frac{1}{4a}\cdot\log\frac{a+u}{a-u}+\text{constant, by (44}b\text{)},$$
$$=\frac{1}{\sqrt{33}}\log\frac{\sqrt{33}-3+4x}{\sqrt{33}+3-4x}+\text{constant.}$$

Ex. 12. $$\int_{0\cdot6}^{1\cdot4}\frac{dx}{3+6x-5x^2}=\frac{1}{5}\int_{0\cdot6}^{1\cdot4}\frac{dx}{0\cdot96-(x-0\cdot6)^2}.$$

Put $u=x-0\cdot6$, then when $x=1\cdot4$, $u=0\cdot8$, and when $x=0\cdot6$, $u=0$; hence the integral becomes
$$\frac{1}{5}\int_0^{0\cdot8}\frac{du}{0\cdot96-u^2}=0\cdot102\left[\log\frac{0\cdot9798+u}{0\cdot9798-u}\right]_0^{0\cdot8}=0\cdot102\log\frac{1\cdot7798}{0\cdot1798}$$
$$=0\cdot102(0\cdot5766-0\cdot5868+2\cdot3026)$$
$$=0\cdot102\times2\cdot2924=\mathbf{0\cdot2338}.$$

40. Variable in Numerator of Integrand. When the numerator also contains x, it must first be reduced, if necessary, by division as in § 6, so that it is of one degree less in x than the denominator. It must then be expressed as the sum of the differential coefficient of the denominator and a constant. Thus, if the fraction is $(px+q)/(ax^2+bx+c)$, then the differential coefficient of ax^2+bx+c is $2ax+b$, and

$$px+q=\frac{p}{2a}\left(2ax+b-b+\frac{2aq}{p}\right)=\frac{p}{2a}(2ax+b)+\left(q-\frac{bp}{2a}\right);$$

$$\therefore \int\frac{px+q}{ax^2+bx+c}\cdot dx=\frac{p}{2a}\int\frac{2ax+b}{ax^2+bx+c}\cdot dx+\left(q-\frac{bp}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

$$=\frac{p}{2a}\cdot\log(ax^2+bx+c)+\left(q-\frac{bp}{2a}\right)\int\frac{dx}{ax^2+bx+c}+A. \quad (45)$$

Ex. 13. *Evaluate the integral* $\int_{1\cdot5}^{3}\frac{4x-3}{2x^2-6x+9}\cdot dx.$

The denominator $=2x^2-6x+9=2(x^2-3x+\frac{9}{2})$

$$=2\{(x-\tfrac{3}{2})^2+\tfrac{9}{4}\}.$$

Hence it contains no real factors.

Since the differential coefficient of the denominator is $4x-6$, and of $4x-3$ is $4x-6+3$,

$$\therefore \int_{1\cdot5}^{3}\frac{4x-3}{2x^2-6x+9}\cdot dx=\int_{1\cdot5}^{3}\frac{4x-6}{2x^2-6x+9}\cdot dx+3\int_{1\cdot5}^{3}\frac{dx}{2x^2-6x+9}.$$

In the first integral, put $u=2x^2-6x+9$, then $du=(4x-6)dx$; and when $x=3$, $u=9$; when $x=1\cdot5$, $u=4\cdot5$.

In the second integral, put $v=x-\frac{3}{2}$, then $dv=dx$, and when $x=3$, $v=\frac{3}{2}$; and when $x=1\cdot5$, $v=0$; hence

$$\int_{1\cdot5}^{3}\frac{4x-3}{2x^3-6x+9}\cdot dx=\int_{4\cdot5}^{9}\frac{du}{u}+\frac{3}{2}\int_{0}^{1\cdot5}\frac{dv}{v^2+(\frac{3}{2})^2}$$

$$=\Big[\log u\Big]_{4\cdot5}^{9}+\frac{3}{2}\cdot\Big[\frac{2}{3}\tan^{-1}\frac{2v}{3}\Big]_{0}^{1\cdot5}$$

$$=\log 2+\frac{\pi}{4}=1\cdot4785.$$

EXERCISES 6C.

Evaluate the indefinite integral $\int \frac{dx}{X}$ in each of the following cases :

1. $X = 16 - x^2$.
2. $X = x^2 - 36$.
3. $X = x^2 + 81$.
4. $X = x^2 - 4x - 21$.
5. $X = 55 - 6x - x^2$.
6. $X = x^2 + 10x + 106$.
7. $X = 9x^2 - 30x - 119$.
8. $X = 27 - 4x(x + 3)$.
9. $X = 25x^2 + 10x + 2$.
10. $X = 4x(x - 5)$.
11. $X = 7x(6 - 7x)$.
12. $X = 5(5x^2 - 2x + 2)$.
13. $X = a^2x^2 + 2abx + b^2 - c^2$.
14. $X = a^2x^2 + 2acx + 2c^2$.
15. $X = ax(2c - ax)$.
16. $X = a^2x^2 - 4abx - 4 - b^4$.

Evaluate the following definite integrals :

17. $\int_0^2 \frac{5x+1}{x^2+4} \cdot dx$.
18. $\int_0^2 \frac{dx}{5 - x^2}$.
19. $\int_0^2 \frac{3x^2}{4 + x^2} \, dx$.
20. $\int_0^{13} \frac{3x+1}{x^2+1} \cdot dx$.
21. $\int^2 \frac{dx}{x^2 + 6x + 34}$.
22. $\int_0^1 \frac{1+x}{3+x^2} \cdot dx$.
23. $16\int_3^6 \frac{x}{x^4 - 16} \cdot dx$.
24. $16\int_{2\cdot 4}^{2\cdot 6} \frac{dx}{x^4 - 16}$.
25. $\int_b^a \frac{dx}{e^x - 25e^{-x}}$, where $a = 2{\cdot}8332$ and $b = 1{\cdot}7918$.
26. Integrate $x\,dx/(a^2 - x^2)^{\frac{3}{2}}$ and $dx/(a^2 + x^2)^{\frac{3}{2}}$. (L.U.)

*27. Prove, by means of the substitution, $t = \tan \frac{x}{2}$, that

$$\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \frac{b + a\cos x}{a + b\cos x}, \quad \text{when } a > b,$$

or

$$= \frac{1}{\sqrt{b^2 - a^2}} \cosh^{-1} \frac{b + a\cos x}{a + b\cos x}, \text{ when } a < b.$$

Use the substitution of Ex. 27 to evaluate the integrals :

28. $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\cos x}$.
29. $\int_0^{\frac{\pi}{2}} \frac{dx}{9\cos x + 12\sin x}$.
30. $\int_{\frac{\pi}{2}}^{\frac{2}{3}\pi} \frac{dx}{\sin x + \tan x}$.
31. $\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + \sin x}$. (L.U.)

Prove, by the same substitution, or otherwise, the following results :

***32.** $\int \frac{(1+\cos x)\,.\,dx}{\sin x\,.\,\cos x} = \log(\sec x - 1)$.

***33.** $\int \frac{\cos x - \sin x}{\sin x\,.\,\cos x}\,dx = \log \frac{(1-\sin x)(1-\cos x)}{\sin x\,.\,\cos x}$.

***34.** Evaluate $\int_0^{\pi} \frac{dx}{1+\cos\alpha\,.\,\cos x}$, where α is constant.

Prove the following results :

35. $\int_0^{\infty} \frac{dx}{(x^2+a^2)^{\frac{5}{2}}} = \frac{2}{3a^4}$. (S.U.)

36. $\int_0^{\pi} \frac{d\theta}{16\cos^2\theta + \sin^2\theta} = \frac{1}{4}$.

37. $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$. (L.U., Sc.)

38. $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$. (M.U.)

***39.** Determine the values of A, B, C, such that

$$\frac{4+5\sin x+\cos x}{1+\sin x+\cos x} = A + \frac{B(\cos x - \sin x)}{1+\sin x+\cos x} + \frac{C}{1+\sin x+\cos x};$$

hence evaluate the definite integral $\int_0^{\frac{\pi}{2}} \frac{4+5\sin x+\cos x}{1+\sin x+\cos x}\,.\,dx$.

41. Integration of Irrational Quadratic Functions. If

$$X \equiv ax^2 + 2bx + c,$$

the integration of $\int \sqrt{X}\,.\,dx$ and of $\int \frac{dx}{\sqrt{X}}$ depends upon the expression of X as an algebraic sum of two squares. It is therefore necessary to establish the standard integrals when X is of the forms $x^2 \pm a^2$, or $a^2 \pm x^2$.

Ex. 14. *Evaluate the integrals* (i) $\int \frac{dx}{\sqrt{X}}$ *and* (ii) $\int \sqrt{X}\,.\,dx$, *when* X *has each of the following forms :*

$$(a)\ X = a^2 - x^2, \quad (b)\ X = x^2 - a^2, \quad (c)\ X = a^2 + x^2.$$

(a) When $X = a^2 - x^2$, put $x = a\sin\theta$, then $dx = a\cos\theta\,.\,d\theta$;

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\,.\,d\theta}{a\cos\theta} = \int d\theta = \theta + A = \sin^{-1}\frac{x}{a} + A,$$

and $$\int\sqrt{a^2-x^2}\,.\,dx=a^2\int\cos^2\theta\,.\,d\theta=\tfrac{1}{2}a^2(\theta+\tfrac{1}{2}\sin 2\theta)+A,$$ from Ex. 4, p. 108,

$$=\tfrac{1}{2}a^2\left(\sin^{-1}\frac{x}{a}+\sin\theta\cos\theta\right)+A$$

$$=\tfrac{1}{2}a^2\left(\sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right)+A$$

$$=\tfrac{1}{2}\mathbf{a}^2\sin^{-1}\frac{\mathbf{x}}{\mathbf{a}}+\tfrac{1}{2}\mathbf{x}\sqrt{\mathbf{a}^2-\mathbf{x}^2}+\mathbf{A}.$$

(b) When $X=x^2-a^2$, put $x=a\cosh z$, then $dx=a\sinh z\,.\,dz$, and $x^2-a^2=a^2(\cosh^2 z-1)=a^2\sinh^2 z$;

$$\therefore\quad\int\frac{dx}{\sqrt{x^2-a^2}}=\int dz=z+A=\cosh^{-1}\frac{\mathbf{x}}{\mathbf{a}}+\mathbf{A}.$$

If the substitution $x=\dfrac{1+t^2}{1-t^2}\,.\,a$ had been made, the integral would become $2\displaystyle\int\frac{dt}{1-t^2}=\log\frac{1+t}{1-t}+B$, by (44),

$$=\log\frac{\mathbf{x}+\sqrt{\mathbf{x}^2-\mathbf{a}^2}}{\mathbf{a}}+\mathbf{B}.$$

Hence, equating the two results:

$$\cosh^{-1}\frac{x}{a}+A=\log\frac{x+\sqrt{x^2-a^2}}{a}+B,$$

which will be identically satisfied for all values of x; putting $x=a$,

$$A=B,$$

so that $$\cosh^{-1}\frac{x}{a}=\log\frac{\mathbf{x}+\sqrt{\mathbf{x}^2-\mathbf{a}^2}}{\mathbf{a}}$$ as proved in Ex. 9, p. 50.

Applying the original substitution to the integral $\int\sqrt{X}\,.\,dx$,

$$\int\sqrt{x^2-a^2}\,.\,dx=a^2\int\sinh^2 z\,.\,dz=\frac{a^2}{4}\int(e^{2z}-2+e^{-2z})\,.\,dz$$

$$=\frac{a^2}{4}(\tfrac{1}{2}e^{2z}-2z-\tfrac{1}{2}e^{-2z})+A=\frac{a^2}{8}\{(e^z-e^{-z})(e^z+e^{-z})-4z\}+A$$

$$=\frac{a^2}{2}(\sinh z\,.\,\cosh z-z)+A=\frac{\mathbf{x}}{2}\sqrt{\mathbf{x}^2-\mathbf{a}^2}-\frac{\mathbf{a}^2}{2}\cosh^{-1}\frac{\mathbf{x}}{\mathbf{a}}+\mathbf{A}.$$

(c) When $X = a^2 + x^2$, put $x = a \sinh z$, then $dx = a \cosh z \, . \, dz$, and $a^2 + x^2 = a^2(1 + \sinh^2 z) = a^2 \cosh^2 z$;

$$\therefore \int \frac{dx}{\sqrt{a^2 + x^2}} = \int dz = z + A = \sinh^{-1}\frac{x}{a} + A.$$

Now the substitution, $x = \dfrac{2at}{1 - t^2}$, rationalizes the integral into

$$2\int \frac{dt}{1 - t^2} = \log \frac{1 + t}{1 - t} + B = \log \frac{1 + t^2}{(1 - t)^2} + B$$

$$= \log \left(\frac{2t}{1 - t^2} + \frac{1 + t^2}{1 - t^2} \right) + B$$

$$= \log \frac{x + \sqrt{a^2 + x^2}}{a} + B.$$

Equating the two results and putting $x = 0$, $A = B$,

$$\therefore \sinh^{-1}\frac{x}{a} = \log \frac{x + \sqrt{a^2 + x^2}}{a}$$ as proved in Ex. 9, p. 50.

Reverting to the original substitution,

$$\int \sqrt{a^2 + x^2} \, . \, dx = a^2 \int \cosh^2 z \, . \, dz = \frac{a^2}{4}\int (e^{2z} + 2 + e^{-2z})dz$$

$$= \frac{a^2}{8}(e^{2z} - e^{-2z}) + \tfrac{1}{2}a^2 z + A$$

$$= \frac{a^2}{8}(e^z - e^{-z})(e^z + e^{-z}) + \tfrac{1}{2}a^2 z + A$$

$$= \tfrac{1}{2}a^2 \sinh z \, . \, \cosh z + \tfrac{1}{2}a^2 z + A$$

$$= \tfrac{1}{2}x\sqrt{a^2 + x^2} + \tfrac{1}{2}a^2 \sinh^{-1}\frac{x}{a} + A.$$

Collecting these results, the standard forms for the integration of irrational quadratic functions are:

$$\left.\begin{aligned} &(a) \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + A, \\ &(b) \int \sqrt{a^2 - x^2} \, . \, dx = \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a} + \tfrac{1}{2}x\sqrt{a^2 - x^2} + A. \end{aligned}\right\} \quad \ldots\ldots (46)$$

$$\left.\begin{aligned}
&(a)\ \int\frac{\mathbf{dx}}{\sqrt{\mathbf{x^2-a^2}}}=\cosh^{-1}\frac{\mathbf{x}}{\mathbf{a}}+\mathbf{A}=\log\frac{\mathbf{x}+\sqrt{\mathbf{x^2-a^2}}}{\mathbf{a}}+\mathbf{A},\\
&(b)\ \int\sqrt{\mathbf{x^2-a^2}}\,.\,\mathbf{dx}=\tfrac{1}{2}\mathbf{x}\sqrt{\mathbf{x^2-a^2}}-\tfrac{1}{2}\mathbf{a^2}\cosh^{-1}\frac{\mathbf{x}}{\mathbf{a}}+\mathbf{A}\\
&\qquad=\tfrac{1}{2}\mathbf{x}\sqrt{\mathbf{x^2-a^2}}-\tfrac{1}{2}\mathbf{a^2}\,.\,\log\frac{\mathbf{x}+\sqrt{\mathbf{x^2-a^2}}}{\mathbf{a}}+\mathbf{A}.
\end{aligned}\right\}\ \ldots\ldots(47)$$

$$\left.\begin{aligned}
&(a)\ \int\frac{\mathbf{dx}}{\sqrt{\mathbf{a^2+x^2}}}=\sinh^{-1}\frac{\mathbf{x}}{\mathbf{a}}+\mathbf{A}=\log\frac{\mathbf{x}+\sqrt{\mathbf{a^2+x^2}}}{\mathbf{a}}+\mathbf{A},\\
&(b)\ \int\sqrt{\mathbf{a^2+x^2}}\,.\,\mathbf{dx}=\tfrac{1}{2}\mathbf{x}\sqrt{\mathbf{a^2+x^2}}+\tfrac{1}{2}\mathbf{a^2}\sinh^{-1}\frac{\mathbf{x}}{\mathbf{a}}+\mathbf{A}\\
&\qquad=\tfrac{1}{2}\mathbf{x}\sqrt{\mathbf{a^2+x^2}}+\tfrac{1}{2}\mathbf{a^2}\log\frac{\mathbf{x}+\sqrt{\mathbf{a^2+x^2}}}{\mathbf{a}}+\mathbf{A}.
\end{aligned}\right\}\ \ldots\ldots(48)$$

42. General Quadratic Function. The general quadratic function,

$$X\equiv ax^2+bx+c,$$

may readily be expressed in the form $v^2\pm\alpha^2$, for

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}=a(v^2\pm\alpha^2),$$

where $$v=x+\frac{b}{2a},\quad \alpha^2=\frac{4ac-b^2}{4a^2},$$

the sign of α^2 depending upon whether $4ac>b^2$ or $4ac<b^2$.

Ex. 15. *Evaluate the integrals* $\int_{1\cdot8}^{3}dx/\sqrt{X}$ and $\int_{1\cdot8}^{3}\sqrt{X}\,.\,dx$, *when* $X\equiv5x^2-6x-3$.

$$\begin{aligned}5x^2-6x-3&=5(x^2-1\cdot2x-0\cdot6)\\
&=5\{(x-0\cdot6)^2-0\cdot36-0\cdot6\}=5\{(x-0\cdot6)^2-0\cdot96\}.\end{aligned}$$

Put $v=x-0\cdot6$ and $\alpha^2=0\cdot96$, then $dv=dx$, and when $x=3$, $v=2\cdot4$, and when $x=1\cdot8$, $v=1\cdot2$, so that

$$\int_{1\cdot8}^{3}\frac{dx}{\sqrt{X}}=\frac{1}{\sqrt{5}}\int_{1\cdot2}^{2\cdot4}\frac{dv}{\sqrt{v^2-\alpha^2}}=\tfrac{1}{5}\,.\,\sqrt{5}\left[\log\frac{v+\sqrt{v^2-\alpha^2}}{\alpha}\right]_{1\cdot2}^{2\cdot4},\ \text{by } (47a),$$

$$=\tfrac{1}{5}\,.\,\sqrt{5}\{\log(2\cdot4+\sqrt{4\cdot8})-\log(1\cdot2+\sqrt{0\cdot48})\},$$

the constant $\log\alpha$ cancelling out,

$$=\tfrac{1}{5}(2\cdot2361)(\log4\cdot591-\log1\cdot893)$$

$$=0\cdot4472(1\cdot5241-0\cdot6382)=0\cdot4472\times0\cdot8859=\mathbf{0\cdot3961}.$$

Similarly, by (47*b*),

$$\int_{1\cdot 8}^{3}\sqrt{X}\,.\,dx=\sqrt{5}\int_{1\cdot 2}^{2\cdot 4}\sqrt{v^2-a^2}\,.\,dv$$

$$=\tfrac{1}{2}\,.\,\sqrt{5}\left[v\sqrt{v^2-a^2}-a^2\log\frac{v+\sqrt{v^2-a^2}}{a}\right]_{1\cdot 2}^{2\cdot 4}$$

$$=\tfrac{1}{2}\sqrt{5}\{2\cdot 4\sqrt{4\cdot 8}-0\cdot 96\log(2\cdot 4+\sqrt{4\cdot 8})$$
$$-1\cdot 2\sqrt{0\cdot 48}+0\cdot 96\log(1\cdot 2+\sqrt{0\cdot 48})\}$$

$$=1\cdot 118\{5\cdot 2584-0\cdot 8314-0\cdot 96(\log 4\cdot 591-\log 1\cdot 893)\}$$

$$=1\cdot 118\,(4\cdot 427-0\cdot 96\times 0\cdot 8859)$$

$$=1\cdot 118(4\cdot 427-0\cdot 850)$$

$$=1\cdot 118\times 3\cdot 577$$

$$=\mathbf{3{\cdot}999 \text{ or } 4 \text{ very nearly.}}$$

Ex. 16. *Evaluate the integrals*

$$(a)\ \int_{3\cdot 5}^{9\cdot 5}\frac{dx}{\sqrt{175+20x-4x^2}},\quad (b)\ \int_{1\cdot 5}^{4\cdot 5}\sqrt{27+12x-4x^2}\,.\,dx.$$

(*a*) $175+20x-4x^2=175-(2x-5)^2+25=200-(2x-5)^2$.

Let $v=2x-5$, $a^2=200$, then $dv=2\,.\,dx$; and when $x=9{\cdot}5$, $v=14$; and when $x=3{\cdot}5$, $v=2$, so that

$$\int_{3\cdot 5}^{9\cdot 5}\frac{dx}{\sqrt{175+20x-4x^2}}=\tfrac{1}{2}\int_{2}^{14}\frac{dv}{\sqrt{a^2-v^2}}=\tfrac{1}{2}\left[\sin^{-1}\frac{v}{a}\right]_2^{14}$$

$$=\tfrac{1}{2}(\sin^{-1}0\cdot 9899-\sin^{-1}0\cdot 1414)$$

$$=\tfrac{1}{2}(1\cdot 4283-0\cdot 1420)$$

$$=\mathbf{0{\cdot}6432.}$$

Note that $\sin^{-1}\dfrac{v}{a}$ *must be expressed in radians.*

(*b*) $27+12x-4x^2=27-(2x-3)^2+9=36-(2x-3)^2$.

Let $v=2x-3$, then $dv=2\,.\,dx$, and when $x=4{\cdot}5$, $v=6$, and when $x=1{\cdot}5$, $v=0$;

$$\therefore\ \int_{1\cdot 5}^{4\cdot 5}\sqrt{27+12x-4x^2}\,.\,dx=\tfrac{1}{2}\int_0^6\sqrt{36-v^2}\,.\,dv$$

$$=\tfrac{1}{4}\left[36\sin^{-1}\frac{v}{6}+v\sqrt{36-v^2}\right]_0^6$$

$$=\tfrac{1}{4}\,.\,18\pi=\mathbf{4{\cdot}5\pi.}$$

Ex 17. *If* $X=4x^2-4x+145$, *calculate the values of*

$$(a)\ \int_3^5 \frac{dx}{\sqrt{X}} \quad \textit{and} \quad (b)\ \int_3^5 \sqrt{X}\,.\,dx.$$

$$4x^2-4x+145=(2x-1)^2+144.$$

Let $v=2x-1$, then $dv=2\,.\,dx$, and when $x=5$, $v=9$, and when $x=3$, $v=5$;

$$\therefore\ (a)\ \int_3^5 \frac{dx}{\sqrt{4x^2-4x+145}}=\tfrac{1}{2}\int_5^9 \frac{dv}{\sqrt{v^2+144}}=\tfrac{1}{2}\left[\log\frac{v+\sqrt{v^2+144}}{12}\right]_5^9$$

$$=\tfrac{1}{2}\{\log 24-\log 18\}=\tfrac{1}{2}\log\tfrac{24}{18}=\tfrac{1}{2}\log\tfrac{4}{3}$$

$$=\tfrac{1}{2}(\log 4-\log 3)=\tfrac{1}{2}(1\cdot3863-1\cdot0986)$$

$$=0\cdot1439$$

$$(b)\ \int_3^5 \sqrt{4x^2-4x+145}\,.\,dx=\tfrac{1}{2}\int_5^9 \sqrt{v^2+144}\,.\,dv$$

$$=\tfrac{1}{4}\left[v\sqrt{v^2+144}+144\log\frac{v+\sqrt{v^2+144}}{12}\right]_5^9$$

$$=\tfrac{1}{4}\{135+144\log 24-65-144\log 18\}$$

$$=\tfrac{1}{4}\{70+144\times 0\cdot2877\}=27\cdot8572.$$

Ex. 18. *Prove that* $4\int_{0\cdot5}^{6\cdot5}\frac{47x-46-8x^2}{\sqrt{2(2x^2-2x+13)}}\,.\,dx=\log 5$

$$2(2x^2-2x+13)=4x^2-4x+26=(2x-1)^2+25=u^2+25,$$

where $\qquad u=2x-1 \quad \text{or} \quad x=\tfrac{1}{2}(u+1).$

Also $\qquad 47x-46-8x^2=39x+6-2(4x^2-4x+26)$

$$=\tfrac{1}{2}(39u+51)-2(u^2+25).$$

Now, since $u=2x-1$, $du=2\,.\,dx$, and when $x=6\cdot5$, $u=12$, and when $x=0\cdot5$, $u=0$; hence

$$4\int_{0\cdot5}^{6\cdot5}\frac{47x-46-8x^2}{\sqrt{2(2x^2-2x+13)}}\,.\,dx=2\int_0^{12}\frac{\tfrac{1}{2}(39u+51)-2(u^2+25)}{\sqrt{u^2+25}}\,.\,du$$

$$=39\int_0^{12}\frac{u\,.\,du}{\sqrt{u^2+25}}+51\int_0^{12}\frac{du}{\sqrt{u^2+25}}-4\int_0^{12}\sqrt{u^2+25}\,.\,du$$

$$=39\left[\sqrt{u^2+25}\right]_0^{12}+51\left[\log\frac{u+\sqrt{u^2+25}}{5}\right]_0^{12}$$

$$-2\left[u\sqrt{u^2+25}+25\log\frac{u+\sqrt{u^2+25}}{5}\right]_0^{12}$$

$$=39(13-5)+51\log 5-2(156+25\log 5)=\log 5.$$

EXERCISES 6D.

Integrate each of the following functions of x.

1. $\sqrt{(16 - x^2)}$. **2.** $1/\sqrt{(16 - x^2)}$. **3.** $\sqrt{(x^2 - 36)}$.

4. $1/\sqrt{(x^2 - 36)}$. **5.** $\sqrt{(x^2 + 81)}$. **6.** $1/\sqrt{(x^2 + 81)}$.

7. $\sqrt{(x^2 - 4x - 21)}$. **8.** $1/\sqrt{(x^2 - 4x - 21)}$. **9.** $\sqrt{(55 - 6x - x^2)}$.

10. $1/\sqrt{(55 - 6x - x^2)}$. **11.** $\sqrt{(x^2 + 10x + 106)}$.

12. $1/\sqrt{(x^2 + 10x + 106)}$. **13.** $\sqrt{(9x^2 - 30x - 119)}$.

14. $1/\sqrt{27 - 4x(x + 3)}$. **15.** $\sqrt{(a^2x^2 + 2abx + b^2 - c^2)}$.

16. $1/\sqrt{a^2x^2 + 2acx + 2c^2}$. **17.** $\sqrt{ax(2c - ax)}$. **18.** $1/\sqrt{x(2c - x)}$.

Evaluate the following definite integrals :

19. $\displaystyle\int_2^6 \frac{dx}{\sqrt{64 - x^2}}$. **20.** $\displaystyle\int_{-6}^4 \frac{dx}{\sqrt{64 - 12x - x^2}}$.

21. $\displaystyle\int_0^{20} \frac{dx}{\sqrt{441 + x^2}}$. **22.** $\displaystyle\int_7^{11\cdot 25} \frac{dx}{\sqrt{x^2 - 49}}$.

23. $\displaystyle\int_{-1}^{0\cdot 5} \frac{dx}{\sqrt{35 - 2x - x^2}}$. **24.** $\displaystyle\int_{-2\cdot 5} \frac{dx}{\sqrt{61 + 10x + x^2}}$.

25. $\displaystyle\int_4^{18} \frac{dx}{\sqrt{x(x + 32)}}$. **26.** $\displaystyle\int_{2\cdot 5}^{8} \frac{dx}{\sqrt{112 - 6x - x^2}}$.

27. $\displaystyle\int_{-3\cdot 5}^{2} \frac{2\,.\,dx}{\sqrt{18 - 7x - x^2}}$. ***28.** $\displaystyle\int_{3b/a}^{7b/2a} \frac{a\,.\,dx}{\sqrt{a^2x^2 - 2abx - 3b^2}}$.

29. $\displaystyle\int_{3\cdot 5}^{8} \sqrt{8 + 7x - x^2}\,.\,dx$. **30.** $\displaystyle\int_{6\cdot 8}^{8\cdot 7} \sqrt{x^2 - 36}\,.\,dx$.

31. $\displaystyle\int_0^1 \frac{dx}{\sqrt{14 - 6x - x^2}}$. **32.** $\displaystyle\int_{-2}^1 \sqrt{5 - 4x - x^2}\,.\,dx$.

33. $\displaystyle\int_9^{13} \sqrt{18x - 65 - x^2}\,.\,dx$. ***34.** $\displaystyle\int_{b/a}^{4b/a} \frac{2\,.\,dx}{\sqrt{8b^2 + 2abx - a^2x^2}}$.

35. $\displaystyle\int_1^8 \sqrt{x^2 + 14x + 113}\,.\,dx$. **36.** $\displaystyle\int_{10}^{24} \sqrt{x(x + 30)}\,.\,dx$.

***37.** Shew by means of the substitution $y = x/(x^2 + 1)$, that

$$\int_0^1 \frac{1 + x^2}{(x^2 + 1)\sqrt{2(x^4 + 1)}}\,.\,dx = \frac{\pi}{8}.$$

***38.** Evaluate $8\int_0^{6\cdot 5} \frac{133 - 4x^2}{\sqrt{169 - 4x^2}} \, . \, dx$.

***39.** A function y of x is defined by the relation $\int y \, . \, dx = \sqrt{a^2 + y^2}$; find the function, it being given that x and y vanish simultaneously.

***40.** If $\int_b^a \frac{dx}{\sqrt{(1 + \cos x)(8 \cos x + 15 \sin x)}} = \frac{\pi}{4}$, and $a = 2 \tan^{-1} 4$, prove that

$$b = 2 \tan^{-1}(\tfrac{15}{8}).$$

***41.** Evaluate $\int_{-1}^{3} \frac{dx}{y}$, where $y^2 = 32 - 7x - x^2$.

***42.** Find the precise relation between x and y when

$$\left(\frac{dx}{dy}\right)^2 = x^2(1 + x^2),$$

it being given that $y = 0$ when $x = 0\cdot 75$.

43. Integration of a Product of Functions by Parts. Let $P(x)$ and $Q(x)$ be two functions of x, which may, for shortness, be denoted by P and Q respectively, then from (35b), p. 76,

$$\frac{d}{dx}(PQ) = P\frac{dQ}{dx} + Q\frac{dP}{dx}.$$

Now, in general, $\frac{dP}{dx}$ will be a function of x; denote it by R, where R stands for a function, $R(x)$, then $dP = R \, . \, dx$,

or
$$P = \int R \, . \, dx,$$

the constant being omitted for the present.

Hence, substituting for P in the above formula,

$$\frac{d}{dx}(PQ) = \frac{dQ}{dx}\int R \, . \, dx + QR;$$

$$\therefore \; QR = \frac{d}{dx}(PQ) - \frac{dQ}{dx}\int R \, . \, dx,$$

or
$$QR \, . \, dx = d(PQ) - \left(\frac{dQ}{dx}\int R \, . \, dx\right)dx.$$

Integrating this gives

$$\int QR \,.\, dx = PQ - \int\left(\frac{dQ}{dx}\int R \,.\, dx\right)dx + A,$$

i.e.

$$\int \mathbf{QR} \,.\, \mathbf{dx} = \mathbf{Q}\int \mathbf{R} \,.\, \mathbf{dx} - \int\left(\frac{\mathbf{dQ}}{\mathbf{dx}}\int \mathbf{R} \,.\, \mathbf{dx}\right)\mathbf{dx} + \mathbf{A}. \quad \text{.............}(49)$$

This is the formula which gives the integral of a product of two functions. Expressed in words, the rule becomes :

The integral of a product of two functions = (First function multiplied by the integral of the second) − Integral of (differential coefficient of first function multiplied by the integral of the second), either function being selected as the first.

Ex. 19. *Evaluate the indefinite integral* $\int x \,.\, \sin^{-1} x \,.\, dx$, *and prove that*

$$2\int_0^1 x \,.\, \sin^{-1} x \,.\, dx = \int_0^{\frac{\pi}{2}} v \,.\, \sin 2v \,.\, dv = \frac{\pi}{4}.$$

Here x could be taken as the first function, because when differentiated it gives a constant. This should, in general, be the chief consideration in choosing the first function, but it is simpler in this case to take $\sin^{-1} x$ as the first function, as will be seen.

Hence

$$\int x \sin^{-1} x \,.\, dx = \sin^{-1} x\int x\, dx - \int\left(\frac{1}{\sqrt{1-x^2}}\int x \,.\, dx\right)dx$$

$$= \tfrac{1}{2}x^2 \sin^{-1} x - \tfrac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}} \,.\, dx$$

$$= \tfrac{1}{2}x^2 \sin^{-1} x + \tfrac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}} \,.\, dx$$

$$= \tfrac{1}{2}x^2 \sin^{-1} x + \tfrac{1}{2}\int\sqrt{1-x^2} \,.\, dx - \tfrac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}$$

$$= \tfrac{1}{2}x^2 \sin^{-1} x + \tfrac{1}{4}(\sin^{-1} x + x\sqrt{1-x^2}) - \tfrac{1}{2}\sin^{-1} x + A$$

$$= \tfrac{1}{4}\{(\mathbf{2x^2-1})\,\mathbf{\sin^{-1} x} + \mathbf{x}\sqrt{\mathbf{1-x^2}}\} + \mathbf{A}.$$

Taking x as the first function :

$$\int x \sin^{-1} x\, dx = x\int \sin^{-1}x \,.\, dx - \int\left(\int \sin^{-1} x \,.\, dx\right)dx\,;$$

thus the evaluation of the integral depends upon the integral $\int \sin^{-1} x \, . \, dx$. To evaluate this, the method may be applied by considering unity as one function, thus taking $\sin^{-1} x$ as the first function :

$$\int \sin^{-1} x \, . \, dx = x \, . \, \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, . \, dx$$

$$= x \, . \, \sin^{-1} x + \int du, \text{ where } u^2 = 1 - x^2,$$

$$= x \, . \, \sin^{-1} x + u$$

$$= x \, . \, \sin^{-1} x + \sqrt{1 - x^2}.$$

Hence, inserting this value,

$$\int x \sin^{-1} x \, . \, dx = x^2 \sin^{-1} x + x\sqrt{1 - x^2} - \int x \sin^{-1} x \, dx - \int \sqrt{1 - x^2} \, . \, dx;$$

$$\therefore \; 2\int x \sin^{-1} x \, . \, dx = x^2 \sin^{-1} x + x\sqrt{1 - x^2} - \tfrac{1}{2} \sin^{-1} x - \tfrac{1}{2} x\sqrt{1 - x^2} + A'$$

$$= \tfrac{1}{2}\{(2x^2 - 1) \sin^{-1} x + x\sqrt{1 - x^2}\} + A';$$

$$\therefore \; \int x \, . \, \sin^{-1} x \, . \, dx = \tfrac{1}{4}\{(2x^2 - 1) \sin^{-1} x + x\sqrt{1 - x^2}\} + A$$

(where $2A = A'$), as before.

Again, $$2\int_0^1 x \sin^{-1} x \, . \, dx = \tfrac{1}{2}\Big[(2x^2 - 1) \sin^{-1} x + x\sqrt{1 - x^2})\Big]_0^1$$

$$= \frac{\pi}{4},$$

and $$\int_0^{\frac{\pi}{2}} v \, . \, \sin 2v \, . \, dv = \Big[v\int \sin 2v \, . \, dv - \int\Big(\int \sin 2v \, . \, dv\Big) dv\Big]_0^{\frac{\pi}{2}}$$

$$= \Big[-\tfrac{1}{2} v \cos 2v + \tfrac{1}{2}\int \cos 2v \, . \, dv\Big]_0^{\frac{\pi}{2}}$$

$$= \Big[-\tfrac{1}{2} v \, . \, \cos 2v + \tfrac{1}{4} \sin 2v\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}.$$

The equality of the integrals may also be shewn by putting $v=\sin^{-1} x$ in the first integral, thus

$$x=\sin v,$$

so that $dx=\cos v \,.\, dv$, and when $x=1$, $v=\frac{\pi}{2}$, and when $x=0$, $v=0$

$$\therefore \quad 2\int_0^1 x \,.\, \sin^{-1}x \,.\, dx=2\int_0^{\frac{\pi}{2}} v \,.\, \sin v \,.\, \cos v \,.\, dv=\int_0^{\frac{\pi}{2}} v \sin 2v \,.\, dv.$$

Hence $$2\int_0^1 x \sin^{-1}x \,.\, dx=\int_0^{\frac{\pi}{2}} \mathbf{v} \,.\, \mathbf{\sin 2v} \,.\, \mathbf{dv}=\frac{\pi}{\mathbf{4}}.$$

Ex. 20. *Assuming that* $\int\frac{dx}{\sqrt{a^2+x^2}}=\log\frac{x+\sqrt{a^2+x^2}}{a}$, *shew that*

$$\int\sqrt{a^2+x^2} \,.\, dx=\tfrac{1}{2}\{x\sqrt{a^2+x^2}+a^2\log(x+\sqrt{a^2+x^2})\}$$

by integrating by parts.

Taking unity as one function, and regarding $\sqrt{a^2+x^2}$ as the first function :

$$\int\sqrt{a^2+x^2} \,.\, dx=x\sqrt{a^2+x^2}-\int\frac{x^2}{\sqrt{a^2+x^2}} \,.\, dx$$

$$=x\sqrt{a^2+x^2}-\int\frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}} \,.\, dx$$

$$=x\sqrt{a^2+x^2}-\int\sqrt{a^2+x^2} \,.\, dx+a^2\int\frac{dx}{\sqrt{a^2+x^2}};$$

$$\therefore \quad 2\int\sqrt{a^2+x^2} \,.\, dx=x\sqrt{a^2+x^2}+a^2\log\frac{x+\sqrt{a^2+x^2}}{a}+A';$$

$$\therefore \quad \int\sqrt{a^2+x^2} \,.\, dx=\tfrac{1}{2}\left(\mathbf{x}\sqrt{\mathbf{a}^2+\mathbf{x}^2}+\mathbf{a}^2\log\frac{\mathbf{x}+\sqrt{\mathbf{a}^2+\mathbf{x}^2}}{\mathbf{a}}\right)+\mathbf{A}.$$

Ex. 21. *Evaluate the integral* $\int\sin^3\theta \,.\, d\theta$; (*a*) *by applying the identity* $4\sin^3\theta=3\sin\theta-\sin 3\theta$: (*b*) *by parts. Shew that the two results agree.*

Hence prove that $$24\int_0^1\frac{t^3}{(1+t^2)^4} \,.\, dt=1.$$

(*a*) From the given identity

$$\int \sin^3\theta \,.\, d\theta = \tfrac{1}{4}\int (3 \sin \theta - \sin 3\theta) \,.\, d\theta = -\tfrac{3}{4} \cos \theta + \tfrac{1}{12} \cos 3\theta + A$$

$$= -\tfrac{3}{4} \cos \theta + \tfrac{1}{12} (4 \cos^3\theta - 3 \cos \theta) + A$$

$$= \tfrac{1}{3} \cos^3\theta - \cos \theta + A.$$

(*b*) Taking $\sin^2\theta$, $\sin \theta$ as two functions,

$$\int \sin^3\theta \,.\, d\theta = \sin^2\theta(-\cos \theta) - 2\int \sin \theta \cos \theta(-\cos \theta) d\theta$$

$$= -\cos \theta(1 - \cos^2 \theta) - 2\int \cos^2 \theta \,.\, d(\cos \theta)$$

$$= \cos^3\theta - \cos \theta - \tfrac{2}{3} \cos^3\theta + A$$

$$= \tfrac{1}{3} \cos^3\theta - \cos \theta + A, \text{ as before.}$$

It should be noticed that the values of the integrals, without modification, are

$$(a) \quad -\tfrac{3}{4} \cos \theta + \tfrac{1}{12} \cos 3\theta + A,$$

$$(b) \quad -\sin^2\theta \,.\, \cos \theta - \tfrac{2}{3} \cos^3\theta + A\ ;$$

and these are identical, as is shewn by putting

$4 \cos^3\theta - 3 \cos \theta$ for $\cos 3\theta$ in (*a*),

and $1 - \cos^2\theta$ for $\sin^2\theta$ in (*b*).

Put $t = \tan \dfrac{\theta}{2}$, then $dt = \frac{1}{2} \sec^2 \dfrac{\theta}{2} \,.\, d\theta$, and $1 + t^2 = \sec^2 \dfrac{\theta}{2}$.

Further, the limits 1, 0, of t give $\dfrac{\pi}{2}$, 0 for θ ;

$$\therefore\ 24\int_0^1 \frac{t^3}{(1+t^2)^4}\, dt = 12 \int_0^{\frac{\pi}{2}} \frac{\tan^3 \dfrac{\theta}{2} \,.\, \sec^2 \dfrac{\theta}{2}}{\sec^8 \dfrac{\theta}{2}} \,.\, d\theta$$

$$= 12\int_0^{\frac{\pi}{2}} \sin^3 \frac{\theta}{2} \,.\, \cos^3 \frac{\theta}{2} \,.\, d\theta$$

$$= \tfrac{3}{2}\int_0^{\frac{\pi}{2}} \sin^3\theta \,.\, d\theta = \tfrac{3}{2}\Big[\tfrac{1}{3} \cos^3\theta - \cos \theta\Big]_0^{\frac{\pi}{2}}$$

(from the above result)

$$= \tfrac{3}{2}(-\tfrac{1}{3} + 1) = 1.$$

Ex. 22. *If S, C denote respectively the integrals*

$$\int e^{ax} \sin bx \,.\, dx, \quad \int e^{ax} \cos bx \,.\, dx,$$

determine, by integration by parts, the values of S and C.

Verify the results independently by use of the identity

$$\cos bx + i \sin bx = e^{ibx},$$

where $i = \sqrt{-1}$.

Taking $\sin bx$, $\cos bx$ respectively as the first function,

$$S = \int e^{ax} \sin bx \,.\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \cdot \int e^{ax} \cos bx \,.\, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \,.\, C, \text{ omitting the constant ;}$$

and $$C = \int e^{ax} \cos bx \,.\, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \cdot \int e^{ax} \sin bx \,.\, dx$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \,.\, S.$$

Hence $$aS + bC = e^{ax} \sin bx,$$

$$bS + aC = -e^{ax} \cos bx.$$

Solving these equations for S and C,

$$S = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx), \quad C = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx).$$

The same results may be readily established by the use of imaginaries, thus, writing as usual i for $\sqrt{-1}$,

$$\cos bx + i \sin bx = e^{ibx};$$

$$\therefore \int e^{ax} (\cos bx + i \sin bx) \,.\, dx = \int e^{ax} \,.\, e^{ibx} \,.\, dx = \int e^{(a+ib)x} \,.\, dx$$

$$= e^{(a+ib)x}/(a+ib) = (a - ib) e^{ax} \,.\, e^{ibx}/(a^2 + b^2)$$

$$= (a - ib) e^{ax} (\cos bx + i \sin bx)/(a^2 + b^2),$$

so that

$$C + iS = e^{ax} \{(a \cos bx + b \sin bx) + i(a \sin bx - b \cos bx)\}/(a^2 + b^2).$$

Equating real and imaginary parts :

$$C = e^{ax}(a \cos bx + b \sin bx)/(a^2 + b^2),$$

and $$S = e^{ax}(a \sin bx - b \cos bx)/(a^2 + b^2),$$

as before.

Exercises 6E.

Integrate each of the following functions by parts :

1. $x \cos nx$. **2.** $e^x \cos x$. **3.** $\log x$.

4. $\tan^{-1} x$. **5.** $x \tan^{-1} x$. (B.U.) **6.** $x \log x$.

7. $x^2 \sin 3x$. **8.** $x^3 \cos 2x$. **9.** $\log x/(1-x)^2$.

10. Shew that if a function $\phi(x)$ can be expressed in the form

$$f(x) + \frac{d}{dx}\{f(x)\},$$

then

$$\int e^x \, . \, \phi(x) \, . \, dx = e^x \, . \, f(x).$$

Use this theorem to evaluate each of the following functions :

11. $\int e^x(2x+3)(2x+1) \, . \, dx$. **12.** $\int e^x(x^3+6) \, . \, dx$.

13. $\int e^x(\sin x + \cos x) \, . \, dx$. **14.** $\int e^{\tan^{-1} t}(1+t+t^2) \, . \, dt/(1+t^2)$.

15. $\int e^x \sec x(1+\tan x) \, . \, dx$. **16.** $\int \frac{e^x}{x}(x \log x + 1) \, . \, dx$.

17. $\int (ea)^x dx$. **18.** $\int_0^1 e^x(x-2)(2x+3)(3x+1)dx$.

19. $\int_3^4 \frac{25-x-x^2}{\sqrt{(5+x)(5-x)}} \, . \, e^x \, . \, dx$.

20. $\int_0^{\sin^{-1} 0 \cdot 6} e^x\{\log(1+\sin x) - \log \cos x + \sec x\} \, . \, dx$.

21. Evaluate, by repeated application of the method of integration by parts, the definite integral

$$\int_0^1 x^2 \, . \, (\log x)^4 \, . \, dx,$$

and verify the result by using the substitution $x = e^{-y}$ to evaluate the integral.

If S_r, C_r, T_r, Se_r, denote the indefinite integrals

$$\int \sin^r x \, . \, dx, \quad \int \cos^r x \, . \, dx, \quad \int \tan^r x \, . \, dx, \quad \int \sec^r x \, . \, dx,$$

respectively, where r is a positive integer, prove the following **reduction formulae** (22 to 25), by the method of integration by parts :

***22.** $n \, . \, S_n = (n-1)S_{n-2} - \cos x \, . \, \sin^{n-1} x$; hence evaluate S_3 and S_5.

***23.** $nC_n = (n-1)C_{n-2} + \cos^{n-1} x \, . \, \sin x$; hence evaluate C_3 and C_5.

***24.** $(n-1)T_n = \tan^{n-1}x - (n-1)T_{n-2}$; hence evaluate T_3, and verify the result by substituting $t = \tan x$ to reduce the integral.

***25.** $(n-1)Se_n = (n-2)Se_{n-2} + \sec^{n-2}x \cdot \tan x$; hence evaluate Se_4 between the limits $\frac{\pi}{4}$ and 0, and verify the result by evaluating the integral independently by means of the substitution $t = \tan x$. (Le.U.)

***26.** If I_n denotes the indefinite integral $\int x^n \cdot \sin x \cdot dx$, shew that

$$I_n = x^{n-1}(n \sin x - x \cos x) - n(n-1)I_{n-2};$$

and evaluate the integral when $n = 3$.

27. Evaluate $\int_0^{\infty} e^{-x^2}x^3 \cdot dx$.

***28.** Shew that if I_n denote the integral

$$\int_0^x x^n(a^2+x^2)^{-\frac{1}{2}}\,dx,$$

when n is positive and greater than unity,

$$nI_n + n(n-1)a^2I_{n-2} = x^{n-1}(x^2+a^2)^{\frac{1}{2}}. \qquad \text{(L.U., Sc.)}$$

***29.** Shew by integration by parts that if I_n denotes the integral

$$\int_0^{\beta} e^{ax}\sin^n x \cdot dx,$$

where $\beta = \tan^{-1}\frac{n}{a}$, then

$$(n^2+a^2)I_n = n(n-1)I_{n-2}. \qquad \text{(L.U., Sc.)}$$

***30.** Shew that $\int_0^{\infty}(e^{-bx}\cos px)^2dx = \frac{1}{4}\left(\frac{1}{b} + \frac{b}{p^2+b^2}\right)$. (L.U.)

***31.** Prove that $n\int_0^{\frac{\pi}{2}}\sin^n x \cdot dx = (n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x \cdot dx$. (L.U.)

***32.** Prove that $\int_0^{\pi}\sin mx \cdot \sin nx \cdot dx = (n \sin m\pi \cos n\pi)/(m^2+n^2)$, where n is an integer and m is not an integer. (L.U.)

***33.** Shew that $\int_0^{\frac{\pi}{2}}\sin^n x \cdot \cos^m x \cdot dx = \frac{n-1}{m+n}\int_0^{\frac{\pi}{2}}\sin^{n-2}x \cdot \cos^m x \cdot dx$.

(L.U., Sc.)

***34.** Prove that $n\int_0^{\frac{\pi}{2}}\cos^n\theta \cdot d\theta = (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}\theta \cdot d\theta$, where n is a positive integer. (L.U., Sc.)

***35.** Prove that if $u_n = \int_0^1 x^n \cos \pi x \,.\, dx$, and $n > 1$,

$$\pi^2 u_n + n + n(n-1)u_{n-2} = 0. \qquad \text{(L.U., Sc.)}$$

36. Find $\int_0^{\frac{\pi}{2}} x^2 \sin x \,.\, dx$. (L.U.)

***37.** Prove that $\int_1^{\infty} \frac{dx}{(x+\sqrt{x^2-1})^n} = \frac{n}{n^2-1}$. (S.U.)

44. Miscellaneous Methods of Integration. Gamma Functions. The definite integral,

$$\int_0^{\frac{\pi}{2}} \sin^p x \,.\, \cos^q x \,.\, dx,$$

where p and q are positive integers, including zero, can be easily evaluated by means of **Gamma functions.**

Let $\Gamma(1+n)$ be a function defined by the relations:

$$\mathbf{\Gamma(1+n) = n \,.\, \Gamma(n)\,;\ \Gamma(1) = 1\,;\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi},} \qquad \text{. . .(50a)}$$

then, when n is a positive integer (see Ex. 9 on p. 139),

$$\mathbf{\Gamma(n+1) = n \,.\, \Gamma(n) = n(n-1) \,.\, \Gamma(n-1) = \ldots = |\underline{n}.} \qquad \text{....(50b)}$$

When n is a positive multiple of $\frac{1}{2}$, write $\frac{1}{2}(2m-1)$ for n, then

$$\Gamma(1+n) = \Gamma\left(1+\frac{2m-1}{2}\right) = \frac{2m-1}{2} \,.\, \Gamma\left(\frac{2m-1}{2}\right), \text{ by (50a)},$$

$$= \frac{2m-1}{2}\Gamma\left(1+\frac{2m-3}{2}\right) = \frac{2m-1}{2} \,.\, \frac{2m-3}{2} \,.\, \Gamma\left(\frac{2m-3}{2}\right).$$

By repeated applications of (29a),

$$\Gamma\left(1+\frac{2m-1}{2}\right) = \frac{2m-1}{2} \,.\, \frac{2m-3}{2} \,.\, \frac{2m-5}{2} \cdots \tfrac{1}{2} \,.\, \Gamma(\tfrac{1}{2}),$$

i.e., $$\mathbf{\Gamma\left(1+\frac{2m-1}{2}\right) = \frac{2m-1}{2} \,.\, \frac{2m-3}{2} \,.\, \frac{2m-5}{2} \cdots \tfrac{1}{2} \,.\, \sqrt{\pi}}. \qquad \text{.......(50c)}$$

With these properties of the Gamma function, the value of the above definite integral is given by the formula

$$\mathbf{\int_0^{\frac{\pi}{2}} \sin^p x \,.\, \cos^q x \,.\, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.} \qquad \text{....(51)}$$

This is true for all values of p and q, but the evaluation of $\Gamma(1+n)$ when n is negative or fractional in any denominator other than 2, is not simple. In practical integration, however, these negative and fractional values are rarely met.

Ex. 23. *Evaluate the definite integrals*

$$(a) \int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, . \, dx, \quad (b) \int_0^{\frac{\pi}{2}} \sin^5 x \, . \, dx,$$

and by means of the substitution, $x = \tan\theta$, shew that

$$\int_0^{\infty} \frac{x^3 \, . \, dx}{(1+x^2)^6} = 0{\cdot}025.$$

(*a*) By (51),

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, . \, dx = \frac{\Gamma(\frac{5}{2}) \, . \, \Gamma(3)}{2\Gamma(\frac{11}{2})} = \frac{\frac{3}{2} \, . \, \frac{1}{2} \, . \, \sqrt{\pi} \, . \, 2}{2 \, . \, \frac{9}{2} \, . \, \frac{7}{2} \, . \, \frac{5}{2} \, . \, \frac{3}{2} \, . \, \frac{1}{2} \, . \, \sqrt{\pi}} = \frac{8}{315}.$$

This integral could also be evaluated without Gamma functions by putting $y = \sin x$, then $dy = \cos x \, . \, dx$, and the limits $\frac{\pi}{2}$, 0 for x become 1, 0 for y; hence,

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, dx = \int_0^1 y^4(1-y^2)^2 dy = \int_0^1 (y^4 - 2y^6 + y^8) dy$$

$$= \Big[\tfrac{1}{5}y^5 - \tfrac{2}{7}y^7 + \tfrac{1}{9}y^9 \Big]_0^1 = \tfrac{1}{5} - \tfrac{2}{7} + \tfrac{1}{9} = \tfrac{8}{315}, \text{ as before.}$$

The substitution $y = \sin x$ will, however, only reduce the integrand when the index of $\cos x$ is odd; similarly, when the index of $\sin x$ is odd an effective substitution is $y = \cos x$, as the next example will shew.

(*b*) Put $y = \cos x$, then $dy = -\sin x \, . \, dx$, and the new limits become 0, 1;

$$\therefore \int_0^{\frac{\pi}{2}} \sin^5 x \, . \, dx = -\int_1^0 (1-y^2)^2 dy = \int_0^1 (1 - 2y^2 + y^4) dy$$

$$= \Big[y - \tfrac{2}{3}y^3 + \tfrac{1}{5}y^5 \Big]_0^1 = 1 - \tfrac{2}{3} + \tfrac{1}{5} = \tfrac{8}{15}.$$

By Gamma functions the work is, however, shorter, for

$$\int_0^{\frac{\pi}{2}} \sin^5 x \,.\, dx = \int_0^{\frac{\pi}{2}} \sin^5 x \,.\, \cos^0 x \,.\, dx = \frac{\Gamma(3)\Gamma(\frac{1}{2})}{2\Gamma(\frac{7}{2})}$$

$$= \frac{2\sqrt{\pi}}{2 \,.\, \frac{5}{2} \,.\, \frac{3}{2} \,.\, \frac{1}{2} \,.\, \sqrt{\pi}} = \frac{8}{15}, \text{ as before.}$$

In the given integral, put $x = \tan\theta$, then $dx = \sec^2\theta \,.\, d\theta$, and the new limits are $\frac{\pi}{2}$ and 0;

$$\therefore \int_0^\infty \frac{x^3 \,.\, dx}{(1+x^2)^6} = \int_0^{\frac{\pi}{2}} \frac{\tan^3\theta}{\sec^{12}\theta} \,.\, \sec^2\theta \,.\, d\theta = \int_0^{\frac{\pi}{2}} \sin^3\theta \,.\, \cos^7\theta \,.\, d\theta$$

$$= \frac{\Gamma(2) \,.\, \Gamma(4)}{2\Gamma(6)} = \frac{3 \,.\, 2}{2 \,.\, 5 \,.\, 3 \,.\, 4 \,.\, 2} = \frac{1}{40} = 0\text{·}025.$$

45. Integrals of the form $\int dx/(P\sqrt{Q})$. The evaluation of the integral

$$\int \frac{dx}{P\sqrt{Q}},$$

where P and Q are functions of x of degree not higher than 2, may be effected by suitable substitutions. These are, in general, as follows:

$$\left.\begin{array}{ll} (a) \text{ When P and Q are both linear, and} \\ (b) \text{ When P is quadratic and Q linear,} \end{array}\right\} \text{put } z^2 = Q. \\ (c) \text{ When P is linear and Q quadratic, put } \frac{1}{z} = P, \\ (d) \text{ When P and Q are both quadratic, put } z \text{ or } z^2 = \frac{Q}{P}. \qquad \ldots\ldots\ldots (52)$$

Ex. 24. *Evaluate the indefinite integral*

$$\int \frac{dx}{(ax+b)\sqrt{cx+d}},$$

a, b, c, d being positive constants, (i) *when $ad > bc$,* (ii) *when $ad = bc$,* (iii) *when $ad < bc$.*

Hence shew that $\quad 2\int_0^{16} \frac{dx}{(4x+1)\sqrt{3x+1}} = 0\text{·}9555.$

Let $y^2 = cx + d$, then $2y \,.\, dy = c \,.\, dx$, and

$$ax + b = \frac{a}{c}(y^2 - d) + b = \frac{1}{c}(ay^2 - ad + bc).$$

Write α^2 for $\dfrac{ad-bc}{a}$, then

(i) when $ad > bc$, $\quad ax+b=\dfrac{a}{c}(y^2-\alpha^2)$;

(ii) when $ad=bc$, $\alpha=0$, $\quad ax+b=\dfrac{a}{c}\cdot y^2$;

(iii) when $ad < bc$, write β^2 for $-\alpha^2$, then

$$ax+b=\frac{a}{c}(y^2+\beta^2).$$

Taking each of these in turn,

(i) $$\int\frac{dx}{(ax+b)\sqrt{cx+d}}=\frac{2}{a}\int\frac{dy}{y^2-\alpha^2}=\frac{1}{a\alpha}\cdot\log\frac{y-\alpha}{y+\alpha}+A$$

$$=\frac{1}{\sqrt{a(ad-bc)}}\log\frac{\sqrt{a(cx+d)}-\sqrt{ad-bc}}{\sqrt{a(cx+d)}+\sqrt{ad-bc}}+A. \quad \text{...(53a)}$$

(ii) The given integral becomes

$$\frac{2}{a}\int\frac{dy}{y^2}=-\frac{2}{ay}+B=B-\frac{2}{a\sqrt{cx+d}}; \quad \text{..................(53b)}$$

(iii) and the integral assumes the form

$$\frac{2}{a}\int\frac{dy}{y^2+\beta^2}=\frac{2}{a\beta}\tan^{-1}\frac{y}{\beta}+C$$

$$=\frac{2}{\sqrt{a(bc-ad)}}\tan^{-1}\sqrt{\frac{a(cx+d)}{bc-ad}}+C. \quad \text{...(53c)}$$

It should be noted that when $ad=bc$,

$$ax+b=\frac{b}{d}(cx+d);$$

$$\therefore \int\frac{dx}{(cx+d)^{\frac{3}{2}}}=-\frac{2}{c(cx+d)^{\frac{1}{2}}}+B, \quad \text{........................(54)}$$

a very important integral.

In the particular case, $a=4$, $b=1$, $c=3$, $d=1$, and $ad > bc$;

$\therefore$ from (53a),

$$2\int_0^{16}\frac{dx}{(4x+1)\sqrt{3x+1}}=\left[\log\frac{\sqrt{12x+4}-1}{\sqrt{12x+4}+1}\right]_0^{16}=\log\tfrac{13}{15}-\log\tfrac{1}{3}$$

$$=\log\tfrac{13}{5}=\log 2{\cdot}6=0{\cdot}9555.$$

In such a case as this, it is much better to work the integral out without reference to the general result, by making the substitution $y^2=3x+1$.

Ex. 25. *Calculate the value of the definite integral*

$$\int_{-1\cdot7}^{-1\cdot5} \frac{dx}{(2x+5)\sqrt{4x^2-9}}.$$

From (32c), let $2x+5=\frac{1}{z}$,

then $$x=\frac{1}{2}\left(\frac{1}{z}-5\right) \quad \text{and} \quad dx=-\frac{dz}{2z^2},$$

also $$4x^2-9=\left(\frac{1}{z}-5\right)^2-9=\frac{1}{z^2}(1-10z+16z^2)$$

$$=\frac{16}{z^2}\{(z-\tfrac{5}{16})^2-(\tfrac{3}{16})^2\},$$

and the limits $-1\cdot5$, $-1\cdot7$ for x become $0\cdot5$ and $0\cdot625$ for z.

Hence $$\int_{-1\cdot7}^{-1\cdot5} \frac{dx}{(2x+5)\sqrt{4x^2-9}}=-\frac{1}{8}\int_{0\cdot625}^{0\cdot5} \frac{dz}{\sqrt{(z-\frac{5}{16})^2-(\frac{3}{16})^2}}$$

$$=-\frac{1}{8}\left[\log \frac{z-\frac{5}{16}+\sqrt{(z-\frac{5}{16})^2-(\frac{3}{16})^2}}{\frac{3}{16}}\right]_{0\cdot625}^{0\cdot5}, \text{ by (17)},$$

$$=\tfrac{1}{8}\log 3=0\cdot1373.$$

EXERCISES 6F.

Evaluate each of the following definite integrals :

1. $\int_0^{\frac{\pi}{2}} \sin^5 x \,.\, \cos^4 x \,.\, dx$. (L.U.) **2.** $\int_0^{\frac{\pi}{2}} \cos^7\theta \,.\, d\theta$.

3. $\int_0^{\frac{\pi}{2}} \sin^4 x \,.\, dx$. **4.** $\int_0^{\frac{\pi}{2}} \cos^{12} x \,.\, \tan^5 x \,.\, dx$.

5. $\int_0^{\frac{\pi}{2}} \sin^2 2\theta \,.\, d\theta$. **6.** $\int_0^{\frac{\pi}{2}} (\cos^9 x \,.\, \tan^4 x+\sin^6 x \cos^3 x)\,.dx$.

7. Evaluate the integral $\int_0^1 x^3(1-x^2)^{\frac{1}{2}} \,.\, dx$, by the substitution $x=\sin\theta$.

***8.** If $I=\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \,.\, \cos^3\theta \,.\, d\theta$, shew, by the use of Gamma functions, that $21I=8$.

9. Defining the Gamma function by the relation

$$\Gamma(n)\,.\,\Gamma(1-n)=\pi \operatorname{cosec} n\pi,$$

where $0<n<1$, establish the following results :

(*a*) $\Gamma(\frac{1}{2})=\sqrt{\pi}$, (*b*) $\Gamma(\frac{1}{4})\,.\,\Gamma(\frac{3}{4})=\pi\sqrt{2}$, (*c*) $\Gamma(\frac{1}{3})\,.\,\Gamma(\frac{2}{3})=\frac{2}{3}\pi\sqrt{3}$.

Calculate the values of the following :

10. $\displaystyle\int_0^{\pi} \sin^4\frac{\theta}{2}\,.\,\cos^3\frac{\theta}{2}\,.\,d\theta.$

11. $\displaystyle\int_0^{\pi} \sin^6\frac{\theta}{2}\,.\,\cos^3\frac{\theta}{2}\,.\,d\theta.$

12. $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\,.\,\sec^6 x\,.\,dx.$

13. $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2x\,.\,\cos 4x\,.\,dx.$

14. $\displaystyle\int_0^{\frac{\pi}{2}} \cos 2x\,.\,\cos 3x\,.\,dx.$

15. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^8 x\,.\,dx.$

16. $\displaystyle\int_0^{\infty} \frac{x^4}{(1+x^2)^5}\,.\,dx.$

17. $\displaystyle\int_0^{\infty} \frac{x^5}{(1+x^2)^6}\,.\,dx.$

18. $\displaystyle\int_0^{\frac{\pi}{4}} \cos^5 x\,.\,\sin^4 x\,.\,dx.$

19. $\displaystyle 12\int_0^{2} \frac{x^2+1}{4x^3+12x+7}\,.\,dx.$

20. $\displaystyle\int_2^{5} \frac{dx}{x\sqrt{8x+9}}.$

21. $\displaystyle\int_0^{2.4} \frac{x^3}{(x^2+1)(x^2+2)}\,.\,dx.$

22. $\displaystyle\int_0^{4} \frac{dx}{(1+2x)\sqrt{2+x^2}}.$

23. $\displaystyle\int_{-2}^{1.6} \frac{dx}{(2x-5)\sqrt{4-x^2}}.$

***24.** Evaluate the integral $\int P\,.\,dx$, where $P^2=\dfrac{9-x}{x^2(4x-9)}$, by putting

$$(9-x)/(4x-9)=z^2.$$

Hence shew that if $I=\displaystyle\int_9^{3.6} P\,.\,dx$, $2\tan I+1=0$.

25. Determine the values of the constants A, B, C, such that

$$\frac{x}{(2x-3)(x^2+2)}=\frac{A}{2x-3}+\frac{Bx+C}{x^2+2};$$

hence evaluate the indefinite integral $\displaystyle\int\frac{x\,.\,dx}{(2x-3)(x^2+2)}$.

26. Evaluate the definite integral $\displaystyle\int_{-0.8}^{1.65}\frac{56\,.\,dx}{(4x+13)\sqrt{5x+4}}$.

27. Shew, by the substitution $t=\tan\frac{x}{2}$, or otherwise, that if X denotes the expression

$$a\cos x+b\sin x,$$

then
$$\int\frac{dx}{X}=\frac{1}{\sqrt{a^2+b^2}}\log\frac{\sqrt{a^2+b^2}-b+a\tan\frac{x}{2}}{\sqrt{a^2+b^2}+b-a\tan\frac{x}{2}},$$

a and b being positive constants.

Hence evaluate
$$\int_0^{\frac{\pi}{2}}\frac{dx}{24\cos x+7\sin x}.$$

***28.** Evaluate the definite integral $\int_0^{\frac{\pi}{2}}\frac{\sin^5 x}{\cos^{\frac{1}{2}}x}\,.\,dx$, by the Gamma functions, using the relation

$$\Gamma(n+1)=n\Gamma(n)$$

for all values of n.

29. Evaluate the integral $\int_0^2\sqrt{\frac{x^5}{2-x}}\,.\,dx$, by taking a new variable θ, where $x=2\sin^2\theta$.

30. Prove that
$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{4\cos^2\theta+9\sin^2\theta}=\frac{\pi}{12}.$$
(L.U., Sc.)

***31.** Evaluate the indefinite integral $\int\frac{x\,.\,dx}{\sqrt{(a^2-x^2)^3(b^2+x^2)}}$ by means of the substitution $y^2=(b^2+x^2)/(a^2-x^2)$; hence find the value of

$$325\,.\int_0^{5}\frac{x\,.\,dx}{\sqrt{(169-x^2)^3(81+x^2)}}.$$

***32.** Prove that
$$\int_0^{\frac{\pi}{2}}\frac{dx}{\sin x+\cos x}=\sqrt{2}\,.\log(1+\sqrt{2}).$$
(L.U., Sc.)

***33.** If $y^2=x^2+a^2$, evaluate the differential coefficients of $\log(x+y)$ and xy with respect to x, and express them in terms of y; hence find the values of

$$(a)\int_0^3\frac{dx}{y},\quad\text{and}\quad(b)\int_0^3 y\,dx,$$

taking a in each case as 4.

34. Evaluate the definite integral, $\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$. (L.U.)

***35.** Find the value of $2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,.\,dx$.

36. Evaluate the integral $4\int_0^a y\sqrt{1+\frac{a}{x}}\,.\,dx$, where $x^2+y^2=a^2$.

37. Prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{1+a^2\cos^2x+b^2\sin^2x}=\frac{\pi}{2\sqrt{(1+a^2)(1+b^2)}}$.

38. Prove that $\int_0^{\frac{\pi}{3}} \sin^4x\,.\,dx=(8\pi-9\sqrt{3})/64$. (L.U.)

39. Prove that $\int_0^{\pi} \cos^4x\,.\,\sin^4x\,.\,dx=3\pi/128$. (L.U., Sc.)

40. Evaluate $\int_{-1}^{1} (1-t^2)^{\frac{1}{2}}t^4\,dt$. (B.U.)

***41.** By putting the quantity under the sign of the square root equal to y^2, obtain the value of

$$\int\sqrt{\frac{a-x}{nx-a}}\cdot\frac{dx}{x}.$$ (L.U., Sc.)

42. Find the value of $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$. (L.U., Sc.)

43. Prove that $\int_0^1 \frac{dx}{(1+x^2)\sqrt{2+x^2}}=\frac{\pi}{6}$.

44. Evaluate the integral $\int_{-1}^{1}\frac{dx}{(3-x)\sqrt{1-x^2}}$. (Li.U.)

***45.** Shew that, if $y\,(x-3)=\sqrt{x^2+10x+9}$, then

$$3y^2+1=4(x+3)^2/(x-3)^2.$$

Use this substitution to evaluate the integral

$$\int_0^1 \frac{dx}{(x-3)\sqrt{x^2+10x+9}}.$$

***46.** Calculate the integral $\int_1^2 \frac{2\,dx}{(2x+1)\sqrt{2x^2+5x+3}}$ by means of the substitution $2x+1=1/z$.

***47.** By using the substitution $z=\sec x+\tan x$, prove that

$$\int_0^{\frac{\pi}{2}}\frac{\sec^2x}{(\sec x+\tan x)^n}\,.\,dx=\frac{n}{n^2-1},$$

n being positive and greater than unity.

CHAPTER VII

EXPANSION OF FUNCTIONS. TAYLOR'S AND MACLAURIN'S THEOREMS. BERNOULLI'S NUMBERS.

46. Expansion in Power Series. The calculation of the numerical values of many functions can only be effected by means of equivalent convergent series It is therefore necessary first to determine the series where such is valid for the range of numerical values required.

The general test for the convergency of an infinite series has already been given in Chapter I. (§ 5), and some algebraic methods of expansion by means of the binomial theorem are illustrated in that chapter. The binomial and exponential series are, however, only particular cases of a general form of expansion which will now be dealt with.

47. Taylor's Theorem. The most general method of expansion is that known as Taylor's theorem, published for the first time in 1715. It may be stated as follows :

If $f(x+h)$ be any function of x which is capable of expansion in a convergent series of positive integral powers of h, then

$$f(x+h)=f(x)+h\cdot\frac{df}{dx}+\frac{h^2}{\lfloor 2}\cdot\frac{d^2f}{dx^2}+\frac{h^3}{\lfloor 3}\cdot\frac{d^3f}{dx^3}+\dots+\frac{h^n}{\lfloor n}\cdot\frac{d^nf}{dx^n}+\dots,$$

provided $f(x)$ and all its derivatives are continuous within the numerical range considered.

If D denotes the operator $\frac{d}{dx}$, then the expansion may be written

$$\mathbf{f(x+h)=\left\{1+hD+\frac{h^2}{\lfloor 2}\cdot D^2+\frac{h^3}{\lfloor 3}\cdot D^3+\dots+\frac{h^n}{\lfloor n}\cdot D^n+\dots\right\}f(x).} \quad\dots\dots(55)$$

Ex. 1. *Deduce the binomial, exponential and sine series from Taylor's theorem.*

(*a*) Write y for x in (55), and let $f(y)=y^n$, so that

$$f(y+h)=(y+h)^n,$$

then if $D_y \equiv \frac{d}{dy}$, $D_y^3 y^n = n(n-1)(n-2)y^{n-3}$,

$$D_y y^n = ny^{n-1},\quad D_y^2 y^n = n(n-1)y^{n-2},$$

and so on;

hence, from Taylor's expansion,

$$(y+h)^n = f(y+h) = y^n + nhy^{n-1}$$
$$+\frac{n(n-1)}{\lfloor 2}\cdot h^2 y^{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3}\cdot h^3 y^{n-3} + \ldots.$$

Now write x for h, and put $y=1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}\cdot x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}\cdot x^3 + \ldots,$$

which agrees with (1).

(*b*) Let $f(y)=e^y$, then $f(y+h)=e^{y+h}$, then if $D_y \equiv \frac{d}{dy}$, $D_y e^y = e^y$, and $D_y^r e^y = e^y$, where r is any positive integer.

Substituting in (55),

$$e^{y+h} = f(y+h) = \left\{1 + h + \frac{h^2}{\lfloor 2} + \frac{h^3}{\lfloor 3} + \ldots\right\} e^y.$$

Write x for h and put $y=0$, and

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots,$$

which is the exponential theorem.

(*c*) Let $\quad f(y)=\sin y$, then $f(y+h)=\sin(y+h)$,

and $\quad D_y \sin y = \cos y$, $D_y^2 \sin y = -\sin y$, $D_y^3 \sin y = -\cos y$.

and so on; hence, from (55),

$$\sin(y+h) = f(y+h)$$
$$= \sin y + h\cos y - \frac{h^2}{\lfloor 2}\sin y - \frac{h^3}{\lfloor 3}\cos y + \frac{h^4}{\lfloor 4}\sin y + \ldots.$$

Now write θ for h and put $y=0$, then

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \ldots,$$

which agrees with (21).

48. Maclaurin's Theorem. If in (55) x be put equal to zero, then, assuming all the derivatives of $f(x)$ remain finite, the expansion becomes

$$f(h) = \left\{1 + hD_0 + \frac{h^2}{\underline{|2}} \cdot D_0^2 + \frac{h^3}{\underline{|3}} \cdot D_0^3 + \ldots + \frac{h^r}{\underline{|r}} \cdot D_0^r + \ldots\right\} f(0),$$

where D_0^r means that $f(x)$ is differentiated r times with respect to x and then x put equal to zero.

Now write x for h, and

$$\mathbf{f(x)} = \left\{\mathbf{1 + xD_0} + \frac{\mathbf{x^2}}{\underline{|\mathbf{2}}} \cdot \mathbf{D_0^2} + \frac{\mathbf{x^3}}{\underline{|\mathbf{3}}} \cdot \mathbf{D_0^3} + \ldots + \frac{\mathbf{x^r}}{\underline{|\mathbf{r}}} \cdot \mathbf{D_0^r} + \ldots\right\} \mathbf{f(0)}. \quad \ldots\ldots\mathbf{(56)}$$

This is *Maclaurin's Theorem*, and is very useful for expanding many functions whose derivatives remain finite at $x = 0$.

Ex. 2. *By means of Maclaurin's theorem, prove that*

$$tan\left(\frac{\pi}{4} + h\right) = 1 + 2h + 2h^2 + \tfrac{8}{3}h^3 + \tfrac{10}{3}h^4 + \ldots.$$

Hence calculate the value of tan 46° to five places of decimals. ($\pi = 3{\cdot}14159$.) (L.U.)

Let
$$t = \tan\left(\frac{\pi}{4} + h\right),$$

then
$$D \equiv \frac{dt}{dh} = \sec^2\left(\frac{\pi}{4} + h\right) - 1 + \tan^2\left(\frac{\pi}{4} + h\right) = 1 + t^2$$

$$D^2 = \frac{d^2t}{dh^2} = 2t \cdot \frac{dt}{dh} = 2t(1 + t^2).$$

$$D^3 = 2(1 + 3t^2) \cdot D = 2(1 + 3t^2)(1 + t^2),$$

$$D^4 = 2(8t + 12t^3) \cdot D = 8t(2 + 3t^2)(1 + t^2), \text{ and so on.}$$

Hence, putting $h = 0$, $t = 1$, and

$$D_0 = 2, \quad D_0^2 = 4, \quad D_0^3 = 16, \quad D_0^4 = 80, \ldots,$$

$\therefore$ from (56),

$$\tan\left(\frac{\pi}{4} + h\right) = 1 + hD_0 + \frac{h^2}{\underline{|2}} \cdot D_0^2 + \frac{h^3}{\underline{|3}} \cdot D_0^3 + \frac{h^4}{\underline{|4}} \cdot D_0^4 + \ldots$$

$$= \mathbf{1 + 2h + 2h^2 + \tfrac{8}{3}h^3 + \tfrac{10}{3}h^4} + \ldots.$$

Instead of substituting in (56), as has been done above, Maclaurin's Theorem is usually applied as follows :

Writing t for $\tan\left(\frac{\pi}{4}+h\right)$ as before, assume that

$$t=\tan\left(\frac{\pi}{4}+h\right)=A_0+A_1h+A_2h^2+A_3h^3+A_4h^4+\dots,$$

where the coefficients of the respective powers of h have to be determined, these being independent of h.

By successive differentiation :

$$\begin{aligned} D &= 1+t^2=A_1+2A_2h+3A_3h^2+4A_4h^3+\dots \\ D^2 &= 2t(1+t^2)=2A_2+6A_3h+12A_4h^2+\dots, \\ D^3 &= 2(1+3t^2)(1+t^2)=6A_3+24A_4h+\dots, \\ D^4 &= 8t(2+3t^2)(1+t^2)=24A_4+\dots. \end{aligned}$$

Putting $h=0$ in each of the above equations,

$$A_0=1,\ A_1=2,\ A_2=2,\ A_3=\tfrac{8}{3},\ A_4=\tfrac{10}{3};$$

hence, $\tan\left(\frac{\pi}{4}+h\right)=1+2h+2h^2+\frac{8}{3}h^3+\frac{10}{3}h^4+\dots$, as before.

To calculate the value of tan 46°, it should be observed that h must be expressed in radians.

Now $\frac{\pi}{4}=45°$, so that if $\frac{\pi}{4}+h=46°$, $h=1°$,

and $\qquad 180°=\pi$ radians $=3{\cdot}14159$ radians;

$\therefore\ 1°=3{\cdot}14159\div180=0{\cdot}017453$ radian.

Hence, taking the series term by term,

$$\begin{aligned} 1 &= 1 \\ 2h &= 0{\cdot}034906 \\ 2h^2 &= 0{\cdot}000609 \\ \tfrac{8}{3}h^3 &= 0{\cdot}000014 \\ \tfrac{10}{3}h^4 &= \underline{0{\cdot}000000} \\ & \quad 1{\cdot}035529 \end{aligned}$$

Hence, correctly to five places of decimals.

$$\tan 46°=\mathbf{1{\cdot}03553}.$$

49. Expansion by the Differentiation and Integration of a Known Series. In the above calculation of tan 46° correctly to five places of decimals, it will be noticed that the fifth term is smaller than 10^{-6}. The next term in the series for h is $\frac{64}{15}h^5$, which becomes $10^{-9}\times 6{\cdot}9$ when $h=0{\cdot}017453$. It is evident,

therefore, that as the series is convergent, the remaining terms have no effect upon a result which is to be correct to five places.

Suppose the terms of a series are continuous functions of x and the series converges within an interval (a, b), *i.e.* $a<x<b$. If now a positive integer p can be chosen so that for every value of n equal to or greater than p, and for every value of x within the interval (a, b), the remainder, after the first n terms of the series are taken, is less than any arbitrarily pre-assigned positive number, the series is said to **converge uniformly** within the interval (a, b).

Such a series can be differentiated or integrated term by term, provided the series obtained in each case is also uniformly convergent within the given range. This gives another method, dependent upon a valid binomial series, of expanding many functions.

Ex. 3. *Assuming the binomial expansion for* $(1-x)^n$, *develop the series for* $\log(1-x)$ *when* $x<1$, *by differentiation, and verify it by Maclaurin's theorem. Hence calculate the value of* $\log_e 0{\cdot}75$ *correctly to four places of decimals.*

Let $y=\log_e(1-x)=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\dots$,

then $\dfrac{dy}{dx}=-\dfrac{1}{1-x}=A_1+2A_2x+3A_3x^2+4A_4x^3+\dots$.

But $-\dfrac{1}{1-x}=-(1-x)^{-1}=-(1+x+x^2+x^3+\dots+x^n+\dots)$,
since $x<1$.

The two series must be identical, so that equating the coefficients of x^{r-1},

$$r\,.\,A_r=-1;$$

$$\therefore\ A_r=-1/r.$$

Hence, by giving r the values 1, 2, 3, ..., successive coefficients are obtained, except A_0.

If, however, x be put equal to zero in the first statement, then $A_0=0$, so that

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots.$$

To verify this by Maclaurin's theorem, assume the same series as in the first statement, then differentiating r times, with respect to x,

$$\frac{d^r y}{dx^r} = -\frac{\underline{|r-1}}{(1-x)^r} = \underline{|r} \cdot A_r + \underline{|r+1} \cdot A_{r+1}x + \ldots.$$

Putting $x=0$,

$$\underline{|r} \cdot A_r = -\underline{|r-1}, \quad \text{or } A_r = -1/r.$$

$\therefore$ By putting $r=1, 2, 3, \ldots$, and finding A_0 as before, the series becomes

$$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots,$$

thus verifying the previous result.

Note that for $x<1$, $1-x<1$; $\therefore \log(1-x)<0$, *i.e.* it is negative.

To calculate $\log_e 0{\cdot}75$, put $x=0{\cdot}25$, $=\frac{1}{4}$, then taking term by term,

$$x = \frac{1}{4} = 0{\cdot}25 \qquad \frac{x^2}{2} = \frac{1}{2 \cdot 4^2} = 0{\cdot}03125$$

$$\frac{x^3}{3} = \frac{1}{3 \cdot 4^3} = 0{\cdot}00520 \qquad \frac{x^4}{4} = \frac{1}{4 \cdot 4^4} = 0{\cdot}00097$$

$$\frac{x^5}{5} = \frac{1}{5 \cdot 4^5} = 0{\cdot}00019 \qquad \frac{x^6}{6} = \frac{1}{6 \cdot 4^6} = 0{\cdot}00004$$

$$\frac{x^7}{7} = \frac{1}{7 \cdot 4^7} = 0{\cdot}00000.$$

The sum of these is $\qquad 0{\cdot}28765$;

$$\therefore \log_e 0{\cdot}75 = -0{\cdot}2877 = \bar{1}{\cdot}7123.$$

Ex. 4. *Expand* $\tan^{-1} x$ *by integration, and deduce that*

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots.$$

By the binomial theorem,

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \ldots.$$

Hence $\displaystyle\int (1+x^2)^{-1} dx = \int (1 - x^2 + x^4 - \ldots + (-1)^n x^{2n} + \ldots) dx$,

or $\qquad \displaystyle\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots + A.$

To find A, put $x=0$, and both sides vanish, except A, so that $A=0$;

$$\therefore \tan^{-1} x = \mathbf{x} - \frac{\mathbf{x}^3}{3} + \frac{\mathbf{x}^5}{5} - \frac{\mathbf{x}^7}{7} + \ldots + (-1)^{\mathrm{n}} \frac{\mathbf{x}^{2\mathrm{n}+1}}{2\mathrm{n}+1} + \ldots.$$

In this series, put $x=1$, then

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots.$$

This is known as *Gregory's series* for π, but it is not suitable for calculating π, as it is too slowly convergent. (See Ex. 21, p. 152.)

50. Expansion by the Formation of a Differential Equation. Sometimes an expansion can be readily effected by means of a differential equation, as the following example illustrates.

Ex. 5. *If* $y=(1-x^2)^{\frac{1}{2}} \cdot \sin^{-1} x$, *shew that*

$$(1-x^2)\frac{dy}{dx} + xy = 1 - x^2.$$

Prove that, if y can be expanded in a series of ascending powers of x, the expansion is

$$x - \frac{x^3}{3} - \frac{2}{3} \cdot \frac{x^5}{5} - \ldots - \frac{2 \cdot 4 \ldots 2n-2}{3 \cdot 5 \ldots 2n-1} \cdot \frac{x^{2n+1}}{2n+1} - \ldots,$$

and obtain the general term. (L.U., Sc.)

Since $$y = (1-x^2)^{\frac{1}{2}} \sin^{-1} x,$$

$$\therefore \frac{dy}{dx} = 1 - \frac{x \sin^{-1} x}{(1-x^2)^{\frac{1}{2}}};$$

$$\therefore (1-x^2)\frac{dy}{dx} + xy = 1 - x^2 - x(1-x^2)^{\frac{1}{2}} \sin^{-1} x + x(1-x^2)^{\frac{1}{2}} \sin^{-1} x$$
$$= \mathbf{1 - x^2}.$$

Let $y = (1-x^2)^{\frac{1}{2}} \sin^{-1} x = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots + A_r x^r + \ldots$, then, if this is an identity, when $x=0$, $A_0=0$, so that the series begins with the term $A_1 x$.

Also $$\frac{dy}{dx} = A_1 + 2A_2 x + 3A_3 x^2 + \ldots.$$

Substitute in the differential equation,

$$(1-x^2)(A_1 + 2A_2 x + 3A_3 x^2 + \ldots) + A_1 x^2 + A_2 x^3 + A_3 x^4 + \ldots = 1 - x^2.$$

This must be identically true for all values of x, hence corresponding coefficients on either side must be equal, *i.e.*

$$A_1=1, \quad 2A_2=0, \quad 3A_3-A_1+A_1=-1,$$

giving $\qquad A_1=1, \quad A_2=0$ and $A_3=-\frac{1}{3}$.

Generally, the coefficient of $x^r=(r+1)A_{r+1}-(r-1)A_{r-1}+A_{r-1}$ and this must vanish for all positive integer values of $r>2$;

$$\therefore\ A_{r+1}=\frac{r-2}{r+1}\cdot A_{r-1}.$$

Since $A_2=0$, all the coefficients vanish for odd integral values of r; replace r by $2p$, then

$$A_{2p+1}=\frac{2(p-1)}{2p+1}\cdot A_{2p-1}.$$

Hence for $p=2, \quad A_5 \quad =\frac{2}{5}A_3=-\frac{2}{5}\cdot\frac{1}{3},$

$$p=3, \quad A_7 \quad =\frac{2\cdot 2}{7}\cdot A_5=-\frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{7},$$

and for $\qquad p=n, \quad A_{2n+1}=\frac{2(n-1)}{2n+1}\cdot A_{2n-1}$

$$=-\frac{2\cdot 4\ldots(2n-2)}{3\cdot 5\ldots\overline{2n+1}},$$

on substituting successive values of coefficients down to A_3;

$$\therefore\ y=x-\frac{x^3}{3}-\frac{2}{3}\cdot\frac{x^5}{5}-\ldots-\frac{2\cdot 4\ldots 2(n-2)}{3\cdot 5\ldots 2n+1}\cdot\frac{x^{2n+1}}{2n+1}-\ldots.$$

51. Bernoulli's Numbers. If the function

$$y=\tfrac{1}{2}x(e^x+1)/(e^x-1)$$

be expanded in ascending powers of x, and the expansion written in the form

$$\frac{x}{2}\cdot\frac{e^x+1}{e^x-1}=1+B_1\frac{x^2}{\lfloor 2}-B_3\frac{x^4}{\lfloor 4}+B_5\frac{x^6}{\lfloor 6}-B_7\frac{x^8}{\lfloor 8}+\ldots, \quad\ldots\ldots\ldots\ldots(57)$$

the coefficients B_{2r-1} $(r=1, 2, 3, \ldots)$ when calculated are called **Bernoulli's Numbers,** after their discoverer, James Bernoulli. These coefficients are of great importance in analysis, and a few are determined in the next example.

Ex. 6. *If* $\frac{1}{2}x \coth \frac{x}{2} = S$, *where* S *denotes the series,*

$$A_0 + A_2x^2 + A_4x^4 + \ldots + A_{2r}x^{2r} + \ldots,$$

prove that $x(e^x + 1) = 2S(e^x - 1)$, *and shew why there are no odd powers of* x *in the series denoted by* S. *Hence, from the definition of Bernoulli's numbers, shew that*

$$B_{2n-1} = (-1)^{n-1}\underline{|2n} \cdot A_{2n},$$

where B_{2n-1} *is the nth Bernoullian number, and calculate the first four of these numbers.*

$$\coth \frac{x}{2} = 1/\tanh \frac{x}{2} = \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)\Big/\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) = (e^x + 1)/(e^x - 1):$$

$$\therefore\ S = \tfrac{1}{2}x \coth \frac{x}{2} = \tfrac{1}{2}x(e^x + 1)/(e^x - 1),$$

so that $$x(e^x + 1) = 2S(e^x - 1).$$

In the above expression for S replace x by $-x$, then

$$S = -\tfrac{1}{2}x(e^{-x} + 1)/(e^{-x} - 1)$$
$$= -\tfrac{1}{2}x(1 + e^x)/(1 - e^{-x}) = \tfrac{1}{2}x(1 + e^x)/(e^x - 1);$$

hence the expression is unaltered, *i.e.* the corresponding series for $-x$ is equal to the series for $+x$; or assuming S to be of the form $A_0 + A_1x + A_2x^2 + \ldots + A_px^p + \ldots$, then for $-x$, the series becomes $A_0 - A_1x + A_2x^2 - \ldots + (-1)^pA_px^p + \ldots$

These two must be identical, so that all the coefficients of the odd powers of x must vanish; hence the series consists only of even powers of x. Such a function is called an **even function,** of which $\cos\theta$ is another example. See (21), p. 49.

Again, since $S = \frac{1}{2}x(e^x + 1)/(e^x - 1)$, therefore by (57)

$$A_0 + A_2x^2 + A_4x^4 + \ldots \equiv 1 + B_1\frac{x^2}{\underline{|2}} - B_3\frac{x^4}{\underline{|4}} + \ldots.$$

Hence, equating the coefficients of x^{2n},

$$(-1)^{n-1}B_{2n-1}/\underline{|2n} = A_{2n},$$

or $$B_{2n-1} = (-1)^{n-1}\underline{|2n} \cdot A_{2n}.$$

In order to calculate B_{2n-1} for $n = 1, 2, 3, 4$, from this relation, it is necessary to determine the first five coefficients of the series for S.

Writing S in the form $x(e^x+1)=2S(e^x-1)$, and substituting the corresponding series for S and e^x,

$$x\left(2+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\right)=2(A_0+A_2x^2+A_4x^4+\ldots)\left(x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\right),$$

which must be identically satisfied for all values of x; hence corresponding coefficients are equal.

Equating the coefficients of x, $\mathbf{A_0=1}$,

and proceeding similarly with the coefficients of x^{2r+1}, the general relation is

$$\frac{A_0}{\underline{|2r+1}}+\frac{A_2}{\underline{|2r-1}}+\frac{A_4}{\underline{|2r-3}}+\ldots+\frac{A_{2r-2}}{\underline{|3}}+A_{2r}=\frac{1}{2\,\underline{|2r}}.$$

For $r=1$, $A_0/\underline{|3}+A_2=1/4$, giving $\mathbf{A_2=\frac{1}{12}}$.

For $r=2$, $A_0/\underline{|5}+A_2/\underline{|3}+A_4=1/(2\,\underline{|4})$, $\therefore\ \mathbf{A_4=-\frac{1}{\underline{|6}}}$.

For $r=3$, $A_0/\underline{|7}+A_2/\underline{|5}+A_4/\underline{|3}+A_6=1/(2\,\underline{|6})$, $\therefore\ \mathbf{A_6=\frac{1}{6\,\underline{|7}}}$.

For $r=4$, $A_0/\underline{|9}+A_2/\underline{|7}+A_4/\underline{|5}+A_6/\underline{|3}+A_8=1/(2\,\underline{|8})$,

$$\therefore\ \mathbf{A_8=-\frac{3}{\underline{|10}}}.$$

Hence

$$\frac{1}{2}x\coth\frac{x}{2}\equiv\frac{1}{2}x\frac{e^x+1}{e^x-1}=1+\frac{1}{6}\cdot\frac{x^2}{\underline{|2}}-\frac{1}{30}\cdot\frac{x^4}{\underline{|4}}+\frac{1}{42}\cdot\frac{x^6}{\underline{|6}}-\frac{1}{30}\cdot\frac{x^8}{\underline{|8}}+\ldots,$$

and by comparing this with (57), or, what comes to the same thing, using the above relation and putting $n=1, 2, 3, 4$ successively,

$$\mathbf{B_1=\tfrac{1}{6},\quad B_3=B_7=\tfrac{1}{30},\quad B_5=\tfrac{1}{42}}.$$

Exercises 7.

Expand each of the following functions in ascending powers of x, and verify the expansions by assuming the series for e^x.

1. $\cos x$. **2.** $\sinh x$. **3.** $\cosh x$.

4. $e^x\sin x$. **5.** $e^x\cos x$. **6.** $\sin x\,.\cosh x$. (L.U., Sc.)

Establish the following results:

7. $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\ldots$, when $x<1$.

8. $\log \cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \dots$.

9. $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots$, when $x < 1$.

10. $\log(1+\cos x) = \log 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 - \dots$.

11. $\log(1+\sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \dots$.

12. Obtain the series for $\log_e(1+x)$ in ascending powers of x, when $x^2 < 1$.

Deduce that if $n > 1$,

$$\log_e(n+1) - \log_e(n-1) = 2\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots\right\},$$

and calculate $\log_e 2$ correct to four figures. (L.U.)

13. From the series for $\log_e(1+x)$, prove that

$$\log_e \frac{n+1}{n-1} = \frac{2n}{n^2+1} + \frac{1}{3}\left(\frac{2n}{n^2+1}\right)^3 + \frac{1}{5}\left(\frac{2n}{n^2+1}\right)^5 + \dots$$

Find $\log_e \frac{4}{3}$ correct to four places of decimals. (L.U.)

14. State the series for the expansion of $\log_e(1+x)$ in ascending powers of x, and find the condition for the convergency of the series.

If a and b are small compared with x, shew that

$$\log_e(x+a) - \log_e x = \frac{a}{b}\left(1 + \frac{b-a}{2x}\right)\{\log_e(x+b) - \log_e x\}. \qquad \text{(L.U.)}$$

15. Expand $\sin^{-1}x$, and deduce from it that

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \dots ;$$

hence calculate π to four places of decimals.

Obtain the expansion of each of the following functions, assuming the validity of the binomial expansion :

16. $\cos^{-1}x$. **17.** $\sinh^{-1}x$. **18.** $\tanh^{-1}x$. **19.** $(\sin^{-1}x)^2$.

20. Expand $\sin(x+h)$ in ascending powers of h; hence shew that, by taking the first three terms only, $\sin 60° 57{\cdot}3'$ can be found with an error less than 4×10^{-7}.

By using the series for $\tan^{-1}x$ obtained in Ex. 4, p. 148, calculate the value of π to five figures from each of the following identities :

21. $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

22. $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$.

23. Prove that $e^x \sec x = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{3}{10}x^5 + \dots$.

24. Shew that

$$e^x \log(1+x) = x + \frac{x^2}{\lfloor 2} + \frac{2x^3}{\lfloor 3} + \frac{9x^5}{\lfloor 5} + \dots .$$

25. If $y = \log(x + \sqrt{1+x^2})$, shew that

$$\frac{dy}{dx} = (1+x^2)^{-\frac{1}{2}};$$

hence, prove, by assuming the binomial expansion to be valid for the values of x concerned, that

$$\log(x + \sqrt{1+x^2}) = x - \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 - \dots .$$

26. If $y = \sin(k \sin^{-1} x)$, prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2 y = 0;$$

hence, shew that

$$y = kx\left\{1 - \frac{k^2-1}{\lfloor 3} \cdot x^2 + \frac{(k^2-1)(k^2-9)}{\lfloor 5} x^4 - \dots\right\}.$$

27. If $(1+x^2)\sin y = 2x$, shew that

$$\frac{dy}{dx} = 2(1+x^2)^{-1};$$

hence, prove that the nth term in the expansion of y is

$$(-1)^{n-1}\frac{2x^{2n-1}}{2n-1}.$$

28. If $y = (1-x^2)^{-\frac{1}{2}} \sin^{-1} x$, shew that

$$(1-x^2)\frac{dy}{dx} - xy = 1.$$

Prove that $\quad (1-x^2)^{-\frac{1}{2}} \sin^{-1} x = x + \tfrac{2}{3}x^3 + \tfrac{8}{15}x^5 + \tfrac{16}{35}x^7,$

accurately as far as x^8 inclusive, for values of x that are small. (L.U., Sc.)

29. If $y = (1+x^2)^{-\frac{1}{2}} \sinh^{-1} x$, prove that

$$(1+x^2)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0.$$

Hence shew that $(1+x^2)^{-\frac{1}{2}} \sinh^{-1} x = x - \tfrac{2}{3}x^3 + \tfrac{2}{3} \cdot \tfrac{4}{5}x^5 - \dots .$ (Lc.U., Sc.)

30. If $y = (\sinh^{-1} x)^2$, prove that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2.$$

Shew that the first three terms of the expansion of $(\sinh^{-1} x)^2$ in powers of x are $x^2 - \tfrac{1}{3}x^4 + \tfrac{8}{45}x^6$. (S.U., Sc.)

***31.** If $y = \sin^{-1}\{(1+x)\sin\alpha\}$, prove that

$$y = \alpha + x\tan\alpha + \tfrac{1}{2}x^2\tan^3\alpha + \dots . \qquad \text{(D.U., Sc.)}$$

***32.** Assuming that y is a function of x which (1) can be expanded in ascending powers of x, and (2) vanishes when $x=0$, and (3) satisfies the equation

$$\frac{d^2y}{dx^2}=b\cdot\frac{dy}{dx}+cy\,;$$

prove that, as far as x^4,

$$y=a_1\left\{x+\tfrac{1}{2}bx^2+\frac{b^2+c}{6}x^3+\frac{b(b^2+2c)}{24}x^4+\dots\right\},$$

where a_1 is undetermined by these conditions. (L.U., Sc.)

***33.** Assuming the expansion,

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^n}{n}+\dots=-\log(1-x),$$

put $x=\cos y+i\sin y=e^{iy}$, where $i=\sqrt{-1}$, and $0<y<2\pi$, then

$$x^n=e^{niy}=\cos ny+i\sin ny\,;$$

hence shew that $C+iS=-\log(2\sin\tfrac{1}{2}y)+\tfrac{1}{2}i(\pi-y)$,

where $C=\cos y+\tfrac{1}{2}\cos 2y+\tfrac{1}{3}\cos 3y+\dots$,

and $S=\sin y+\tfrac{1}{2}\sin 2y+\tfrac{1}{3}\sin 3y+\dots$.

Hence, by equating real and imaginary parts, shew that

$$(a)\ C=-\log(2\sin\tfrac{1}{2}y)\quad\text{and}\quad(b)\ S=\tfrac{1}{2}(\pi-y).$$

***34.** By integrating the identity (b) of Ex. 30, *i.e.*

$$\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\dots=\tfrac{1}{2}(\pi-x),$$

where $0<x<2\pi$, between the limits $x=\pi$ and $x=0$, shew that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}.$$

***35.** Denoting the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$ by s, prove that

$$(a)\ \frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots+\frac{1}{2n^2}+\dots=\tfrac{1}{4}s\,;$$

$$(b)\ \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{(2n-1)^2}+\dots=\tfrac{3}{4}s.$$

Hence, from the result in Ex. 34, shew that $s=\pi^2/6$.

***36.** If $\quad y=\tfrac{1}{2}x\coth\frac{x}{2}=A_0+A_2x^2+A_4x^4+\dots+A_{2r}x^{2r}+\dots,$

shew, by using the exponential equivalent of the hyperbolic function, that

$$(a)\ 2(e^x-1)\frac{dy}{dx}=e^x(1+x-2y)+1\,;$$

$$(b)\ 2(e^x-1)\frac{d^3y}{dx^3}+6e^x\left(\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)=e^x(3+x-2y).$$

Hence, by putting $x=0$ in each of these relations, find the values of A_0 and A_2, and by repeated differentiation determine A_4, A_6, ...

***37.** Using the series already developed in Exs. 33 or 34, for $\frac{x}{2}\coth\frac{x}{2}$, prove by replacing x with ix, where $i=\sqrt{-1}$, that

$$\frac{x}{2}\cdot\cot\frac{x}{2}=1-\frac{B_1}{|\underline{2}}x^2-\frac{B_3}{|\underline{4}}x^4-\ldots-\frac{B_{2n-1}}{|\underline{2n}}\cdot x^{2n}-\ldots.$$

***38.** From the identity

$$\sin x=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\times\ldots\times\left(1-\frac{x^2}{n^2\pi^2}\right)\times\ldots,$$

shew by logarithmic differentiation that

$$x\cot x=1-\frac{2x^2}{\pi^2}\left(1-\frac{x^2}{\pi^2}\right)^{-1}-\frac{2x^2}{2^2\pi^2}\left(1-\frac{x^2}{2^2\pi^2}\right)^{-1}-\ldots.$$

Hence, by expanding each of the binomial expressions, shew that the coefficient of x^{2n} is

$$-\frac{2}{\pi^{2n}}\cdot S_{2n},$$

where
$$S_{2n}=1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\ldots.$$

Then, by comparing this with the corresponding coefficient of the series for $x\cot x$ deduced from Ex. 35, prove that

$$B_{2n-1}=\frac{2\,|\underline{2n}}{(2\pi)^{2n}}\cdot S_{2n}.$$

***39.** Calculate the first five numbers of Bernoulli by the methods of either Ex. 36 or Ex. 37, and shew that their values are

$$B_1=\tfrac{1}{6},\quad B_3=B_7=\tfrac{1}{30},\quad B_5=\tfrac{1}{42},\quad B_9=\tfrac{5}{66}.$$

Hence, by using the formula for B_{2n-1} of the last example, prove that

$$S_2=\frac{\pi^2}{6},\quad S_4=\frac{\pi^4}{90},\quad S_6=\frac{\pi^6}{945},\quad S_8=\frac{\pi^8}{9450}\quad\text{and}\quad S_{10}=\frac{\pi^{10}}{93555}.$$

From these values, prove that

$$2\,.\,S_2\,.\,S_8=3\,.\,S_4\,.\,S_6.$$

***40.** Assuming the expansion for $\log(1-y)$, and using the results of Ex. 39, shew, by substituting $1-y$ for x, that

$$\int_1^0\frac{\log x}{1-x}=\frac{\pi^2}{6}.$$

Obtain the same result by means of the substitution $x=e^{-y}$, and expanding by the binomial theorem.

CHAPTER VIII

NUMERICAL AND GRAPHICAL SOLUTION OF EQUATIONS

52. Practical Approximate Methods. Many equations required in practice can only be solved approximately. These equations in general fall into two groups :

(α) **Algebraic Equations** of the type $x^m + ax + b = 0$, where $m > 2$.

(β) **Transcendental Equations,** such as $x + a \log x = b$; $e^x + ax = c$; $x + a \sin x + b \log x = c$, etc.

Equations of type (α) can be solved algebraically for $m = 3$ and for $m = 4$, but the process is often long and difficult, and a graphical method simpler and shorter. For $m > 4$, and the transcendental equations, no algebraic method is, in general, available.

The graphical method may be applied in two ways : (i) by plotting a single locus, and (ii) by plotting two loci on the same sheet and finding their points of intersection. The roots thus determined are only rough approximations, and analytical methods may be applied to obtain values from these correct to any desired degree of accuracy.

53. Evaluation and Expansion of a Polynomial by Horner's Method. In the case of algebraic equations, before plotting on squared paper, it is usually necessary to reduce them to a simpler form, and this is quickly and conveniently done by an easy and compact method devised by W. G. Horner, which is illustrated in the following example.

Ex. 1. *If* $f(x) \equiv 2x^5 - 25x^4 + 108x^3 - 189x^2 + 118x - 20$, *shew how to evaluate* $f(a)$, *where* a *is an actual number, by Horner's method, and prove the result. Extend the method to obtain the expanded form of* $f(x+a)$, *and verify it by Taylor's theorem.*

Determine the value of a which will cause the term in x^4 to vanish, and by using this value obtain the transformed function.

It is obvious that by putting $x=a$,

$$f(a)=2a^5-25a^4+108a^3-189a^2+118a-20.$$

When a is a number, the value of $f(a)$ entails the calculation of a^5, a^4, a^3, a^2, and the multiplication of these by the respective coefficients. This may, however, be obviated by the following simple process.

Write down the coefficients in proper order in a horizontal row; multiply the coefficient of x^5 by a, and, writing the product under the coefficient of x^4, add it to this coefficient. Call this sum A_4. Now multiply A_4 by a, and add it to the coefficient of x^3. Call the result A_3, and repeat the process until A_0 is found. This is then the value of $f(a)$. Following this method the work appears as follows:

$$\begin{array}{cccccc}
2 & -25 & 108 & -189 & 118 & -20 \\
 & 2a & A_4a & A_3a & A_2a & A_1a \\
\hline
2 & A_4 & A_3 & A_2 & A_1 & \mathbf{A_0}
\end{array}$$

To prove that A_0 is the value of $f(a)$, work back from A_0 to 2, thus:

$$\begin{aligned}
A_0 &= A_1a-20=a(A_2a+118)-20=a^2(A_3a-189)+118a-20 \\
&= a^3(A_4a+108)-189a^2+118a-20 \\
&= a^4(2a-25)+108a^3-189a^2+118a-20 \\
&= 2a^5-25a^4+108a^3-189a^2+118a-20 \\
&= \mathbf{f(a)}.
\end{aligned}$$

Let the method now be extended until there are only two terms remaining, thus:

$$\begin{array}{cccccc}
2 & -25 & 108 & -189 & 118 & -20 \\
 & 2a & A_4a & A_3a & A_2a & A_1a \\
\hline
2 & A_4 & A_3 & A_2 & A_1 & \mathbf{A_0} \\
 & 2a & B_4a & B_3a & B_2a & \\
\hline
2 & B_4 & B_3 & B_2 & \mathbf{B_1} & \\
 & 2a & C_4a & C_3a & & \\
\hline
2 & C_4 & C_3 & \mathbf{C_2} & & \\
 & 2a & E_4a & & & \\
\hline
2 & E_4 & \mathbf{E_3} & & & \\
 & 2a & & & & \\
\hline
2 & \mathbf{F_4} & & & &
\end{array}$$

To calculate each of the numbers in thicker type, it is readily seen that

$$F_4 = 2a + 2a + 2a + 2a + 2a - 25 = 10a - 25.$$

$$\begin{aligned} E_3 &= a(E_4 + C_4 + B_4 + A_4) + 108 \\ &= a(8a - 25 + 6a - 25 + 4a - 25 + 2a - 25) + 108 \\ &= 20a^2 - 100a + 108. \end{aligned}$$

$$\begin{aligned} C_2 &= a(C_3 + B_3 + A_3) - 189 \\ &= a^2(C_4 + 2B_4 + 3A_4) + 324a - 189 \\ &= a^2(6a - 25 + 8a - 50 + 6a - 75) + 324a - 189 \\ &= 20a^3 - 150a^2 + 324a - 189. \end{aligned}$$

Similarly,

$$B_1 = 10a^4 - 100a^3 + 324a^2 - 378a + 118,$$

and, as already proved, $A_0 = f(a)$.

Now $f(x+a) \equiv 2x^5 + F_4x^4 + E_3x^3 + C_2x^2 + B_1x + A_0$, as will be verified by Taylor's theorem.

From (55),

$$f(a+x) = \left\{1 + xD + \frac{x^2}{\lfloor 2}D^2 + \frac{x^3}{\lfloor 3} . D^3 + \frac{x^4}{\lfloor 4} . D^4 + \frac{x^5}{\lfloor 5} . D^5\right\} f(a).$$

Taking each term of the expansion in turn :

$$f(a) = 2a^5 - 25a^4 + 108a^3 - 189a^2 + 118a - 20 = A_0,$$

$$x . Df(a) = (10a^4 - 100a^3 + 324a^2 - 378a + 118)x = B_1x,$$

$$\begin{aligned} \frac{x^2}{\lfloor 2} . D^2f(a) &= (40a^3 - 300a^2 + 648a - 378)\frac{x^2}{\lfloor 2} \\ &= (20a^3 - 150a^2 + 324a - 189)x^2 = C_2x^2, \end{aligned}$$

$$\begin{aligned} \frac{x^3}{\lfloor 3} . D^3f(a) &= (120a^2 - 600a + 648)\frac{x^3}{\lfloor 3} \\ &= (20a^2 - 100a + 108)x^3 = E_3x^3, \end{aligned}$$

$$\frac{x^4}{\lfloor 4} . D^4f(a) = (240a - 600)\frac{x^4}{\lfloor 4} = (10a - 25)x^4 = F_4x^4,$$

$$\frac{x^5}{\lfloor 5} . D^5f(a) = 240 . \frac{x^5}{\lfloor 5} = 2x^5,$$

which agrees with Horner's process as applied above, thus

$$f(x+a) = \mathbf{2x^5} + \mathbf{F_4x^4} + \mathbf{E_3x^3} + \mathbf{C_2x^2} + \mathbf{B_1x} + \mathbf{A_0}.$$

Now, for F_4 to vanish, $10a - 25 = 0$, or $a = 2{\cdot}5$.

Finally, $f(x+2{\cdot}5)$ must be evaluated. This may very conveniently be done by Horner's method; thus, proceeding as above, the following table is easily constructed:

2	-25	108	-189	118	-20
	5	-50	145	-110	20
2	-20	58	-44	8	**0**
	5	$-37{\cdot}5$	$51{\cdot}25$	$18{\cdot}125$	
2	-15	$20{\cdot}5$	$7{\cdot}25$	**26·125**	
	5	-25	$-11{\cdot}25$		
2	-10	$-4{\cdot}5$	**−4**		
	5	$-12{\cdot}5$			
2	-5	**−17**			
	5				
2	**0**				

so that the transformed function becomes

$$f(x+2{\cdot}5) \equiv \mathbf{2x^5 - 17x^3 - 4x^2 + 26{\cdot}125x}.$$

It should be noted that $x=2{\cdot}5$ is a root of the equation $f(x)=0$.

54. Analytical Solution of a Cubic Equation. The most general form of an equation of the third degree may be written

$$x^3+px^2+qx+r=0,$$

for should there be a numerical coefficient of x^3 other than unity, the equation may be divided throughout by that coefficient.

If x be now replaced by $x-\frac{1}{3}p$, the above equation may by Horner's process be reduced to the form $x^3+ax+b=0$, where a and b are real numbers. This is, therefore, taken as the general form of the cubic for purposes of solution.

The algebraical solution of this equation was first published by Cardan in 1545. He replaced x by $u+v$, so that the transformed equation became $u^3+v^3+(3uv+a)x+b=0$, and then imposed the further condition that u and v should be chosen so that the coefficient of x should vanish; this gives $3uv+a=0$ and $u^3+v^3+b=0$, from which u and v may be found. The solution of $x^2+ax+b=0$ then becomes:

$$\mathbf{x}=\left\{-\frac{\mathbf{b}}{\mathbf{2}}+\sqrt{\frac{\mathbf{b}^2}{\mathbf{4}}+\frac{\mathbf{a}^3}{\mathbf{27}}}\right\}^{\frac{1}{3}}+\left\{-\frac{\mathbf{b}}{\mathbf{2}}-\sqrt{\frac{\mathbf{b}^2}{\mathbf{4}}+\frac{\mathbf{a}^3}{\mathbf{27}}}\right\}^{\frac{1}{3}}. \qquad \text{............}\mathbf{(58a)}$$

The three values of x satisfying the equation $x^3 + ax + b = 0$, are $p + q$, $\omega p + \omega^2 q$, $\omega^2 p + q\omega$, where p and q are the arithmetical cube roots of $-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$, when $\frac{b^2}{4} + \frac{a^3}{27} > 0$, and ω, ω^2 are the complex cube roots of unity. ....(58c)

(*b*) Dividing out by the coefficient of x^3, this equation becomes

$$x^3 - \tfrac{5}{4}x + \tfrac{1}{2} = 0,$$

and on comparing with (58*a*), $a = -\frac{5}{4}$ and $b = \frac{1}{2}$,

but
$$\frac{b^2}{4} + \frac{a^3}{27} = \frac{1}{16} - \frac{125}{64 \cdot 27} = -\frac{17}{64 \cdot 27},$$

which is thus negative.

Applying De Moivre, as in (58*b*),

$$\beta = \sqrt{\frac{17}{64 \cdot 27}} = \sqrt{\frac{51}{64 \cdot 81}} = \frac{7 \cdot 1411}{72},$$

and $\alpha = -\dfrac{b}{2} = -\dfrac{1}{4}$: the De Moivre expression for $\alpha + i\beta$ must, therefore, have a negative cosine.

But $\theta = \tan^{-1}\beta/\alpha = \tan^{-1}(-0{\cdot}39671) = -21^\circ\ 38'$.

Now $\cos(-21^\circ\ 38')$ is positive, so that the supplement of the angle must be taken;

$$\therefore\ \theta = 180^\circ - 21^\circ\ 38' = 158^\circ\ 22'.$$

Hence from (58*b*), the values of x are given by

$$2\sqrt{\frac{5}{12}} \cdot \cos\frac{158^\circ\ 22' + 2m\pi}{3}, \quad (m = 0, 1, 2).$$

Taking each value in turn:

For $m = 0$, $x = \sqrt{\frac{5}{3}} \cdot \cos 52^\circ\ 47' = 1{\cdot}2910 \times 0{\cdot}6048 = 0{\cdot}7808.$

For $m = 1$, $x = 1{\cdot}2910 \cos 172^\circ\ 47' = -1{\cdot}2910 \times 0{\cdot}9921$
$= -1{\cdot}2806.$

For $m = 2$, $x = 1{\cdot}2910 \cos 292^\circ\ 47' = 1{\cdot}2910 \times 0{\cdot}3873 = 0{\cdot}5.$

The accuracy of these values may be judged by the fact that $x = 0{\cdot}5$ completely satisfies the equation, and with this value, $4x^3 - 5x + 2$ immediately breaks up into $(2x - 1)(2x^2 + x - 2)$. The roots of $2x^2 + x - 2 = 0$ are $\frac{1}{4}(-1 \pm = \sqrt{17}) = 0{\cdot}7808$ or $-1{\cdot}2808$.

55. Algebraical Solution of a Quartic. An equation of the fourth degree—called a Quartic or Biquadratic—has been solved algebraically both by Ferrari, a pupil of Cardan, and Euler. In each case, however, the solution depends upon the determination of a real root of a cubic equation, and therefore, as a general rule, it is simpler to employ a graphical method of solution for all quartics arising from practical problems. Ferrari's method is illustrated in the following example, where a root of the cubic is easily found. In the majority of practical cases, however, this is not so, and the student should employ the methods illustrated in Examples 6 and 8 (pp. 166 and 170).

Ex. 4. *Prove that, by expressing the left-hand side of the biquadratic equation, $ax^4+4bx^3+6cx^2+4dx+e=0$, as the difference of two squares, the solution of the equation can be made to depend upon the solution of a cubic equation.*

Hence solve the equation, $2x^4+6x^3-3x^2+2=0$. (L.U., Sc.)

Divide the given equation throughout by a, which is obviously not zero, and write it in the form

$$x^4+2fx^3+gx^2+2hx+l=0,$$

where $\quad f=2b/a, \quad g=6c/a, \quad h=2d/a, \quad$ and $\quad l=e/a.$

Now suppose it is possible to choose numbers, p, q, r, s, such that

$$x^4+2fx^3+gx^2+2hx+l\equiv(x^2+px+q)^2-(rx+s)^2,$$

i.e.

$$\equiv x^4+2px^3+(p^2+2q-r^2)x^2 +2(pq-rs)x+q^2-s^2;$$

then, on comparing corresponding coefficients,

$$p=f, \quad p^2+2q-r^2=g, \quad pq-rs=h, \quad \text{and} \quad q^2-s^2=l.$$

Eliminate r and s,

$$(pq-h)^2=(p^2+2q-g)(q^2-l),$$

which, on putting in the value of p, becomes

$$2q^3-gq^2-2(l-hf)q+gl-lf^2-h^2=0.$$

Now every cubic equation has at least one real root, hence this root may be obtained—sometimes by inspection, though rarely in practical cases. Denote it by q'; then r and s are determinate,

and $$(x^2+fx+q')^2-(rx+s)^2=0,$$

or $$\{x^2+(f+r)x+q'+s\}\{x^2+(f-r)x+q'-s\}=0,$$

so that $x^2+(f+r)x+q'+s=0$, or $x^2+(f-r)x+q'-s=0$,

the solutions of which are

$$x=\tfrac{1}{2}\{-f-r\pm\sqrt{(f+r)^2-4(q'+s)}\},$$

or $$x=\tfrac{1}{2}\{-f+r\pm\sqrt{(f-r)^2-4(q'-s)}\},$$

thus giving the four values of x satisfying the given quartic.

In the particular equation given, after division throughout by 2,

$$2p=3,\quad p^2+2q-r^2=-\tfrac{3}{2},\quad pq-rs=0,\quad q^2-s^2=1,$$

so that the resulting cubic becomes

$$8q^3+6q^2-8q-15=0,$$

a root of which is $q=\frac{5}{4}$, and this with $p=\frac{3}{2}$ gives $r=\frac{5}{2}$ and $s=\frac{3}{4}$; hence the original equation becomes

$$(x^2+\tfrac{3}{2}x+\tfrac{5}{4})^2-(\tfrac{5}{2}x+\tfrac{3}{4})^2=0,$$

which on factorisation gives

$$x^2+4x+2=0,\quad \text{or}\quad x^2-x+\tfrac{1}{2}=0;$$

$$\therefore\ x=-2\pm\sqrt{2}\quad\text{or}\quad \tfrac{1}{2}(1\pm i);$$

hence, the complete set of values satisfying the equation is

$$x=-\mathbf{0{\cdot}5858},\quad -\mathbf{3{\cdot}4142},\quad \tfrac{1}{2}(\mathbf{1\pm i}).$$

56. Approximate Solution by Horner's Method. The method of evaluating and expanding a polynomial, illustrated in Ex. 1, p. 156, may be conveniently employed to determine to any practical degree of accuracy the real roots of an algebraic equation of any degree. The process will now be exemplified.

Ex. 5. *Obtain, by Horner's method, the real roots of*

$$x^4-7x^3-17x^2+30=0$$

to three places of decimals.

Denote the function $x^4-7x^3-17x^2+30$ by $f(x)$, then using Horner's method of evaluation, $f(0)=30$, $f(1)=7$, $f(2)=-78$; hence one root lies between 1 and 2.

Let $1+\alpha$ be a first approximation to this root, where $\alpha<1$, so that $x=1+\alpha$. Replace x in $f(x)$ by $1+\alpha$ by Horner's process:

$$\begin{array}{rrrrr}
1 & -7 & -17 & 0 & 30 \\
 & 1 & -6 & -23 & -23 \\
\hline
1 & -6 & -23 & -23 & \mathbf{7} \\
 & 1 & -5 & -28 & \\
\hline
1 & 5 & -28 & \mathbf{-51} & \\
 & 1 & -4 & & \\
\hline
1 & -4 & \mathbf{-32} & & \\
 & 1 & & & \\
\hline
\mathbf{1} & \mathbf{-3} & & &
\end{array}$$

Hence, the transformed equation $f(1+\alpha)=0$ becomes

$$\alpha^4-3\alpha^3-32\alpha^2-51\alpha+7=0.$$

Since α is relatively small, terms of higher order than the first may be neglected, so that the first approximate correction is given by

$$-51\alpha+7=0, \quad \text{from which} \quad \alpha=0{\cdot}137.$$

Again, let $x=1+\alpha+\beta$ be a second approximation, where $\beta<\alpha$. Replace α in $f(1+\alpha)=0$ by $0{\cdot}137+\beta$.
Proceeding as before, and using a slide rule,

$$\begin{array}{rrrrr}
1 & -3 & -32 & -51 & 7 \\
 & 0{\cdot}137 & -0{\cdot}39 & -4{\cdot}41 & -7{\cdot}595 \\
\hline
1 & -2{\cdot}863 & -32{\cdot}39 & -55{\cdot}44 & \mathbf{-0{\cdot}595} \\
 & 0{\cdot}137 & -0{\cdot}37 & -4{\cdot}49 & \\
\hline
1 & -2{\cdot}726 & -32{\cdot}76 & \mathbf{-59{\cdot}93} &
\end{array}$$

Since accuracy to three places only is required, there is no need to proceed further, because terms of higher order than the first will not affect the result.

Hence β is given by

$$-59{\cdot}93\beta-0{\cdot}595=0, \text{ from which } \beta=-0{\cdot}00993;$$

$$\therefore\ x=1+\alpha+\beta=1+0{\cdot}137-0{\cdot}00993=\mathbf{1{\cdot}1270}.$$

If greater accuracy is needed the complete transformed equation $f(0{\cdot}137+\beta)=0$ must be determined, and then β replaced by γ, where $x=1+\alpha+\beta+\gamma$; the evaluation of γ will then give a third approximation, and the whole process may be repeated until the desired degree of accuracy is attained.

For the second root, it is found by trial to lie between 8 and 9; hence, by repeating the above process, it is easily found that

$$x = 8.8730.$$

The equation $f(x) = 0$ may readily be solved by the algebraic method of Ex. 3. The student should, therefore, use this method to shew that $f(x) \equiv (x^2 - 10x + 10)(x^2 + 3x + 3)$, and then check the above roots by solving the quadratic which has real roots.

57. Approximate Location of Roots by Graeffe's Method. All the real roots of an algebraic equation may be located approximately by a process of successive transformation of the given equation into one whose roots are the $2n$th powers of those to be determined, n being a positive integer. The method, due to Graeffe in 1837, depends upon the fact that when the roots are unequal the differences between them are increased by successive transformation until, to a first approximation, only the largest remains significant. A practical outline of the process is given in the following example.

Ex. 6. *If the roots of the equation*

$$x^3 + a_1x^2 + b_1x + c_1 = 0$$

are the squares of the roots of

$$x^3 + ax^2 + bx + c = 0,$$

calculate the coefficients a_1, b_1, c_1, in terms of a, b, c.

Hence, by successive transformation locate approximately the roots of $x^3 - 6x + 4 = 0$.

Let α, β, γ be the roots of $x^3 + ax^2 + bx + c = 0$, then

$$x^3 + ax^2 + bx + c \equiv (x - \alpha)(x - \beta)(x - \gamma)$$

$$\equiv x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma.$$

Equating corresponding coefficients:

$$\alpha + \beta + \gamma = -a,$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = b,$$

$$\alpha\beta\gamma = -c.$$

Now the roots of $x^3 + a_1x^2 + b_1x + c_1 = 0$ are α^2, β^2, γ^2, so that $\alpha^2 + \beta^2 + \gamma^2 = -a_1$, $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = b_1$ and $\alpha^2\beta^2\gamma^2 = -c_1$.

But $a^2 = (\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha) = -a_1+2b$;

$$\therefore\ a_1 = -(\mathbf{a}^2 - 2\mathbf{b}).$$

Similarly, $b^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\gamma(\alpha+\beta+\gamma) = b_1+2ac$;

$$\therefore\ b_1 = \mathbf{b}^2 - 2\mathbf{ac}.$$

And finally, $c^2 = \alpha^2\beta^2\gamma^2 = -c_1$; $\therefore\ c_1 = -\mathbf{c}^2$.

Hence, to transform the equation $x^3+ax^2+bx+c=0$, whose roots are α, β, γ, into one whose roots are α^2, β^2, γ^2, the coefficients a, b, c must be replaced respectively by $-(a^2-2b)$, b^2-2ac, and $-c^2$.

This process is quite simple to carry out with numerical coefficients by the use of a slide rule: thus, with the given equation, successive coefficients are best calculated as follows:

	a	b	c	
1	0	-6	4	Coefficients of given equation.
1	0	36	16	Coefficients squared
	12	0		$-(a^2-2b) = -12$; $b^2-2ac = 36$.
1	-12	36	-16	Coeffs. of 1st transformed equation.
1	144	1296	256	Coefficients squared.
	-72	-384		
1	-72	912	-256	Coeffs. of 2nd transformed equation.
1	5184	$8{\cdot}318 \times 10^5$	$6{\cdot}552 \times 10^4$	
1	-3360	$7{\cdot}95 \times 10^5$	$-6{\cdot}552 \times 10^4$	
1	$1{\cdot}129 \times 10^7$	$6{\cdot}322 \times 10^{11}$	$4{\cdot}29 \times 10^9$	
	$-0{\cdot}159 \times 10^7$	$-0{\cdot}004 \times 10^{11}$		
1	$\times 0{\cdot}97 \times 10^7$	$6{\cdot}318 \times 10^{11}$	$-4{\cdot}29 \times 10^9$	

Since the terms corresponding to $2b$ and $2ac$ are becoming very small in comparison with the square terms, a^2, b^2, the transformation need not proceed further.

Hence the equation

$$x^3 - 0{\cdot}97 \times 10^7 \,.\, x^2 + 6{\cdot}318 \times 10^{11} \,.\, x - 4{\cdot}29 \times 10^9 = 0$$

has for its roots α^{16}, β^{16}, γ^{16}, where α, β, γ are the roots of $x^3-6x+4=0$: so that

$$\alpha^{16}+\beta^{16}+\gamma^{16} = 0{\cdot}97 \times 10^7,$$

$$\alpha^{16}\beta^{16}+\beta^{16}\gamma^{16}+\gamma^{16}\alpha^{16} = 6{\cdot}318 \times 10^{11},$$

and

$$\alpha^{16}\beta^{16}\gamma^{16} = 4{\cdot}29 \times 10^9.$$

If $\alpha > \beta > \gamma$, then as a first approximation β^{16} and γ^{16} may be considered relatively small compared with α^{16}; hence

$$\alpha^{16} = 0{\cdot}97 \times 10^7, \text{ giving } \alpha = 2{\cdot}73;$$

and $\alpha^{16}(\beta^{16}+\gamma^{16})=6{\cdot}318\times10^{11}$; rejecting γ as small compared with β, $\beta^{16}=6{\cdot}318\times10^{11}\times\alpha^{-16}=6{\cdot}318\times10^{4}\div0{\cdot}97$, giving $\beta=2$ approximately.

Finally, $\gamma^{16}=4{\cdot}29\times10^{9}\times(\alpha\beta)^{-16}$ giving $\gamma=0{\cdot}73$.

By substitution $\mathbf{x}=\mathbf{2}$ is a root, and the other roots are then found to be $-\mathbf{2{\cdot}7321}$ and $\mathbf{0{\cdot}7321}$.

The signs of the approximate roots must be determined by substitution, and when all are incommensurable Horner's method may be applied to find further approximations to any desired degree of accuracy.

58. Graphical Solution of an Algebraic Equation. In determining the approximate roots of an algebraic equation by a graphical method, it is generally simpler to split the equation into the two equivalent simultaneous equations, one of which should be linear if possible, and then plot both on the same sheet. The abscissae of the points of intersection give approximate solutions to the given equation. Nearer approximations may then be found, either by plotting on a larger scale in the neighbourhood of a point of intersection, or by correcting analytically by means of the binomial theorem. The latter is usually much shorter and more satisfactory. For a small root, a check may be made by using the following expansion due to Bertrand.

A numerically small root of the equation, $\mathbf{x^{m+1}+ax-b=0}$, is given by the expansion

$$\mathbf{x}=\frac{b}{a}-\left(\frac{b}{a}\right)^{m+1}\cdot\frac{1}{a}+\frac{2m+2}{a^2\,\underline{|2}}\cdot\left(\frac{b}{a}\right)^{2m+1}-\frac{(3m+2)(3m+3)}{a^3\,\underline{|3}}\left(\frac{b}{a}\right)^{3m+1}+\ldots$$

provided this series is convergent for the numerical values of a, b, and m. ....(60)

Ex. 7. *Obtain graphically the two real roots, lying between* 1 *and* -3, *of the equation*

$$x^5-29x+12=0,$$

correctly to five significant figures. Check the smaller numerical root by Bertrand's expansion.

The equation may be written $x^5=29x-12$, in which form it is easily seen to be equivalent to the two simultaneous equations, (i) $y=x^5$ and (ii) $y=29x-12$, which is linear.

The first is easily plotted between the given limits, and since the second is a straight line, two points, as far apart as possible, in order to ensure greater accuracy, will suffice to determine the position of the line.

In Fig. 7 the two loci are shewn, and it will be observed that the straight line intersects the quintic curve in two points. The abscissae of these points will thus give first approximations to the two roots required. Reading from the graphs, these values are $-2{\cdot}42$ and $0{\cdot}42$.

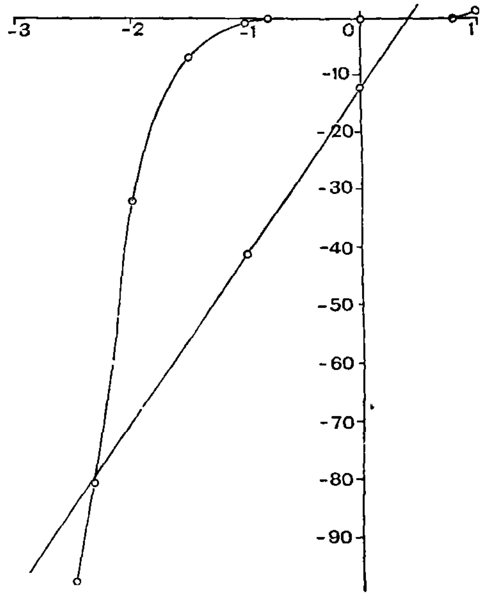

FIG. 7. Graphical solution of an algebraic equation.

Having now located the roots, further approximations to any degree of accuracy may be found by Horner's process illustrated in Ex. 1. Thus let $x=0{\cdot}42+\alpha$, then replacing x by $0{\cdot}42+\alpha$ in the given equation :

1	0	0	0	−29	12
	0·42	0·1764	0·07406	0·03	−12·167
1	0·42	0·1764	0·07406	−28·97	− **0·167**
	0·42	0·3528	0·2223	0·12	
1	0·84	0·5292	0·29636	**−28·85**	

This will be sufficient to give an approximate value for α, since higher powers beyond the first may be neglected.

Hence $$-28{\cdot}85\alpha=0{\cdot}167,$$

so that $$\alpha=-0{\cdot}005788\,;$$

$$\therefore\ x=0{\cdot}42-0{\cdot}005788=\mathbf{0{\cdot}4142}\text{ correct to four places.}$$

Proceeding in precisely a similar manner with the other root, it is found that $x=-2{\cdot}42+0{\cdot}005782=-\mathbf{2{\cdot}4142}$ to five significant figures.

To check this, put $a=-29$, $b=-12$, and $m=4$ in Bertrand's expansion (60); then taking term by term:

$$b/a=12/29 \qquad =0\cdot413793$$

$$-b^{m+1}/a^{m+2}=-(-12)^5/(-29)^6=0\cdot0004183$$

$$(2m+2)b^{2m+1}/(a^{2m+3}\,|\,2)=5(-12)^9/(-29)^{11}=0\cdot0000021$$

$$-(3m+2)(3m+3)\,.\,b^{3m+1}/(a^{3m+4}\,|\,3)$$

$$=-35(-12)^{13}/(-29)^{16}=0\cdot0000000$$

$$0\cdot4142134$$

so that, to five significant figures, $x=0\cdot4142$, as previously found.

The accuracy of the result may be judged from the fact that $x^5-29x+12\equiv(x^2+2x-1)(x^3-2x^2+5x-12)$, and the two roots required are given by $x^2+2x-1=0$, which on solving gives $x=-1\pm\sqrt{2}=-2\cdot4142$ or $0\cdot4142$. In practice it is not often that a quintic can thus be factorised, but this one is purposely chosen here to illustrate the reliability of a graphical result carefully obtained.

59. Special Case of the Quartic. For a biquadratic equation with numerical coefficients, it is generally possible to obtain the roots as the abscissae of the points of intersection of a parabola and a circle. The method has the advantage that no term in the equation need be removed, and, in addition, both curves are easy to draw. The following example illustrates the process:

Ex. 8. *By means of a substitution of the form*

$$y=x^2+\tfrac{1}{2}px+m,$$

shew that the roots of the equation

$$x^4+px^3+qx^2+rx+s=0$$

may be obtained by means of the intersections of a circle and a parabola.

Find approximately the real roots of the equation

$$x^4+4x^3+5x^2+4x-5=0. \qquad \text{(L.U.)}$$

Since $\quad y=x^2+\tfrac{1}{2}px+m,$

$$\therefore\; y^2=x^4+px^3+(\tfrac{1}{4}p^2+2m)x^2+pmx+m^2.$$

But $\quad 0=x^4+px^3+\qquad qx^2+rx\quad+s.$

$\therefore$ By subtraction,

$$y^2 = (\tfrac{1}{4}p^2 + 2m - q)x^2 + (pm - r)x + m^2 - s = 0,$$

or $$(q - \tfrac{1}{4}p^2 - 2m)x^2 + y^2 + (r - pm)x + s - m^2 = 0.$$

This equation will represent a circle if the coefficients of x^2 and y^2 are both positive and equal.

Hence, let m be chosen so that $q - \frac{1}{4}p^2 - 2m = 1$,

i.e. $$m = \tfrac{1}{2}(q - \tfrac{1}{4}p^2 - 1)\ ;$$

then the equation may be written

$$x^2 + y^2 + gx + c = 0,$$

where $$g = r - pm = r - \tfrac{1}{2}p(q - \tfrac{1}{4}p^2 - 1),$$

and $$c = s - m^2 = s - \tfrac{1}{4}(q - \tfrac{1}{4}p^2 - 1)^2.$$

The centre and radius of the circle may be found by expressing the equation in the form

$$(x + \tfrac{1}{2}g)^2 + y^2 = \tfrac{1}{4}g^2 - c,$$

so that the centre is the point $(-\frac{1}{2}g, 0)$, and the radius is $\sqrt{\frac{1}{4}g^2 - c}$.

The circle will, therefore, be real if $\frac{1}{4}g^2 > c$.

Hence the given biquadratic is equivalent to the two equations,

$$\text{(i)}\ y = x^2 + \tfrac{1}{2}px + \tfrac{1}{2}(q - \tfrac{1}{4}p^2 - 1),$$

or $$y + \tfrac{1}{2}(1 + \tfrac{3}{8}p^2 - q) = (x + \tfrac{1}{4}p)^2,$$

representing a parabola whose vertex is the point

$$(-\tfrac{1}{4}p,\ \tfrac{1}{2}q - \tfrac{3}{16}p^2 - \tfrac{1}{2}),$$

and $$\text{(ii)}\ (x + \tfrac{1}{2}g)^2 + y^2 = \tfrac{1}{4}g^2 - c,$$

representing a circle of radius $\sqrt{\frac{1}{4}g^2 - c}$ whose centre is the point $(-\frac{1}{2}g, 0)$. The points of intersection, therefore, give the roots of these equations, and consequently those of the given quartic.

For the particular case, $p = 4$, $q = 5$, $r = 4$, $s = -5$, so that $m = \frac{1}{2}(5 - 4 - 1) = 0$, $g = r = 4$ and $c = s = -5$; hence the parabola is $y = x^2 + 2x$, or $y + 1 = (x + 1)^2$, its vertex being the point $(-1, -1)$, and the circle is $(x + 2)^2 + y^2 = 9$, the centre being $(-2, 0)$, and radius 3.

The two loci are shewn in Fig. 8, where the abscissae of the points of intersection are approximately 0·59 and $-2·96$.

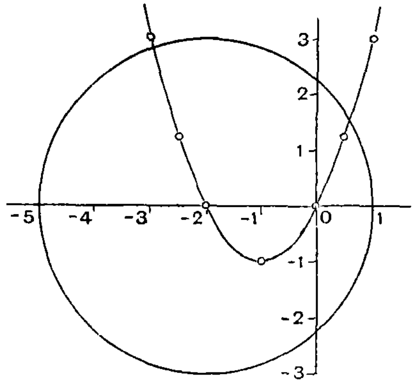

FIG. 8. Solution of a quartic equation.

Applying Horner's method to find a further approximation, by replacing x with $0·59 + \alpha$ in the given equation :

1	4	5	4	-5
	0·59	2·708	4·547	5·044
1	4·59	7·708	8·547	0·044
	0·59	3·056	6·352	
1	5·18	10·764	14·899	

so that α is given by $14·899\alpha = -0·044$;

$$\therefore \alpha = -0·002953,$$

and $x = 0·59 - 0·002953 = 0·5870$ correct to three places of decimals.

In a similar way, the correction for the second root is found as $-0·000041$, so that $x = -2·960$.

60. Transcendental Equations and the Method of Correcting an Approximately Located Root. If $f(x) = 0$ be a transcendental equation whose roots are required, it is best to plot the function $y = f(x)$ and thus obtain the approximate location of the roots. These must then be corrected to the necessary degree of accuracy, but neither Horner's method, nor its equivalent, the binomial expansion, is in general available. Expansion in the neighbourhood of the root must be effected by Taylor's theorem (55), provided that both $f(x)$ and $D_x f(x)$ are continuous at the point where $x =$ the root. The method will now be considered practically.

Ex. 9. *Shew that if* α *be an approximation to a root of an equation* $F(x)=0$, $\alpha - F(\alpha)/F'(\alpha)$ *is, in general, a closer approximation,* $F(x)$ *and* $F'(x)$ *being finite and continuous in the neighbourhood of the root.* (L.U.)

Find graphically an approximate value of the root of the equation

$$x + 20 \sin x = 14{\cdot}5,$$

and proceed to obtain a value correct to four significant figures.

Let β be a first correction to the root α, where $\beta < \alpha$, then approximately

$$F(\alpha+\beta)=0.$$

But by Taylor's theorem (55),

$$F(\alpha+\beta) = F(\alpha) + \beta \, . \, F'(\alpha) + \frac{\beta^2}{\underline{|2}} \, . \, F''(\alpha) + \dots ,$$

where $F'(\alpha)$ indicates the result of differentiating $F(x)$ with respect to x and then replacing x by α.

Hence, since

$$F(\alpha+\beta)=0,$$

$$\therefore \; F(\alpha) + \beta \, . \, F'(\alpha) + \frac{\beta^2}{\underline{2}} \, . \, F''(\alpha) + \dots = 0.$$

Now as β is small, the higher powers beyond the linear term may be neglected, so that β is given by the equation

$$F(\alpha) + \beta \, . \, F'(\alpha) = 0,$$

from which $\beta = -F(\alpha)/F'(\alpha)$.

Hence, a closer approximation is given by

$$x = \alpha + \beta = \alpha - F(\alpha)/F'(\alpha).$$

If this is not the desired degree of accuracy, the process may be repeated, until this is attained.

Let $y = x + 20 \sin x - 14{\cdot}5$; then it is evident that x is expressed in radians. Hence the following table may easily be calculated.

x Radians.	x Degrees.	$\sin x$	$20 \sin x$	y
0·5236	30°	0·5	10	$-3{\cdot}9764$
0·6981	40°	0·6428	12·856	$-0{\cdot}9459$
0·7418	42·5°	0·6756	13·512	$-0{\cdot}2462$
0·7854	45°	0·7071	14·142	0·4274
0·8290	47·5°	0·7373	14·046	1·0750
0·8727	50°	0·7660	15·320	1·6927

On plotting these values for x and y and drawing a smooth curve through the points, it will be seen that the approximate value of x where $y=0$ is 43·3 or 0·7557 radian.

To correct this, let $x=0{\cdot}7557+h$ where h is a small quantity, then from Taylor's theorem, neglecting all terms in h except the linear term,

$$f(0{\cdot}7557+h)=f(0{\cdot}7557)+h\,.f'(0{\cdot}7557)=0\,;$$

$$\therefore\ h=-f(0{\cdot}7557)/f'(0{\cdot}7557)$$

$$=-(0{\cdot}7557+20\sin 43{\cdot}3^\circ-14{\cdot}5)/(1+20\cos 43{\cdot}3^\circ)$$

$$=0{\cdot}0283/15{\cdot}556=0{\cdot}0018\,;$$

$\therefore\ x=0{\cdot}7557+0{\cdot}00182=$ **0·7575**, correct to four significant figures.

Ex. 10. *Find a positive root, less than one radian, correct to four significant figures of the equation* $\cos x+4\log_{10}x-0{\cdot}485=0$.

Let $y=\cos x+4\log_{10}x-0{\cdot}485$, then the following table is readily constructed.

Values of x		$\cos x$	$\log_{10} x$	$4\log_{10} x$	$-0{\cdot}485$	y
Radians.	Degrees					
0·2	11° 28′	0·9801	1·3010	3·2040	−0·485	2·3009
0·3	17° 11′	0·9554	1·4771	3·9084	−0·485	-1·6212
0·4	22° 55′	0·9211	$\bar{1}$·6021	2·4084	−0·485	1·1555
0·5	28° 39′	0·8776	$\bar{1}$·6990	2·7960	0·485	0·7614
0·6	34° 23′	0·8253	1·7782	1·1128	−0·485	0·5469
0·7	40° 6′	0·7649	1·8451	1·3804	−0·485	0·3397
0·8	45° 50′	0·6968	$\bar{1}$·9031	1·6124	−0·485	0·1758
0·9	51° 34′	0·6216	$\bar{1}$·9542	$\bar{1}$·8168	−0·485	−0·0466
1·0	57° 18′	0·5402	0	0	−0·485	+0·0552
1·2	68° 45′	0·3625	0·0792	0·3168	−0·485	+0·1943
1·4	80° 13′	0·1699	0·1461	0·5844	−0·485	+0·2693

Plotting the graph from these figures, Fig. 10 is obtained, and reading off the value of x when $y=0$, an approximate solution, $x=0{\cdot}94$, is determined. Let $0{\cdot}94+\alpha$ be a nearer approximation to the root; then applying Taylor's theorem, and rejecting all terms in powers of α above the first :

$$\alpha = -f(0{\cdot}94)/f'(0{\cdot}94)$$
$$= -(\cos 53^\circ 51' + 4\log 0{\cdot}94 - 0485)\bigg/\left(-\sin 53^\circ 51' + \frac{4}{0{\cdot}94}\log_{10} e\right)$$
$$= -(-0{\cdot}00257)/1{\cdot}0406$$
$$= 0{\cdot}0024696.$$

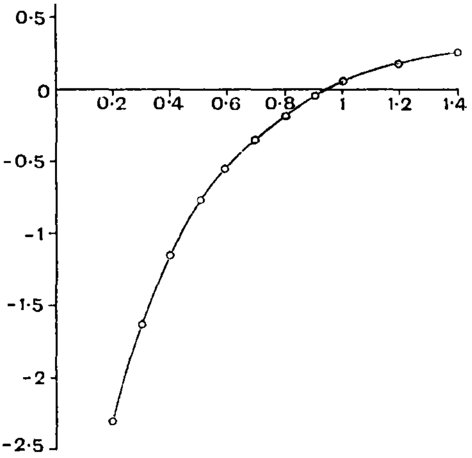

FIG. 9. Graphical solution of a transcendental equation.

Hence, the root correct to four significant figures is $x = \mathbf{0{\cdot}9425}$.

EXERCISES 8.

1. There is a root of $x^3 + 5x - 11 = 0$ between 1 and 2; find it, using squared paper, accurately to four significant figures.

2. There is a root of $x^3 - 10x^2 + 40x - 35 = 0$ which lies between 1 and 2; find it correctly to three significant figures.

3. Plot the function $y = 4x^3 - 109x + 210$, between $x = 6$ and $x = -6$; hence determine the three roots of the equation

$$4x^3 - 109x + 210 = 0.$$

4. Solve graphically the equation

$$x^3 - 16x - 24 = 0,$$

and check one of the roots by Bertrand's expansion.

5. Determine three roots of the equation

$$x^5 - 4x + 1 = 0,$$

and check the smaller positive root from Bertrand's expansion.

6. If $x/y = e^{a\theta}$, where $a = 0{\cdot}3$ and $\theta = 2{\cdot}848$, find x in terms of y; hence, if $x - y = 55{\cdot}35$ also, calculate the actual values of x and y.

7. If $y = \dfrac{2}{x} + 5\log_{10} x - 2{\cdot}7$, find the values of y when x has the values 2, 2·5, 3. Plot these values of x and y on squared paper, and draw the probable curve on which these points lie. State approximately what value of x will cause y to be zero.

8. If $f(x)$ denotes the expression $x^4 - 8x^3 - x^2 + 68x + 60$, find, by Horner's method, the expression for $f(x+2)$: hence, solve the equation $f(x) = 0$.

9. Give any method for the solution of a quartic equation. Solve the quartic $x^4 + 3x^3 - x^2 - 13x - 10 = 0$. (Le.U., Sc.)

10. Solve the equation $x^4 + 96x = 80$. (D.U., Sc.)

11. Solve $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$. (Br.U.)

12. Draw the graphs of $10\cos x + 8$ and $x\sin x$ from $x = 0$ to $x = 2\pi$, and find the solutions of the equation $10\cos x + 8 = x\sin x$ which fall within the above interval. (S.U.)

13. Find, by any method, the least positive root of the equation

$$3\tan x = 3x + 1$$

as accurately as you can. (L.U.)

14. Draw a rough graph of e^x, and by plotting more accurately in the neighbourhood of the required value of x, solve the equation

$$e^{3(x-1)} = 2x + 1. \qquad \text{(L.U.)}$$

15. Obtain the three solutions lying between 2 and -2 of the equation

$$x^5 - 5x + 1 = 0.$$

16. Solve the equation $s = c\cosh\dfrac{x}{c}$ for c, when $s = 489{\cdot}4$ and $x = 22$.

17. Find accurately to three significant figures, a value of x to satisfy the equation

$$0{\cdot}5x^{1{\cdot}5} - 12\log_{10} x + 2\sin 2x = 2{\cdot}06.$$

18. Find a value of x correct to four significant figures, which satisfies the equation

$$4\log_{10} x + \tan 2x = 0{\cdot}5873.$$

19. Find, with three significant figures accurate, a root of

$$5x^{\frac{3}{2}} + x\log_{10} x - 8{\cdot}868 = 0.$$

20. From a circle of radius 5 inches, it is required to cut off a segment whose area is 2·915 square inches; find the size of the angle subtended at the centre of the circle by the arc of the segment. Hence, find the length of the bounding chord.

21. The area of a segment of a circle of radius 9·8 inches is 4·802 square inches; find the angle subtended at the centre by the arc of the segment.

22. Solve the equation $4 \log_{10} x + 5x = 38{\cdot}8$.

23. Determine a value of x between 0 and 1 which satisfies the equation $5x + 9 \sin x = 1{\cdot}464$.

24. Solve the equation $3x + 5 \cos x = 5{\cdot}3$,

the value of x lying between 0 and 1.

Shew that, by replacing $\cos x$ by the first two terms of its equivalent expansion, the resulting equation is $25x^2 - 30x + 3 = 0$.

Explain why only one root of this quadratic agrees approximately with that found by the graphical method.

25. A circle is bisected areally by the circumference of another circle whose centre lies on its circumference. If r, u are the radii, and 2α, 2β, the angles of the segments on either side of the common chord of the given circle and the bisecting circle respectively, prove that

$$2 \sin 2\beta - 4\beta \cos 2\beta = \pi.$$

Solve this equation graphically, and hence shew that

$$u^2 : r^2 = 67 : 50$$

very nearly.

26. $ABCD$ is a square whose side is ten inches. With O, the mid-point of AB as centre, a circle is described cutting the sides AD, BC in E, F respectively. Find the angle EOF when the arc EF bisects the area of the square; hence calculate the radius of the circle.

27. Find a positive value of x satisfying the equation

$$x^2 - 5 \log_{10} x = 5{\cdot}8.$$

28. A cylindrical oil tank 16 ft. long has its axis horizontal, and its diameter 10 ft. Plot a graph from which a scale in feet may be constructed along a vertical end diameter shewing the contents in steps of 100 gallons. Give the reading for 500 gallons.

29. Trace the curve $y = e^{x-1}$.

Find graphically an approximate value of the root of the equation $3 - x = e^{x-1}$, and obtain a more accurate value by the use of Taylor's series. (L.U.)

30. Shew graphically that the equation $2^x = 3x$ has two real roots, and find them correctly to three places of decimals. (L.U.)

31. Determine two roots, correctly to four significant figures, of the equation

$$x^5 - 101x - 140 = 0.$$

32. Draw the graph of $\cosh x$.

Shew that the equation $4x = \cosh x$ has two real roots, and find them approximately. (L.U.)

33. Find, in any manner, correct to four places of decimals, the positive root of the equation $x^3 - 3x^2 - x - 2 = 0$. (L.U.)

34. On a sheet of squared paper draw two vertical axes, OY, $O'Y'$, one unit apart, and graduate them upwards and downwards from a central horizontal line OO'. Now plot the hyperbola $y(x-1) = x^2$, and draw any line from O intersecting the curve at P and the $O'Y'$ axis in K where the scale reading is $-k$. At P write the number $+k$, and by a similar process graduate the curve. Let any line be drawn from a point a on the OY axis to a point $-b$ on the $O'Y'$ axis, intersecting the hyperbola at the point u. Shew that u is a solution of the quadratic $u^2 + au + b = 0$; hence write down the solutions of $x^2 + 4x + 2 = 0$ and $x^2 - 3x - 3 = 0$.

35. In the equation $x^3 - 3px + q = 0$, put $x = \frac{1}{2}\sqrt{p}\cos\theta$, and by comparison with the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, shew that

$$2\sqrt{p}\cos\theta, \quad 2\sqrt{p}\cos\left(\theta + \frac{2\pi}{3}\right), \quad \text{and} \quad 2\sqrt{p}\cos\left(\theta + \frac{4\pi}{3}\right)$$

are its roots, provided q^2 is less than $4p^3$. Apply this method of solution to the equation $x^3 - 6x + 4 = 0$.

36. Plot the function $15(y + \sinh x) = 16x$, between $x = 0$ and $x = 1$; hence, solve the equation $15 \sinh x = 16x$.

37. The smallest root in absolute value of the equation

$$a + bx + cx^2 + dx^3 = 0$$

is given by Whittaker's series:

$$-\frac{a}{b} - \frac{a^2c}{b\begin{vmatrix} b & c \\ a & b \end{vmatrix}} - \frac{a^3\begin{vmatrix} c & d \\ b & c \end{vmatrix}}{\begin{vmatrix} b & c \\ a & c \end{vmatrix} \cdot \begin{vmatrix} b & c & d \\ a & b & c \\ 0 & a & b \end{vmatrix}} \cdots$$

Apply this to find the smallest root of $8x^3 - 17x^2 + 42x - 5 = 0$.

CHAPTER IX

DETERMINATION OF LAWS FROM EXPERIMENTAL DATA. SMALL ERRORS

61. Construction of an Empirical Function. In many practical problems the functional relation between two varying quantities cannot be determined from theoretical considerations, and, as a result, an approximate relationship called an **Empirical Function** —has to be constructed from experimental data. The general plan employed in this process is to derive from the observations made, by plotting on squared paper, a straight line from which approximate values of any constants required may be determined and a functional relationship thus constructed. It must be borne in mind, however, that all data which are purely experimental are liable to be slightly in error, and therefore constants derived from them will only be approximately true. Indeed, the complete function when determined will only be an approximation to the true relationship between the varying quantities. The following example will illustrate the method of constructing such a function from experimental observations.

Ex. 1. *A steamship at a speed of v knots uses an Indicated Horse Power P. The following table gives a series of values of v and P, and it is supposed that a relation of the form $P=av^n$, where a and n are constants, expresses the law connecting P and v. Investigate if this is so, and determine the most probable values of a and n. From it, find the value of P when $v=26$.*

v	10	12	14	16	18	20
P	1066	1912	3216	4951	7361	10355

If $\quad P = av^n$, then $\log P = \log a + n \log v$.

Let $\quad y = \log P, \quad x = \log v, \quad$ and $c = \log a$;

$\therefore$ the equation becomes $y = nx + c$, which represents a straight line of slope θ, where $n = \tan\theta$, and c = intercept on the axis of y. To test the truth of this assumption, the equation must be plotted for the observed values. Hence the following table:

$x = \log v$	1	1·0792	1·1461	1·2041	1·2553	1·3010
$y = \log P$	3·0277	3·2814	3·5073	3·6947	3·8670	4·0151

In Fig. 10 these values are plotted, and it is evident that the points lie practically on a straight line, the slight discrepancies being most probably due to experimental errors.

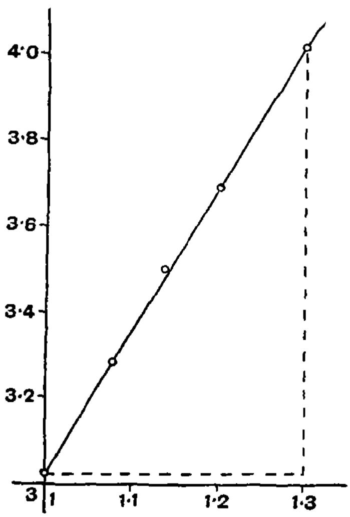

FIG. 10.

To find n, it is obvious from the graph that

$$n = \tan\theta$$
$$= (4·0151 - 3·0277)/(1·3010 - 1)$$
$$= 0·9945 \div 0·4786 = \mathbf{3·28};$$

$\therefore \quad y = 3·28x + c.$

Since $y = 3·0277$ when $x = 1$,

$\therefore \quad \log a = c = 3·0277 - 3·28 = \bar{1}·7477$,

so that $\qquad a = 0·5594$,

the probable value of which will be

$$a = \mathbf{0·56}.$$

Hence the complete law is

$$\mathbf{P = 0·56v^{3·28}}.$$

To find P when $v = 26$, $x = 1·415$, $c = \bar{1}·7477$; hence from the linear equation

$$y = \bar{1}·7477 + 3·28 \times 1·415 = 4·3877 = \log(2·442 \times 10^4);$$

$$\therefore \quad P = \mathbf{2·442 \times 10^4}.$$

The value of y might have been obtained at once from the graph.

62. Common Types of Equations. Mechanical or physical considerations of a particular problem will often lead to a clue of the form of probable equation to be tried, but many cases occur

where no hint is obtainable as to the functional dependence of the variables. The solution, therefore, becomes a matter of trial, and the following types are given as a guide in such cases, these being the most commonly met with in practice.

No	Type of Equation; Variables, u, v	Substitutions for Change of Variables, etc.	Linear Equation
i.	$v=b+\frac{a}{u}$	$y=v$, $x=\frac{1}{u}$	$y=ax+b$
ii.	$v=au^n$	$y=\log v$, $x=\log u$, $c=\log a$	$y=nx+c$
iii.	$v=be^{au}$	$y=\log_e v$, $x=u$, $c=\log_e b$	$y=ax+c$
iv.	$v=au^2+bu$	$xy=v$, $x=u$	$y=ax+b$ if u is not 0

(61)

Ex. 2. *A pendulum was partially immersed in a medium which damped its vibrations. For each vibration the amplitude v and time t were carefully observed as follows:*

v	10	7·6	5·8	4·4	3·3	2·6	1·9
t	0	8	17	25	34	41	50

Find the relation connecting v and t, allowing for slight errors of observation.

From a consideration of damped vibrations, it is known that the amplitude v is connected with the time by the exponential law; hence, the probable form of equation to take is $v=be^{-at}$.

To express this relation in linear form, take natural logarithms, so that

$$\log_e v=\log_e b-at.$$

Now let $y=\log_e v$, $c=\log_e b$, then

$$y=c-at.$$

From this construct the following table for plotting. Natural logarithms should be read directly from Napierian tables.

$y=\log_e v$	2·3026	2·0281	1·7579	1·4816	1·1939	0·9555	0·6419
t	0	8	17	25	34	41	50

These points are shown in Fig. 11, from which it is evident that they lie approximately on a straight line, the slight discrepancies being probably due to errors of observation.

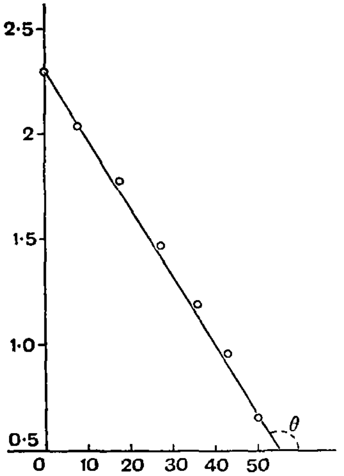

FIG 11

To find the numerical value of a, it is obvious from the linear equation that

$$a = -\tan\theta$$
$$= \tan(\pi-\theta) = (2\cdot3026 - 0\cdot6119)/50$$
$$= 0\cdot033214,$$

so that, correct to three figures, a may be taken as 0·033.

Finally, for c, when $t=0$,

$$y = c = 2\cdot3026\,;$$

$\therefore$ $\log_e b = c = 2\cdot3026$, giving $b = 10$.

$\therefore$ Complete relationship is

$$v = 10e^{-0\cdot33t}.$$

63. Approximate Integral Function Laws. In some cases experimental data may be represented approximately by a finite power series, called an **integral function.** For this to be done where it is possible, the ordinates must be known at equal intervals along the horizontal axis, and if these are not given they may be determined by plotting. The values must then be tested by the method of differences.

Ex. 3. *Determine the necessary conditions that a series of observations connecting two variables may be represented approximately by a rational integral function, and assuming the conditions fulfilled, shew how the coefficients may be calculated.*

Illustrate the method in the case of the following table of experimental observations.

x	12	14	16	18	20	22	24	26
y	217·38	343·97	512·56	729·15	999·73	1330·34	1726·96	2195·54

Let x_r, y_r ($r=1, 2, 3, \dots n$) be n pairs of corresponding observations which are given, and suppose that the functional relationship representing them is $y=f(x)\equiv a+bx+cx^2+dx^3+ex^4$,

where a, b, c, d, e are numerical coefficients to be determined later.

Suppose, further, that the given values of x increase in the positive direction by a constant quantity h, then

$$y_r = f(x_r) = f\{x_1 + (r-1)h\}.$$

Now the difference between two consecutive ordinates is called the first order of differences. This is

$$y_{r+1} - y_r = f(x_r + h) - f(x_r) = a_1 + b_1 x_r + c_1 x_r^2 + d_1 x_r^3,$$

where

$$a_1 = f(h),\ b_1 = h(2c + 3dh + 4eh^2),\ c_1 = 3h(d + 2eh),\ \text{and}\ d_1 = 4eh.$$

Similarly,

$$y_{r+2} - y_{r+1} = a_1 + b_1(x_r + h) + c_1(x_r + h)^2 + d_1(x_r + h)^3.$$

The second order of differences is

$$\begin{aligned} &y_{r+2} - y_{r+1} - (y_{r+1} - y_r) \\ &= y_{r+2} - 2y_{r+1} + y_r = (b_1 + c_1 h + d_1 h^2)h + h(2c_1 + 3d_1 h)x_r + 3d_1 h x_r^2 \\ &= a_2 + b_2 x_r + c_2 x_r^2. \end{aligned}$$

Similarly,

$$y_{r+3} - 2y_{r+2} + y_{r+1} = a_2 + b_2(x_r + h) + c_2(x_r + h)^2.$$

Proceeding in the same manner to form the third order of differences:

$$\begin{aligned} y_{r+3} - 3y_{r+2} + 3y_{r+1} - y_r &= h(b_2 + c_2 h) + 2c_2 h x_r, \\ y_{r+4} - 3y_{r+3} + 3y_{r+2} - y_{r+1} &= h(b_2 + c_2 h) + 2c_2 h(x_r + h); \end{aligned}$$

and finally, the fourth order of differences becomes

$$y_{r+4} - 4y_{r+3} + 6y_{r+2} - 4y_{r+1} + y_r = 2c_2 h^2,$$

which is independent of x_r, and therefore constant.

It is obvious, therefore, that if $n > 5$ this will be true for all the values of r available from the data, so that a series of equal constant numbers is thus derivable.

This is the characteristic property of a series of corresponding pairs of numbers satisfying a rational integral function law, for whilst it is here shown to be true for a function of the fourth degree, the analysis is easily extended to the general case. It should be noted that the coefficients of y_r, y_{r+1}, ..., are those of the binomial expansion; thus, for the kth order of differences, the coefficients are those of the expansion of $(1-u)^k$.

In practice, however, functions of a higher degree than three are rarely met with.

The conditions, therefore, that a series of observations may be represented by a rational integral function, $y=f(x)$, are :

(i) **The values of y must be given, or determined by plotting, at equal intervals along the x-axis.**

(ii) **In calculating successive orders of differences of these values, a stage is ultimately reached when a series of constants is obtained.**

The above investigation shews that for a function of the fourth degree the equal constants appear after the formation of the **fourth** order of differences. This is obvious from the fact that the calculation of each successive order of differences reduces the degree of x by unity. Hence, if a series of equal numbers appears on forming the pth order of differences, the integral function is of the pth degree.

The coefficients a, b, c, d, e in $f(x)$ may readily be calculated from the equations expressing the successive differences, beginning with the last and working backwards. The process is best illustrated by a numerical example ; thus, taking the given table of observations and noting that the ordinates are equidistant from each other, after the first, differences are best found by tabulating as follows :

Abscissae x		Ordinates y		Differences.		
				1st Order	2nd Order	3rd Order
x_1	12	y_1	217·38			
				126·59		
x_2	14	y_2	343·97		42·00	
				168·59		6·00
x_3	16	y_3	512·56		48·00	
				216·59		5·99
x_4	18	y_4	729·15		53·99	
				270·58		6·04
x_5	20	y_5	999·73		60·03	
				330·61		5·98
x_6	22	y_6	1330·34		66·01	
				396·62		5·95
x_7	24	y_7	1726·96		71·96	
				468·58		
x_8	26	y_8	2195·54			

From these figures it is evident that, allowing for slight errors of observation, the third order of differences is practically constant, and equal to 6 approximately ; hence the integral function law will be of the form

$$y=a+bx+cx^2+dx^3.$$

Forming the three orders of differences in terms of the coefficients, these become

(1) $y_2 - y_1 = bh + ch^2 + dh^3 + h(2c + 3dh)x_1 + 3hdx^2$,

(2) $y_3 - 2y_2 + y_1 = 2h^2(c + 3dh) + 6dh^2x_1$,

(3) $y_4 - 3y_3 + 3y_2 - y_1 = 6dh^3$, where $h = 2$.

Taking the differences from the table, and noting that h is 2,

$$6dh^3 = 6\ ; \quad \therefore\ d = \mathbf{0{\cdot}125}.$$

With this value of d, (2) becomes

$$8c + 6 + 36 = 42, \quad \text{giving } c = \mathbf{0}.$$

Hence from (1),

$$2b + 108 + 18 + 1 = 126{\cdot}59, \quad \text{giving } b = -\mathbf{0{\cdot}2}.$$

Inserting these values in the function, and taking any corresponding pair of values of x and y, $a = 3{\cdot}75$ approximately.

$\therefore$ The required equation is

$$\mathbf{y = 3{\cdot}75 - 0{\cdot}2x + 0{\cdot}125x^3}.$$

64. Other Types of Equational Laws. Occasionally in practice curves are met with which may be approximately represented by the following equational forms :

$$y = a + bx^n, \quad y = a + be^{cx}, \quad y = ax^n + bk^{cx}.$$

When n is a positive integer, these equations may generally be derived by the method of differences. The first form can be dealt with as in Ex. 3. The application of the method to the two remaining forms is illustrated in the following examples.

Ex. 4. *The following values of P and t are the observed results of an experiment, and theory suggests that they probably follow a law of the form, $P = a + be^{mt}$. Investigate whether this is so, and if found approximately true, determine the most probable values of* a, *b and m.*

t	1·4	1·8	2·2	2·6	3·0	3·4	3·8
P	1·73	1·84	1·98	2·15	2·35	2·60	2·90

It will be observed that the values of t are in arithmetical

progression, the common interval being 0·4. Denote this interval by h, then if P_r, t_r be the rth pair of values of P and t,

$$P_r = a + be^{mt_r},$$

$$P_{r+1} = a + be^{m(t_r+h)}:$$

$$\therefore\ P_{r+1} - P_r = b\{e^{m(t_r+h)} - e^{mt_r}\} = b(e^{mh} - 1)e^{mt_r}.$$

Taking Napierian logarithms,

$$\log(P_{r+1} - P_r) = \log b(e^{mh} - 1) + mt_r,$$

or, writing y for $\log(P_{r+1} - P_r)$, and c for $\log b(e^{mh} - 1)$, the equation becomes

$$y = mt + c,$$

which represents a straight line.

To test this, the following table must be constructed from the given data:

r	t_r	P_r	$P_{r+1} - P_r$	$y = \log(P_{r+1} - P_r)$
1	1·4	1·73		
			0·11	−2·2073
2	1·8	1·84		
			0·14	−1·9661
3	2·2	1·98		
			0·17	−1·7720
4	2·6	2·15		
			0·20	−1·6095
5	3·0	2·35		
			0·25	−1·3863
6	3·4	2·60		
			0·30	−1·2040
7	3·8	2·90		

On plotting the values of y and t, a close approximation to a straight line is obtained; hence, allowing for slight errors of observation, the equation $P = a + be^{mt}$ will represent the given values. It remains, therefore, to find the values of the constants a, b and m.

Let θ be the angle made by the straight line with the axis of t, measured in the positive direction, then

$$m = \tan\theta = (2·2073 - 1·2040)/(3·4 - 1·4) = 1·0033 \div 2 = 0·5016$$
$$= 0·50 \text{ approximately.}$$

Also, when $t = 3·4$, $y = -1·2040$, so that from $y = mt + c$

$$c = -1·2040 - 1·7 = -2·9040.$$

But $c = \log b(e^{mh} - 1) = \log b + \log (e^{0\cdot 2} - 1) = \log b - 1{\cdot}5078$;

$$\therefore \log b = c + 1{\cdot}5078 = -2{\cdot}9040 + 1{\cdot}5078 = -1{\cdot}3962$$
$$= 0{\cdot}9064 - 2{\cdot}3026 = \log 2{\cdot}476 - \log 10 = 0{\cdot}2476;$$
$$\therefore b = 0{\cdot}25 \text{ approximately.}$$

Finally, taking $t = 3{\cdot}4$ when $P = 2{\cdot}60$, and substituting in

$$P = a + be^{mt},$$

$a = 2{\cdot}60 - 0{\cdot}25e^{1\cdot 7} = 2{\cdot}60 - 1{\cdot}3684 = 1{\cdot}2316 = 1{\cdot}23$ approximately.

Hence, the complete equation becomes

$$\mathbf{P = 1{\cdot}23 + 0{\cdot}25e^{0\cdot 5t}}.$$

Ex. 5. *The following values of Q and z were obtained in a laboratory, and theoretical considerations shewed that the law connecting these quantities was most probably of the form*

$$Q = az^2 + b \,.\, 10^x.$$

Test this, and if found true, due allowance being made for observational errors, determine the values of a and b.

z	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
Q	1·616	2·185	2·969	3·995	5·296	6·917	8·917	11·343

The method of differences is here directly applicable since the ordinates z are equidistant over the range given, the equal intervals being 0·1. Denote 0·1 by h, then if Q_r, z_r be a pair of corresponding values,

$$Q_r = az_r^2 + b \,.\, 10^{z_r},$$

and
$$Q_{r+1} = a(z_r + h)^2 + b \,.\, 10^{z_r + h};$$

$$\therefore Q_{r+1} - Q_r = 2ahz_r + ah^2 + b(10^h - 1)10^{z_r}.$$

Similarly, $Q_{r+2} - Q_{r+1} = 2ah(z_r + h) + ah^2 + b(10^h - 1) \,.\, 10^{z_r + h}$;

$$\therefore Q_{r+2} - 2Q_{r+1} + Q_r = 2ah^2 + b(10^h - 1)^2 \,.\, 10^{z_r}.$$

Similarly, proceeding to form the third order of differences,

$$Q_{r+3} - 3Q_{r+2} + 3Q_{r+1} - Q_r = b(10^h - 1)^3 \,.\, 10^{z_r}.$$

Taking common logarithms, and writing y for

$$\log (Q_{r+3} - 3Q_{r+2} + 3Q_{r+1} - Q_r),$$

and c for $\log b + 3\log(10^h - 1) = \log b - 1{\cdot}7601$, the equation becomes

$$y = z + c,$$

which represents a straight line inclined to either axis at an angle of 45°. To test this, the following table of differences must be constructed :

r	z_r	Q_r	DIFFERENCES			y
			1st Order	2nd Order	3rd Order	
1	0·1	1·616				
			0·569			
2	0·2	2·185		0·215		
			0·784			
3	0·3	2·969		0·242	0·027	−1·5686
			1·026			
4	0·4	3·995		0·275	0·033	−1·4815
			1·301			
5	0·5	5·296		0·320	0·045	−1·3468
			1·621			
6	0·6	6·917		0·372	0·052	−1·2840
			1·993			
7	0·7	8·910		0·440	0·068	−1·1675
			2·433			
8	0·8	11·343				

Plotting y and z, remembering that the first value of y corresponds to $x_1 = 0{\cdot}1$, and so on, an approximate straight line is obtained.

To find c, choose any pair of values, and substitute in the equation $y = z + c$; thus, taking the last pair,

$$c = -1{\cdot}1675 - 0{\cdot}5 = -1{\cdot}6675\ ;$$

$$\therefore\ \log b = -1{\cdot}6675 + 1{\cdot}7601 = 0{\cdot}0926\ ;$$

$$\therefore\ b = 1{\cdot}235, \quad \text{or} \quad 1{\cdot}24 \text{ approximately.}$$

Taking the initial equation, and selecting any pair of values of z and Q, the value of a may be determined ; thus, choosing $Q = 5{\cdot}296$, and $z = 0{\cdot}5$,

$$5{\cdot}296 = a(0{\cdot}5)^2 + 1{\cdot}24 \times 10^{0\cdot5}$$

$$= 0{\cdot}25a + 3{\cdot}921\ ;$$

$$\therefore\ a = (5{\cdot}296 - 3{\cdot}921) \div 0{\cdot}25$$

$$= 1{\cdot}375 \times 4 = 5{\cdot}500.$$

Hence, the complete equation becomes

$$Q = 5{\cdot}5z^2 + 1{\cdot}24 \times 10^z.$$

65. Effect of Small Errors of Observation. All observed measurements are only approximate, but if made as accurately as possible the difference between the absolute and observed values will, in general, be of little or no practical importance. It is when calculations are carried out with these observed values that errors are liable to increase to such an extent as to affect the ultimate result very materially. It is, therefore, very important in all practical work to consider how the accuracy of a calculated value is influenced by small errors in the data used.

If u be an approximate number known correctly to n places of decimals, then the **limit of the error of u is defined as $5 \times 10^{-n-1}$**, and if ∂u denote this limit, **the quotient $\partial u/u$ is defined as the relative error of u.**

Ex. 6. *If $y=f(x)$, shew that, for a measured value a of x, subject to an error $+\partial a$, then the approximate relative error in the corresponding value of y is $f'(a) \,.\, \partial a/f(a)$, where $f'(a)$ denotes the operation of differentiating $f(x)$ with respect to x and then writing a for x.*

Deduce that the relative error in a product or quotient of two numbers is equal respectively to the sum or difference of the limiting errors of the numbers. Illustrate by finding how many significant figures in the following calculations are trustworthy, assuming that the given numbers are correct to the number of figures stated :

$$\text{(i)}\ 4{\cdot}63 \times 8{\cdot}723, \quad \text{(ii)}\ \frac{3{\cdot}87 \times 18{\cdot}26}{7{\cdot}86}.$$

Let u be the value of y, when $x=a$, so that $u=f(a)$, and suppose that $u+\partial u$ is the true value of u when $x=a+\partial a$, then

$$u+\partial u=f(a+\partial a).$$

$\therefore$ By subtraction,
$$\begin{aligned}\partial u &= f(a+\partial a)-f(a)\\ &= f(a)+\partial a \,.\, f'(a)+\ldots-f(a), \text{ by (55)},\\ &= \partial a \,.\, f'(a) \text{ approximately,}\end{aligned}$$

since ∂a is small.

Hence the relative error of u

$$=\frac{\partial u}{u}=\frac{\partial a \,.\, f'(a)}{f(a)}.$$

Let $y=zw$, then if ∂y, ∂z, ∂w are the limiting errors in y, z, w respectively,

$$\partial y=(z+\partial z)(w+\partial w)-zw=z\,.\,\partial w+w\,.\,\partial z \text{ approximately;}$$

$$\therefore\ \frac{\partial y}{y}=\frac{\partial z}{z}+\frac{\partial w}{w}.$$

In a similar manner it may be shewn that if $y=z/w$, then

$$\frac{\partial y}{y}=\frac{\partial z}{z}-\frac{\partial w}{w}.$$

It is easy to shew by logarithmic differentiation that similar results are true for any number of approximate values; hence—

The relative error of a product is equal to the sum of the relative errors of the respective factors, and the relative error of a quotient is equal to the difference of the relative errors of dividend and divisor respectively.

(i) Let $y=4{\cdot}63\times8{\cdot}723$,

limit of error of $4{\cdot}63=0{\cdot}005$, and of $8{\cdot}723=0{\cdot}0005$;

$$\therefore\ \frac{\partial y}{y}=\frac{0{\cdot}005}{4{\cdot}63}+\frac{0{\cdot}0005}{8{\cdot}723}=0{\cdot}00108+0{\cdot}000057$$
$$=0{\cdot}001137;$$

$$\therefore\ \partial y=0{\cdot}001137\times4{\cdot}63\times8{\cdot}723=0{\cdot}0459.$$

Hence the product will only be true to one place.

(ii) Let $y=\dfrac{3{\cdot}87\times18{\cdot}26}{7{\cdot}86}$, then

$$\frac{\partial y}{y}=\frac{0{\cdot}005}{3{\cdot}87}+\frac{0{\cdot}005}{18{\cdot}26}-\frac{0{\cdot}005}{7{\cdot}86}$$
$$=0{\cdot}005(0{\cdot}2584+0{\cdot}0548-0{\cdot}1272)=0{\cdot}00093;$$

$$\therefore\ \partial y=0{\cdot}00093\times3{\cdot}87\times18{\cdot}26\div7{\cdot}86=0{\cdot}0836.$$

Hence, since $\partial y>0{\cdot}05$, the first place in the value of y will be unreliable, so that the result will only be true to the nearest unit.

Ex. 7. *The horizontal pressure p at any depth h in a granular material with a horizontal upper face is given by the formula*

$$p(1+\sin\theta)=wh(1-\sin\theta),$$

where w is the weight per unit volume of material and θ is the natural angle of slope. Shew that for small errors ∂w, $\partial\theta$ in the measured values of w and θ, the resultant error ∂p in p is

$$p(\partial w/w-2\sec\theta\,.\,\partial\theta).$$

If $w=114$ to the nearest unit, $h=6$, and $\theta=0{\cdot}5236$ radian, this value being correct to four figures, shew that the limiting percentage error in the calculated value of p is $0{\cdot}43$ approximately.

It is evident that

$$p+\partial p = h(w+\partial w)\{1-\sin(\theta+\partial\theta)\}/\{1+\sin(\theta+\partial\theta)\}$$

and $$p = hw(1-\sin\theta)/(1+\sin\theta),$$

from which ∂p may be obtained, and the result simplified by using Taylor's theorem, as in Ex. 6.

It is, however, much shorter and simpler to differentiate logarithmically, thus

$$\log p = \log h + \log w + \log(1-\sin\theta) - \log(1+\sin\theta);$$

$$\therefore \frac{\partial p}{p} = \frac{\partial w}{w} - \frac{\cos\theta\,.\,\partial\theta}{1-\sin\theta} - \frac{\cos\theta\,.\,\partial\theta}{1+\sin\theta}$$

$$= \frac{\partial w}{w} - 2\sec\theta\,.\,\partial\theta;$$

$$\therefore \partial p = p(\partial w/w - 2\sec\theta\,.\,\partial\theta).$$

$0{\cdot}5236$ radian $=30^\circ$, and $\sin 30^\circ = 0{\cdot}5$.

$\therefore$ With the given values, $p=(114\times 6\times 0{\cdot}5)/1{\cdot}5 = 228$.

Now the limit of the error in $w=0{\cdot}5$;

$$\therefore \frac{\partial w}{w} = \frac{0{\cdot}5}{114} = 0{\cdot}004386,$$

and the limit of the error in $\theta = 0{\cdot}00005$;

$$\therefore 2\sec\theta\,.\,\partial\theta = (4\times 0{\cdot}00005)/\sqrt{3} = 0{\cdot}000116;$$

$$\therefore \partial p/p = 0{\cdot}004270.$$

$\therefore$ Percentage error $=100\,.\,\partial p/p = 0{\cdot}427 = 0{\cdot}43$ approximately.

Exercises 9.

1. Experiments on the specific heat K_v of a gas at constant volume produced the following results, T being the absolute temperature:

K_v	0·1377	0·1384	0·1392	0·1400	0·1408	0·1418	0·1427	0·1444
T	273	288	305	321	336	355	373	408

Shew that K_v and T are connected approximately by a linear law, and determine this law.

2. At the following draughts in sea water, a particular vessel has the following displacements :

Draught, h ft. - -	15	12	9	6·3
Displacement, T tons	2098	1512	1018	586

Plot log T and log h on squared paper, and determine a simple law connecting T and h.

If one ton of sea water measures 35 cu. ft., find the law connecting h and the displacement V cu. ft.

3. In some experiments in towing a canal boat, the following observations were made, P being the pull in pounds and v the speed of the boat in miles per hour :

v	1·68	2·43	3·18	3·60	4·03
P	76	160	240	320	370

Plot log v and log P on squared paper, and derive the approximate formula connecting P and v.

4. The following table gives corresponding values of two quantities x and y, which should be connected by the approximate law $x^n y = c$. Investigate if this is so, and if found true, determine the values of the constants n and c :

x	37·36	31·34	26·43	19·08	16·33	14·04
y	10·16	12·26	14·70	20·80	24·54	28·83

5. The following tests were made upon a condensing steam turbine electric generator. There are probably errors of observation as the measurement of steam is troublesome :

Output in kilowatts, K -	1190	995	745	498	247	0
Weight W lb. of steam consumed per hour - -	23120	20040	16630	12560	8320	4065

Find a simple approximate law connecting K and W.

6. The following data were obtained from a test of an experimental dam acting as a weir. The length L of the dam is 3 feet:

Quantity Q in cubic feet per second - - -	2·34	4·23	6·38	9·12	14·04
Head H in inches - -	4·80	7·21	9·61	12·12	16·20

Shew that the formula connecting Q and H is of the form $Q = CLH^n$, and determine the constants C and n. (L.U.)

7. It is thought that the following observed values of x and y follow the law $y = Ae^{bx}$. Test if this is so, and if found approximately true, find the values of A and b:

x	1·0	1·5	2·0	2·5	3 0	3 5	4·0	4·5
y	13·28	15·04	17·53	19·80	23·11	26·00	30·50	34·40

8. The following quantities measured in a laboratory are thought to follow the law $y = ab^{-x}$. Verify this, and determine the constants a and b:

x	0·1	0·2	0·4	0·6	1·0	·5	2·0
y	350	316	120	6·3	12 86	2·57	0·425

9. A small needle suspended in a magnetic field of strength H makes n vibrations per minute, and it is supposed that

$$H = an^2 + b.$$

Shew that this law is approximately true from the following observed values, and determine the constants a and b:

H	0	0 0353	0·0738	0·1156	0·1605	0·2085	0·2596
n	32	35	38	41	44	47	50

10. In a platinum resistance thermometer the resistance R at a temperature $t°$ C. is given in the following table of experimental observations:

R	2·297	2·572	2·822	3·110	3·374	3·635	3·883	4·149
t	−75	−50	−25	0	25	50	75	100

Find the complete relationship between R and t in the form

$$R = a + bt + ct^2.$$

11. The following values of Q and t were observed during an experiment, and theory suggested that there might be a law

$$Q = at^2 + b \, . \, 10^t.$$

Test this, and if found approximately true, find the values of a and b.

Q	3·407	4·516	6·014	7·959	10·42	13·50	17·30	21·96
t	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8

12. If a, b, c are numbers which may be slightly inaccurate by percentage amounts not exceeding $\pm\alpha$, $\pm\beta$ and $\pm\gamma$ respectively, shew that, in computing $a^l b^m / c^n$, the answer is liable to a percentage error confined within the limits $\pm(l\alpha + m\beta - n\gamma)$.

The diameter d and depth h of a cylindrical bin measure 22·2 in. and 28·5 in. Find how many lb. of grain the bin will hold when just full, the specific volume s of the grain being estimated at 7·2 cu. in. to the lb.

If the accuracy of d, h, s is true only within the limits $\pm 0{\cdot}1$ in., $\pm 0{\cdot}2$ in. and $\pm 0{\cdot}02$ in. respectively, find the actual and the percentage error in the above result.

13. The gunners' rule is that one halfpenny subtends an angle of one minute at a distance of one hundred yards. Taking the diameter of a halfpenny as one inch, find the percentage error in this rule.

14. The formula

$$\phi = 1{\cdot}056 \log_e \frac{t}{273} + 9 \times 10^{-7}(\tfrac{1}{2}t^2 - 503t) + 0{\cdot}0902$$

is replaced by $\phi = \log_e \dfrac{t}{273}$ as an approximation. Find the percentage error in doing this when $t = \theta + 273$ and $\theta = 53$.

15. If $k = 8{\cdot}392 \times 65{\cdot}25 \div 7{\cdot}528$, each number being correct to the number of figures given, find how many figures in the value of k are reliable.

16. The area of a triangle is calculated from the angles A and C and the side b. If a small error ∂A be made in measuring A, shew that the percentage error in the area is approximately

$$100\partial A \, . \, \sin C / \{\sin A \, . \, \sin (A + C)\}.$$

What is the percentage error if, in addition, there are errors Δb and ΔC in the other parts ? (L.U.)

17. In a triangle ABC the sides b, c and the angle A are measured. If small errors e and ϕ are made in measuring the sides and angle respectively, shew that the error in the calculated value of a is

$$(\cos B + \cos C)e + b\phi \sin C.$$

If $b = c = 4$ in., $A = \pi/3$, $e = 0{\cdot}1$ in., $\phi = 0{\cdot}01$ radian, find the error in a and verify by trigonometrical calculations. (S.U.)

18. Two sides of a triangle are measured and found to be 32·5 in. and 24·2 in., the included angle being 57°; find the area of the triangle. If the true lengths of the sides are really 32·6 in. and 24·1 in., what is the percentage error in the area ?

19. The perpendicular distance x of a point P from a fixed base line AB is estimated from the measurements of $AB = a$, the angle $PAB = \alpha$ and the angle $PBA = \beta$. Prove that

$$x \sin(\alpha + \beta) = a \sin \alpha \sin \beta.$$

Explain how to find the percentage error in x due to small errors in the measurements of a, α and β, and verify that when $\alpha = \beta = 45°$, the percentage error in x, due to an error of $+1'$ in each of the measurements α and β, is 0·06 nearly. (L.U.)

***20.** The efficiency n of the teeth of a pair of screw wheels is given by the formula

$$\eta = \frac{\cos(\theta_2 + \phi) \cdot \cos \theta_1}{\cos(\theta_1 - \phi) \cdot \cos \theta_2},$$

where θ_1, θ_2 are the screw angles and ϕ is the angle of friction. Shew that, for a small error of $\partial\phi$ in ϕ, the corresponding error in η is given by

$$\partial\eta = -\sin(\theta_1 + \theta_2) \sec^2(\theta_1 - \phi) \cdot \cos \theta_1 \cdot \sec \theta_2 \cdot \partial\phi.$$

Hence calculate the percentage error in η when $\theta_1 = 34·8°$, $\theta_2 = 55·2°$, and coefficient of friction $\mu = 0·084$, correct to three places.

***21.** The rate of flow Q of water per second over a sharp-edged notch of length l, the height of the surface of nearly still water above the sill being h, is given by the formula

$$Q = c(l - \tfrac{1}{5}h)h^{\frac{3}{2}}.$$

Shew that for a small error ∂h in the measurement of h, the error ∂Q in Q is

$$\tfrac{1}{2}(3l - h)h^{\frac{1}{2}} \cdot \partial h.$$

Sometimes an approximate formula, $Q = clh^{\frac{3}{2}}$, is used to find Q; shew that for any given values of l and h, the percentage error in using this formula is

$$\frac{100h}{5l - h}.$$

CHAPTER X

TWO-DIMENSIONAL GEOMETRY

66. Coordinates. If OX, OY be two mutually perpendicular axes, the position of any point P in their plane may be defined by its distances from OX, OY respectively. These distances are called the **Cartesian Coordinates** of P, and are denoted by (x, y).

If, however, only one axis of reference, OX, be used, P may be defined by the distance OP and the angle XOP. These are called the **Polar Coordinates** of P, and are denoted by (r, θ).

It is obvious that the relations between the two sets of coordinates are

$$\mathbf{x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = y/x.} \qquad \text{.............(62)}$$

Ex. 1. *Find an expression for the area of a triangle in terms of the coordinates of its vertices.*

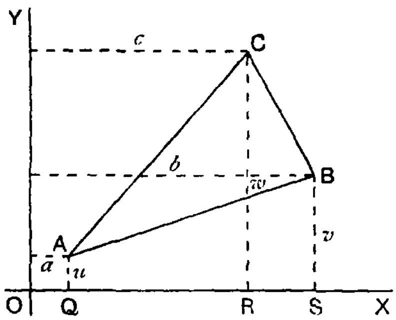

FIG. 12. Area of a triangle.

The points P_1, P_2, P_3, P_4 have coordinates (2, 1), (5, 2), (6, 6), *and* (9, 0). *Find the areas of the quadrilaterals $P_1P_2P_3P_4P_1$ and $P_1P_2P_4P_3P_1$. If P_1P_3 and P_2P_4 intersect in P, shew the connection between the area of $P_1P_2P_4P_3P_1$ and those of P_1P_2P and PP_4P_3.* (L.U.)

(i) Let A, (a, u), B, (b, v), C, (c, w), be the vertices of the triangle, with their respective coordinates; then from Fig. 12 $QACR$ is a trapezium.

$\therefore$ Area of $QACR = \frac{1}{2} . QR . (QA + RC) = \frac{1}{2}(c-a)(u+w)$.

Similarly, the area of $RCBS = \frac{1}{2}(b-c)(v+w)$, and the area of $QABS = \frac{1}{2}(b-a)(u+v)$.

$\therefore$ Area of $\triangle ABC$ = area $QACR$ + area $RCBS$ − area $QABS$

$$= \tfrac{1}{2}(c-a)(u+w) + \tfrac{1}{2}(b-c)(v+w) - \tfrac{1}{2}(b-a)(u+v)$$

$$= \tfrac{1}{2}\{a(v-w) + b(w-u) + c(u-v)\}$$

$$= \tfrac{1}{2}\begin{vmatrix} a & b & c \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}. \qquad \text{......(63)}$$

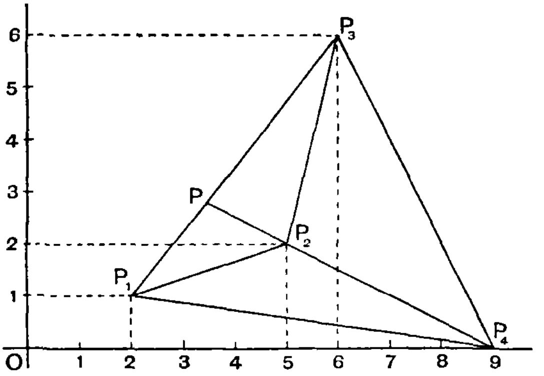

FIG. 13. Area of a quadrilateral.

(ii) The points P_1, P_2, P_3, P_4 are shewn in Fig. 13; then by joining them as shewn:

$$\text{Area of } \triangle P_1P_2P_4 = \tfrac{1}{2}\begin{vmatrix} 2 & 9 & 5 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 5.$$

$$\text{Area of } \triangle P_2P_3P_4 = \tfrac{1}{2}\begin{vmatrix} 5 & 9 & 6 \\ 2 & 0 & 6 \\ 1 & 1 & 1 \end{vmatrix} = 9.$$

$$\text{Area of } \triangle P_1P_2P_3 = \tfrac{1}{2}\begin{vmatrix} 2 & 5 & 6 \\ 1 & 2 & 6 \\ 1 & 1 & 1 \end{vmatrix} = 5{\cdot}5.$$

$\therefore$ Area of $P_1P_2P_3P_4P_1 = \triangle P_1P_2P_4 + \triangle P_2P_3P_4 = 5 + 9 = \mathbf{14}$,
and area of $P_1P_2P_4P_3P_1 = \triangle P_1P_2P_3 + \triangle P_2P_3P_4 = 5{\cdot}5 + 9 = \mathbf{14{\cdot}5}$.

Produce P_4P_2 to intersect P_1P_3 in P, then

$$\text{area of } P_1P_2P_4P_3P_1 = \triangle P_1P_2P + \triangle PP_2P_3 + \triangle P_2P_3P_4$$
$$= \triangle P_1P_2P + \triangle PP_4P_3.$$

67. The Straight Line. Every equation of the first degree in x and y represents a straight line ; for let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three arbitrary points on the locus $ax+by+c=0$, then

$$ax_1+by_1+c=0,$$
$$ax_2+by_2+c=0,$$
$$ax_3+by_3+c=0.$$

Eliminating a, b, c from these equations,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

But by (63), $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$ = twice the area of the triangle whose vertices are the three given points.

Hence, since this area is zero, the three points must be collinear ; that is, the equation $ax+by+c=0$ represents a straight line.

Ex. 2. (i) *Shew that the equation of the line joining the points* (x_1, y_1), (x_2, y_2) *is* $x(y_1-y_2)-y(x_1-x_2)+x_1y_2-x_2y_1=0$, *whether the axes are rectangular or oblique.*

(ii) *ABCD, A′BC′D′ are two parallelograms having a common angle at B ; prove that DD′, AC′ and A′C are concurrent.* (L.U.)

(i) Let $ax+by+c=0$ be the equation of the straight line passing through the given points, and let (x, y) be any other point on the straight line, then the area of the triangle whose vertices are (x, y), (x_1, y_2), (x_2, y_2) is zero,

i.e.
$$\tfrac{1}{2}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

or
$$x(y_1-y_2)-y(x_1-x_2)+x_1y_2-x_2y_1=0.$$

If the axes were inclined at any angle ω, then the distances (x, y) of any point would be measured in directions parallel to the axes. Thus in Fig. 14, if OX, OY are the oblique axes, and P any point, $OL = MP = x$, and $LP = OM = y$. Let OY' be drawn perpendicular to OX, and let $ON = x'$, $OK = y'$, then

$$x' = OL + LN = x + y \cos \omega \quad \text{and} \quad y' = OK = NP = y \sin \omega.$$

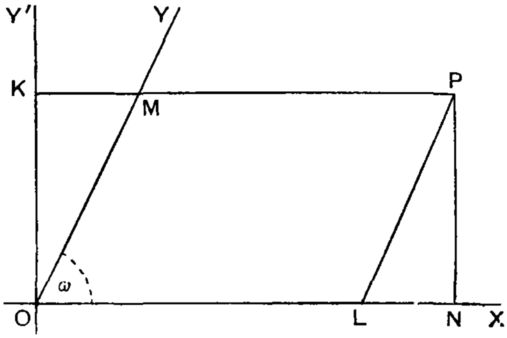

FIG. 14. Oblique axes.

Hence the straight line $ax' + by' + c = 0$ through P referred to the rectangular axes becomes $ax + (a \cos \omega + b \sin \omega)y + c = 0$ referred to the oblique axes.

Since $a \cos \omega + b \sin \omega$ is constant, it may be written B, so that the equation becomes $ax + By + c = 0$.

If, therefore, (x_1, y_1), (x_2, y_2) are points on this line,

$$ax + By + c = 0,$$

$$ax_1 + By_1 + c = 0,$$

$$ax_2 + By_2 + c = 0.$$

Eliminating the constants a, B, c,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

or
$$x(y_1 - y_2) - y(x_1 - x_2) + x_1 y_2 - x_2 y_1 = 0,$$
as before.

Hence the equation is independent of the angle at which the axes are inclined, provided the proper interpretation be given to the coordinates.

(ii) Take DC, DA as the axes of x and y respectively, and let the coordinates of the vertices be

$$A\,(0, v_1),\qquad B\,(u_1, v_1),\qquad C\,(u_1, 0),\qquad D\,(0, 0),$$
$$A'\,(u_2, v_1),\qquad C'\,(u_1, v_2),\qquad D'\,(u_2, v_2)\,;$$

then the straight line through D, D' is

$$v_2x - u_2y = 0,$$

the line through A, C' is

$$x(v_1 - v_2) + u_1y - u_1v_1 = 0,$$

and the line through A', C is

$$v_1x - y(u_2 - u_1) - u_1v_1 = 0.$$

If these straight lines are concurrent the above equations must have a common root, *i.e.* they must be consistent. The condition for this is, by (10),

$$\begin{vmatrix} v_2 & -u_2 & 0 \\ v_1 - v_2 & u_1 & -u_1 \\ v_1 & u_1 - u_2 & -u_1 \end{vmatrix} = 0.$$

Subtracting the first row from the third, the determinant becomes $\begin{vmatrix} v_2 & -u_2 & 0 \\ v_1 - v_2 & u_1 & -u_1 \\ v_1 - v_2 & u_1 & -u_1 \end{vmatrix}$, which is zero, since two rows are identical.

$\therefore$ the lines are concurrent at the point whose coordinates are the common solution of the above equations.

68. Forms of the Linear Equation. From § 67, it is evident that the **general equation** of a straight line may be written in the form

$$\mathbf{ax + by + c = 0.} \qquad \text{.........(64a)}$$

This equation may, however, assume several useful forms ; thus, dividing throughout by b and writing m for $-a/b$, and n for $-c/b$, it becomes

$$\mathbf{y = mx + n,} \qquad \text{.........(64b)}$$

which is known as the **tangent form,** since m is the gradient, *i.e.* the tangent of the angle of slope, ϕ, and n is the intercept cut off on the y-axis.

Again, dividing the general equation throughout by $-c$, and writing α for $-c/a$ and β for $-c/b$, it becomes

$$\frac{\mathbf{x}}{\alpha}+\frac{\mathbf{y}}{\beta}=\mathbf{1}, \qquad \textbf{(64c)}$$

which is known as the **intercept form,** since α, β are the intercepts cut off by the line on the x- and y-axes respectively (Fig. 15).

Further, from (64a) and (64c), the intercepts cut off on the x- and y-axes respectively are $-c/a$ and $-c/b$. Now if p be the length of the perpendicular from the origin to the line, and ψ the angle it makes with the x-axis, then

$$p=-c/b\,.\,\sin\psi=-c/a\,.\,\cos\psi,$$

or $\qquad a=-c\cos\psi/p$

and $\qquad b=-c\sin\psi/p.$

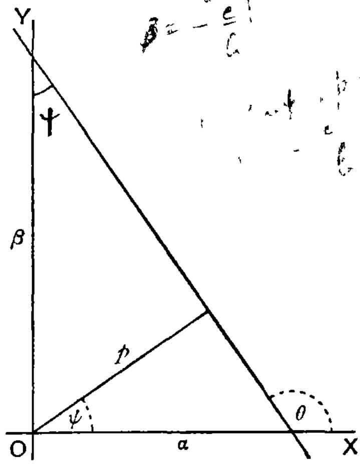

FIG. 15. The straight line.

Substituting these values in (64a), and multiplying out by p, the equation becomes

$$\mathbf{x}\cos\psi+\mathbf{y}\sin\psi=\mathbf{p}. \qquad \textbf{(64d)}$$

This is known as the **perpendicular form.**

Finally, changing x, y into polar coordinates (r, θ), by (62), the perpendicular form is transformed into

$$\mathbf{r}\cos(\theta-\psi)=\mathbf{p}, \qquad \textbf{(64e)}$$

which is the **polar equation** of a straight line. (See Fig. 15.)

Ex. 3. *Find the angles between the straight lines $ax+by+c=0$ and $px+qy+s=0$. Deduce the conditions which must be fulfilled for the lines to be* (i) *parallel and* (ii) *perpendicular.*

Calculate the length of the perpendicular from the point (h, k) to the line $ax+by+c=0$; hence find the distance between the lines

$$6x+1{\cdot}75y+2{\cdot}5=0 \quad \textit{and} \quad 4{\cdot}8x+1{\cdot}4y+11=0.$$

(a) Writing each of the equations in the form

$$y=m_1x+n_1, \quad y=m_2x+n_2,$$

where $m_1 = -a/b$, $m_2 = -p/q$, $n_1 = -c/b$, and $n_2 = -s/q$, it is clear that since m_1, m_2 are the respective gradients, if θ_1, θ_2 are the corresponding slopes of the lines,

$$m_1 = \tan\theta_1 \quad \text{and} \quad m_2 = \tan\theta_2.$$

Let θ be the acute angle between the given lines, then, from Fig. 16,

$$\theta = \theta_2 - \theta_1 ;$$

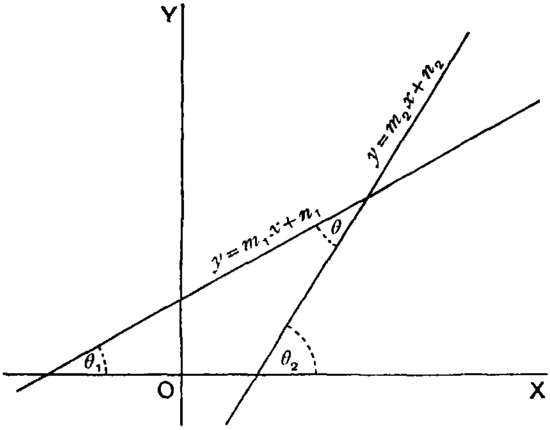

FIG. 16. Angle between two straight lines

$$\therefore \tan\theta = \tan(\theta_2 - \theta_1) = (\tan\theta_2 - \tan\theta_1)/(1 + \tan\theta_1 \cdot \tan\theta_2)$$
$$= (m_2 - m_1)/(1 + m_1m_2) = (aq - bp)/(bq + ap) ;$$

$$\therefore \boldsymbol{\theta = \text{principal value of } \tan^{-1}\left(\frac{aq - bp}{bq + ap}\right).} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(65a)}$$

Since the sum of the angles of intersection on one side of either line is 180°, the larger angle between the given lines is $180° - \theta$.

When $\theta = 0$, the lines are either coincident or parallel. But $\theta = 0$ when $\tan^{-1}\{(aq - bp)/(bq + ap)\} = 0$, *i.e.* when $aq = bp$, or $a/b = p/q$, so that $m_1 = m_2$.

If, in addition, $n_1 = n_2$, the two equations are identical, and the lines thus coincident. Hence, as long as n_1, n_2, *i.e.* c/b and s/q, are unequal, the lines will be parallel when

$$\mathbf{a/b = p/q.}$$

Similarly, the lines are perpendicular when $\theta = 90°$, *i.e.* when

$$\tan^{-1}\{(aq - bp)/(bq + ap)\} = \infty .$$

This will be the case when $bq+ap=0$, provided $aq-bp$ is neither zero nor infinite.

Hence the lines are mutually perpendicular when

$$\mathbf{a/b = -q/p.}$$

From this analysis it follows that **two straight lines**

$$\mathbf{y = m_1x + n_1, \quad y = m_2x + n_2}$$

are parallel when

$$\mathbf{m_1 = m_2 \quad and \quad n_1, n_2 \ are \ unequal,} \quad \ldots\ldots\ldots\ldots\ldots\mathbf{(65b)}$$

and are mutually perpendicular when

$$\mathbf{m_1m_2 = -1.} \quad \ldots\ldots\ldots\ldots\ldots\mathbf{(65c)}$$

(*b*) Let $ax'+by'+c'=0$ be the line through the point (h, k) parallel to $ax+by+c=0$. Turning each of these equations into the perpendicular form (64*d*),

$$p = x \cos\psi + y \sin\psi,$$
$$p' = x' \cos\psi + y' \sin\psi.$$

Now the actual distance between these lines is obviously equal to the difference in the lengths of the perpendiculars p, p'; hence,

$$d = p - p' = (x - x')\cos\psi + (y - y')\sin\psi$$
$$= (x - h)\cos\psi + (y - k)\sin\psi,$$

since the line $p' = x'\cos\psi + y'\sin\psi$ passes through the point (h, k).

But from § 68, $a = -c\cos\psi/p'$ and $b = -c\sin\psi/p'$;

$\therefore\ p'^2 = c^2/(a^2+b^2)$, $\cos\psi = -a/\sqrt{a^2+b^2}$, and $\sin\psi = -b/\sqrt{a^2+b^2}$.

Substituting these values in the above value of d,

$$d = \{(h-x)a + (k-y)b\}/\sqrt{a^2+b^2}$$
$$= \mathbf{(ha + bk + c)/\sqrt{a^2+b^2}}, \quad \text{since } ax+by+c=0.$$

It should be observed that the numerator is the left-hand expression of $ax+by+c=0$, in which x and y are replaced by h and k; hence **the perpendicular distance of the point (h, k) from the line ax + by + c = 0 is given by**

$$\mathbf{d = \frac{ah + bk + c}{\sqrt{a^2+b^2}}.} \quad \ldots\ldots\ldots\ldots\ldots\mathbf{(66)}$$

Multiplying the first of the given equations by 4 and the second by 5, these become

$$24x + 7y + 10 = 0,$$
$$24x + 7y + 55 = 0,$$

from which it is obvious that the lines are parallel. Now the distance between them will be equal to the difference in the lengths of the perpendiculars to them from the origin.

Length of perpendicular from (0, 0) to $24x + 7y + 10 = 0$ is, by (66),

$$10/\sqrt{24^2 + 7^2} = 10/25 = 0{\cdot}4.$$

Length of perpendicular from (0, 0) to $24x + 7y + 55 = 0$ is

$$55/25 = 2{\cdot}2.$$

$\therefore$ Distance between the lines $= 2{\cdot}2 - 0{\cdot}4 =$ **1·8 units.**

69. Two Straight Lines. Let $hx + ky + l = 0$, $px + qy + s = 0$ be two straight lines, then the equation

$$(hx + ky + l)(px + qy + s) = 0$$

or $$phx^2 + (hq + kp)xy + kqy^2 + (lp + hs)x + (lq + ks)y + ls = 0$$

will represent both these straight lines. This quadratic is a particular form of the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

which will be denoted by $F(x, y) = 0$ for brevity.

It is evident, therefore, that $F(x, y) = 0$ will only represent two straight lines when the function $F(x, y)$ is resolvable into real rational factors. The next example shews the condition that must be satisfied to render this possible.

Ex. 4. *Find the condition that*

$$ax^2 + 2hxy + by^2 + 2gx^2 + 2fy + c = 0$$

may represent two straight lines.

What additional condition is required in order that the lines may be parallel?

Find the distance between the pair of lines represented by

$$x^2 + 2\sqrt{3} \,.\, xy + 3y^2 - 3x - 3\sqrt{3}y - 4 = 0.$$

(L.U., Sc.)

Arrange the given equation as a quadratic in y,

$$by^2 + 2(hx + f)y + (ax^2 + 2gx + c) = 0;$$

hence, by solving,

$$by = -(hx + f) \pm \sqrt{(hx + f)^2 - b(ax^2 + 2gx + c)}.$$

Now, if the given equation represents two straight lines, $F(x, y)$ must be resolvable into two rational factors, so that the expression under the square root sign, *i.e.*

$$(hx+f)^2 - b(ax^2+2gx+c)$$

or

$$(h^2-ab)x^2+2(hf-bg)x+f^2-bc,$$

must be a perfect square. The condition for this is

$$(hf-bg)^2=(f^2-bc)(h^2-ab)$$

or

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

This left-hand expression is called the **discriminant** of the equation $F(x, y)=0$, and is generally denoted by the symbol Δ. It may be expressed in a convenient determinant form given below.

Hence **the general equation $\mathbf{F(x, y)=0}$ will represent two straight lines when the discriminant vanishes,** *i.e.*

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \equiv abc+2fgh-af^2-bg^2-ch^2=0. \qquad \text{(67a)}$$

To deduce the conditions that must be satisfied in addition, when the lines are parallel, divide $F(x, y)=0$ throughout by b, which is assumed to be greater than zero, and write H, A, G, F, C for h/b, a/b, g/b, f/b, c/b respectively; then if the resulting equation represents two straight lines, these will be of the form $y=mx+n_1$, $y=mx+n_2$, since they are also to be parallel; hence the identity

$$y^2+2Hxy+Ax^2+2Gx+2Fy+C \equiv (y-mx-n_1)(y-mx-n_2)$$
$$=y^2-2mxy+m^2x^2+(n_1+n_2)mx-(n_1+n_2)y+n_1n_2.$$

Hence, on comparing corresponding coefficients,

$$m=-H, \quad m^2=A, \quad (n_1+n_2)m=2G, \quad n_1+n_2=-2F, \quad n_1n_2=C.$$

Eliminating m, n_1, n_2 from the first four of these relations,

$$H^2=A \quad \text{and} \quad FH=G,$$

i.e.

$$h^2=ab \quad \text{and} \quad fh=gb.$$

$\therefore$ **If $\Delta=0$, the lines will be parallel when**

$$\frac{a}{h}=\frac{h}{b}=\frac{g}{f}. \qquad \text{(67b)}$$

The given equation satisfies both conditions (67*a*) and (67*b*).

To resolve the function $F(x, y)$ into factors, it should be observed that when the lines are parallel, the three terms of the second degree must be a perfect square.

Now $$x^2 + 2\sqrt{3}xy + 3y^2 = (x + \sqrt{3}y)^2.$$

$$\therefore \text{ Let } x^2 + 2\sqrt{3}xy + 3y^2 - 3x - 3\sqrt{3}y - 4$$
$$\equiv (x + 3\sqrt{y} + \alpha)(x + \sqrt{3}y + \beta)$$
$$= x^2 + 2\sqrt{3}xy + 3y^2 + (\alpha + \beta)(x + \sqrt{3}y) + \alpha\beta\ ;$$
$$\therefore \alpha + \beta = -3, \quad \alpha\beta = -4, \text{ giving } \alpha = 1, \quad \beta = -4\ ;$$
$$\therefore \text{ the lines are } x + \sqrt{3}y + 1 = 0, \quad x + \sqrt{3}y - 4 = 0.$$

To find their distance apart, take a convenient point on one of the lines, *e.g.* (4, 0) on the latter, then the perpendicular distance from this point to the other line is, by (66),

$$(4 + 1)/\sqrt{1} + 3 = 2{\cdot}5 \text{ units.}$$

70. Conic Sections. Let AB, CD, Fig. 17, be two straight lines intersecting each other at an angle ϕ. Take any two points A, B on opposite sides of O, the point of intersection, and draw AC, BD perpendicular to CD. Now suppose the figure makes a complete revolution about CD, then the solid generated is a double right circular cone, which is called a **complete cone,** of which $AA'B'B$ is the elevation, CD is the **axis,** and AB a **generating line.**

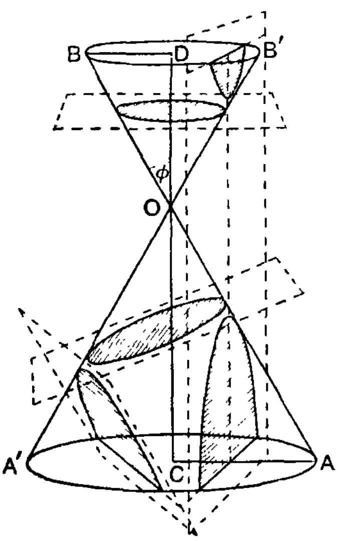

FIG. 17. Sections of a right circular conic.

If this cone be cut by any plane, the curve of intersection is called a **conic.** It will be shewn later (§ 102) that when the cutting plane is

(i) parallel to a generating line, the curve is a **parabola ;**

(ii) inclined to the axis at an angle greater than ϕ, and cuts both generating lines OA, OA' on the same side of the vertex O,

the curve is an **ellipse,** of which the circle is a particular case when the plane is perpendicular to CD;

(iii) parallel to the axis and cuts both parts of the complete cone, the curve is a **hyperbola.**

It should be observed that when the cutting plane parallel to a generator becomes tangential to the curved surface of the cone, the section becomes a pair of coincident straight lines; similarly, when the plane parallel to the axis passes through the axis, the section is a pair of intersecting straight lines.

71. Definition of a Conic as a Locus. It is generally more convenient to define a conic as the locus of a point which moves so that the ratio of its distances from a fixed point and a fixed straight line is constant. The fixed point is called the **focus,** the fixed straight line, the **directrix,** and the constant ratio, the **eccentricity** of the curve, which is denoted by e.

For a parabola, $e=1$; for an ellipse, $e<1$; for a circle, $e=0$, and for a hyperbola, $e>1$.

Ex. 5. *Find the general polar equation of a conic referred to the focus as pole and any line through the focus as the axis. Shew, by transferring to Cartesian coordinates, that the general equation $F(x, y)=0$ represents also a conic, and discuss briefly the reduction of this equation, and the conditions to be satisfied that it may represent the respective curves.*

Let O (Fig. 18) be the focus, HK the directrix, and OX' any

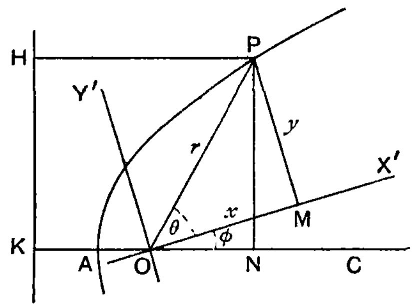

FIG. 18. The conic as a locus.

arbitrary axis through O, making an angle ϕ with the perpendicular to HK through O.

Suppose P be any point on the locus, whose polar coordinates are (r, θ), then, by definition,

$$OP/PH = e,$$

or
$$r = e\,.\,PH = e\,.\,NK = e\,.\,(KO + ON)$$
$$= e\,.\,KO + er\cos(\theta - \phi).$$

Now let the value of r be l, when $\theta - \phi = \frac{\pi}{2}$, then

$$l = e\,.\,KO,$$

so that
$$r = l + er\cos(\theta - \phi),$$

or
$$l/r = 1 - e\cos(\theta - \phi)\ , \qquad (68)$$

which is the general polar equation of a conic.

It may be here remarked that very often the polar equation is given as $l/r = 1 + e\cos(\theta - \phi)$. This is due to the fact that θ is taken as the supplement to $\angle POX'$, so that it is measured by a negative rotation of OP. It is, however, more consistent to retain the positive rotational measurement of θ, and especially so when transformations to Cartesian coordinates have to be made.

Take OX' as the x-axis and OY', the perpendicular to it at O, as the y-axis; then if (x, y) are the Cartesian coordinates of P, from (62),
$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2.$$

Substituting in (68), and writing p, q for $\cos\phi$, $\sin\phi$ respectively, which are constant,

$$l/r = 1 - e\{px/r + qy/r\};$$
$$\therefore\ l = r - e(px + qy),$$

or
$$r = l + e(px + qy).$$

$\therefore$ By squaring,

$$x^2 + y^2 = l^2 + 2el(px + qy) + e^2(px + qy)^2,$$

or
$$x^2(1 - e^2p^2) - 2e^2pqxy + y^2(1 - e^2q^2) - 2elpx - 2elqy - l^2 = 0,$$

which may be written,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

i.e.
$$F(x, y) = 0.$$

The same form of equation would be obtained if P were referred to any other axes parallel respectively to OX', OY'.

Hence the general equation of the second degree, $F(x, y) = 0$, also represents a conic.

To find the simplest forms of the equations for the respective curves, let the rectangular axes be turned through an angle ϕ in the negative direction, so that OX' is coincident with ON and OY' is parallel to the directrix KH; then, in (68), ϕ becomes zero, so that $p=1$ and $q=0$; the above equation thus becomes

$$x^2(1-e^2)+y^2-2elx-l^2=0.$$

Hence, for a **parabola,** $e=1$, and

$$y^2-2lx-l^2=0,$$

or

$$y^2=l(2x+l).$$

If the curve cuts the axis of x in A, then $AO=e\,.\,AK$, *i.e.* $AO=AK$ for $e=1$.

When $y=0$, $x=-\frac{1}{2}l$, from the above equation, so that $OK=l$. Hence, changing the origin from O to A, the equation becomes

$$\mathbf{y^2=2lx}, \quad \text{......(69a)}$$

where l = distance of focus from directrix.

This is the simplest form of the equation to a parabola.

For an **ellipse,** $e<1$, so that $1-e^2$ is positive. Writing a for this coefficient,

$$ax^2+y^2-2elx-l^2=0,$$

i.e.

$$a\left(x-\frac{el}{a}\right)^2+y^2=\frac{l^2}{a}(a+e^2)=\frac{l^2}{a}, \text{ since } a=1-e^2.$$

Change the origin to a point C on the x-axis, where $OC=el/a$, and write

$$a\alpha^2=\beta^2=l^2/a,$$

then the equation becomes

$$\frac{\mathbf{x}^2}{\alpha^2}+\frac{\mathbf{y}^2}{\beta^2}=\mathbf{1}, \quad \text{......(69b)}$$

which is the simplest form of the equation to an ellipse.

This equation may be written, on solving for y, in the form

$$y=\pm\frac{\beta}{\alpha}\sqrt{\alpha^2-x^2},$$

by which it is obvious that for every value of x there are two values of y, equal in magnitude but opposite in sign; hence the curve is symmetrical about the x-axis. Also the maximum value of x is α and the minimum value is $-\alpha$, and in each case the value of y is zero; hence the segment cut off on the x-axis by the curve has a total length of 2α and is called the **major axis**.

Similarly, by writing the equation in the form

$$x = \pm\frac{\alpha}{\beta}\sqrt{\beta^2 - y^2},$$

it is clear that the curve is symmetrical about the y-axis, and the maximum and minimum values of y are $\pm\beta$ respectively, x being zero in each case, so that the segment cut off on the y-axis has a length 2β, which is called the **minor axis**. The origin is therefore symmetrically situated within the curve, and is called the **centre** of the conic.

Further, $\alpha^2 = l^2/a^2$, and $\beta^2 = l^2/a$.

$\therefore$ By division, $\beta^2/\alpha^2 = a = 1 - e^2$;

$$\therefore \mathbf{e^2 = 1 - \beta^2/\alpha^2.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(69c)}$$

This gives the eccentricity in terms of the semi-axes.

Turning now to the **hyperbola**, $e > 1$, so that $1 - e^2$ becomes negative; writing $a = e^2 - 1$, which is positive, the equation

$$x^2(1 - e^2) + y^2 - 2elx - l^2 = 0$$

becomes $$ax^2 - y^2 + 2elx + l^2 = 0,$$

or $$a(x + el/a)^2 - y^2 = l^2(e^2 - a^2)/a = l^2/a;$$

hence, changing the origin to the point $(-el/a, 0)$, the equation takes the form

$$\mathbf{\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1,} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(69d)}$$

where $$a\alpha^2 = \beta^2 = l^2/a.$$

This is the simplest form of the equation of a hyperbola.

By writing this equation in the form

$$y = \pm\frac{\beta}{\alpha}\sqrt{x^2 - \alpha^2},$$

it is obvious, as in the case of the ellipse, that the curve is symmetrical about the axis of x. For any value of x between $\pm\alpha$, $x^2 - \alpha^2$ is negative, and the corresponding value of y is imaginary; hence no part of the curve lies between these values of x, and the axis whose length is 2α lies outside the curve.

Again, writing the equation $x = \pm\frac{\alpha}{\beta}\sqrt{y^2 + \beta^2}$, it is seen that the curve is symmetrical about the y-axis, and that there are no limits to the value of y; hence, the curve consists of two branches each infinite in extent, with the origin equidistant from the two vertices. This point is thus symmetrically situated with

respect to the two branches of the curve, and is therefore called the **centre**.

Proceeding as in the case of the ellipse,

$$a\alpha^2 = \beta^2 = l^2/a,$$

from which $$\mathbf{e^2 = 1 + \beta^2/\alpha^2}, \quad \text{.................................(69e)}$$

thus giving the eccentricity of the curve.

The general reduction of the general equation $F(x, y) = 0$ may be effected as follows.

Let (x', y') be the coordinates of P (Fig. 18) referred to OC, and the perpendicular to it through O, as x and y axes respectively; then

$$x' = ON = r \cos(\theta - \phi) = r \cos\theta \,.\, \cos\phi + r \sin\theta \,.\, \sin\phi$$
$$= x \cos\phi + y \sin\phi, \text{ by (62).}$$

Similarly, $y' = NP = r \sin(\theta - \phi) = y \cos\phi - x \sin\phi$.

Solving these equations for x, y, the relations between the two sets of coordinates, when the axes are turned through an angle ϕ, are

$$\mathbf{x = x' \cos\phi - y' \sin\phi, \qquad y = x' \sin\phi + y' \cos\phi}. \quad \text{............(70)}$$

Now, from the above analysis, it is evident that a value of ϕ may be found which will make the coefficient of xy vanish; thus substituting the above expressions for x in $F(x, y) = 0$, the coefficient of xy is

$$2(b - a) \sin\phi \cos\phi + 2h(\cos^2\phi - \sin^2\phi).$$

or $$(b - a) \sin 2\phi + 2h \cos 2\phi,$$

and this will vanish if

$$\tan 2\phi = 2h/(a - b), \quad \text{or} \quad \phi = \tfrac{1}{2} \tan^{-1}\frac{2h}{a - b}.$$

Hence, ϕ may be determined, and $F(x, y) = 0$ may now be written

$$a_1x^2 + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

or $$a_1\left(x + \frac{g_1}{a_1}\right)^2 + b_1\left(y + \frac{f_1}{b_1}\right)^2 = \frac{g_1^2}{a_1} + \frac{f_1^2}{b_1} - c_1;$$

and, changing the origin to the point $(-g_1/a_1, -f_1/b_1)$, and writing k for $g_1^2/a_1 + f_1^2/b_1 - c_1$, the equation reduces to

$$a_1x^2 + b_1y^2 = k.$$

If $k = 0$, the equation will represent two straight lines; if k is not zero, then $$a_1x^2/k + b_1y^2/k = 1,$$

and the equation will represent an ellipse if k/a_1, k/b_1 are both positive, and a hyperbola if k/b_1 is negative.

Finally, if a_1 is zero, the former equation may be written

$$\left(y+\frac{f_1}{b_1}\right)^2 = -\frac{2g_1}{b_1}\left(x-\frac{f_1{}^2}{2b_1g_1}-\frac{c_1}{2g_1}\right),$$

which, on changing the origin to the point $\left(-\frac{f_1}{b_1}, \frac{f_1{}^2}{2b_1g_1}+\frac{c_1}{2g_1}\right)$, assuming g_1 is not zero, becomes

$$y^2 = 2kx,$$

where $k = -g_1/b_1$. This, therefore, is a parabola.

Ex. 6. *Find the curve represented by the equation*

$$57x^2 - 48xy + 43y^2 - 18x - 124y + 58 = 0.$$

In the practical reduction of a given equation with numerical coefficients, it is better to remove first the terms in x and y by changing the origin, and then turn the axes through a certain angle to remove the term in xy.

Thus in the given equation, take (ξ, η) as a new origin, then

$$57(x+\xi)^2 - 48(x+\xi)(y+\eta) + 43(y+\eta)^2 - 18(x+\xi) - 124(y+\eta) + 58 = 0,$$

i.e. $$57x^2 - 48xy + 43y^2 + (114\xi - 48\eta - 18)x + (86\eta - 48\xi - 124)y + 57\xi^2 - 48\xi\eta + 43\eta^2 - 18\xi - 14\eta + 58 = 0.$$

Now choose ξ, η so that

$$114\xi - 48\eta - 18 = 0, \quad \text{and} \quad 86\eta - 48\xi - 124 = 0.$$

The solution of these equations gives

$$\xi = 1, \quad \eta = 2.$$

Substituting these values in the above equation,

$$57x^2 - 48xy + 43y^2 = 75.$$

Now replace x and y by the relations given in (70),

$$57(x \cos\phi - y \sin\phi)^2 - 48(x\cos\phi - y\sin\phi)(x\sin\phi + y\cos\phi) + 43(x\sin\phi + y\cos\phi)^2 = 75.$$

When the coefficient of xy is zero,

$$-28 \cos\phi \,.\, \sin\phi = 48(\cos^2\phi - \sin^2\phi),$$

or $$-14 \sin 2\phi = 48 \cos 2\phi;$$

$$\therefore \tan 2\phi = -\tfrac{24}{7},$$

so that $$\cos 2\phi = -\tfrac{7}{25} \quad \text{and} \quad \sin 2\phi = \tfrac{24}{25};$$

$$\therefore \cos^2\phi = \tfrac{1}{2}(1+\cos 2\phi) = \tfrac{9}{25}, \ \sin^2\phi = \tfrac{16}{25}, \text{ and } \sin\phi\cos\phi = \tfrac{12}{25}.$$

Substituting these values in the above equation, it becomes

$$25x^2+75y^2=75,$$

or

$$\mathbf{x^2/3+y^2=1.}$$

which is an ellipse whose semi-axes are $\sqrt{3}$ and 1.

72. The Parabola. It has already been shewn that the simplest form of the equation to a parabola is $y^2=2lx$, where l is the distance of the focus from the directrix. Since, however, $e=1$ for this conic, the vertex of the curve bisects the distance between the focus and directrix, it is usually more convenient to denote this length by $2a$. Then, as the vertex is the origin, the distance between the origin and the focus is a, so that $l=2a$, and the **equation becomes $\mathbf{y^2=4ax}$.** This is the standard form of the equation, and is usually sufficient for most practical problems.

It should be observed that when $x=a$,

$$y^2=4a^2, \text{ giving } y=\pm 2a=\pm l,$$

so that $2l$ is the length of the double ordinate through the focus. This focal chord is called the **latus rectum**, so that l denotes the length of the semi-latus rectum.

Ex. 7. *$OABCD$... is a polygon with its vertices on a series of equidistant vertical lines; if, when lines are drawn through O parallel to AB, BC, CD, ... to meet the vertical through A in B_1, C_1, D_1, ..., $AB_1=B_1C_1=C_1D_1=\dots$, prove that the vertices of the polygon lie on a parabola.* (L.U.)

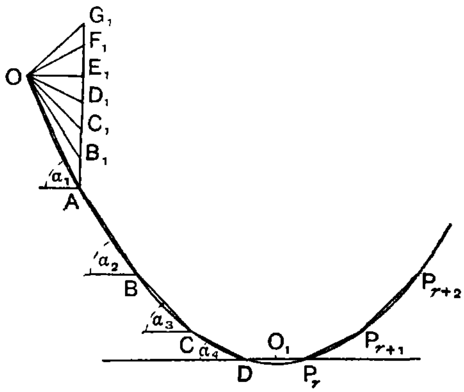

FIG. 19. The parabola.

The polygon is shewn in Fig. 19. Suppose the rth side is horizontal, and let α_1, α_2, α_3, ... be the inclinations of the sides OA, AB, BC, ... to the horizontal; then, since the horizontal projections of the sides are equal,

$$OA\cos\alpha_1=AB\cos\alpha_2=BC\cos\alpha_3=\dots=DP_r\,.$$

Let $DP_r = a$, and each of the equal segments, AB_1, B_1C_1, ... $= b$.

Take O_1, the mid-point of DP_r as origin, and DP_r produced as the axis of x; then the coordinates of P_r are $(\frac{1}{2}a, 0)$; the coordinates of P_{r+1} are $(\frac{3}{2}a, b)$; the coordinates of P_{r+1} are

$$(\tfrac{5}{2}a, b+2b) = (\tfrac{5}{2}a, 3b).$$

Hence, if (x, y) are the coordinates of the nth vertex after P_r,

$$x = \tfrac{1}{2}a + na = \tfrac{1}{2}(2n+1)a$$

and $$y = (1+2+3+\ldots+n)b = \tfrac{1}{2}n(n+1)b.$$

Eliminating n from these equations,

$$x^2 = \frac{2a^2}{b}\left(y + \frac{b}{8}\right),$$

which represents a parabola whose axis is the y-axis, and whose vertex lies at a distance $b/8$ below O_1.

Suppose the number of sides of the polygon be increased indefinitely, so that it becomes a continuous curve. Then since the horizontal projections of the sides are equal, this curve will represent the form assumed by a chain suspended from two points when its weight is uniformly distributed horizontally.

That the curve is still a parabola may be proved independently as follows.

Take any position of the chain KP_{r+2}, beginning with the lowest point K, which may conveniently be chosen as a new origin. Let w = weight of chain per unit horizontal length, T_0 = the tension at K, and T = tension along the tangent at P_{r+2}. Suppose the slope at P_{r+2} be θ, and the coordinates of this point be (x, y), then resolving horizontally and vertically,

$$T\cos\theta = T_0 \quad \text{and} \quad T\sin\theta = wx.$$

$\therefore$ By division, $$\tan\theta = wx/T_0,$$

or replacing $\tan\theta$ by $\frac{dy}{dx}$, $$\frac{dy}{dx} = \frac{wx}{T_0},$$

which on integration gives

$$y = \frac{w}{2T_0}\cdot x^2, \quad \text{or} \quad x^2 = ky, \quad \text{where} \quad k = 2T_0/w.$$

The constant of integration is zero, since at K, $x=0$ and $y=0$. Hence the curve is a parabola.

This represents approximately the case of a suspension bridge with a uniformly loaded horizontal roadway, the weight of the suspending chains being neglected.

Ex. 8. *A particle is projected on a horizontal plane with velocity V, and its direction makes an angle θ with the horizontal; shew that, when subject only to the acceleration of gravity, its path is a parabola, and find* (i) *its greatest elevation,* (ii) *its time of flight, and* (iii) *its horizontal range.*

Deduce the value of θ which will give maximum range.

Prove also that the velocity acquired by the particle at any point in its path is equal in magnitude to that which it would acquire in falling to that point from the directrix.

Let P (Fig. 20) be any point on the path after t seconds from the time of projection. Take O as origin, then $ON=x$ and

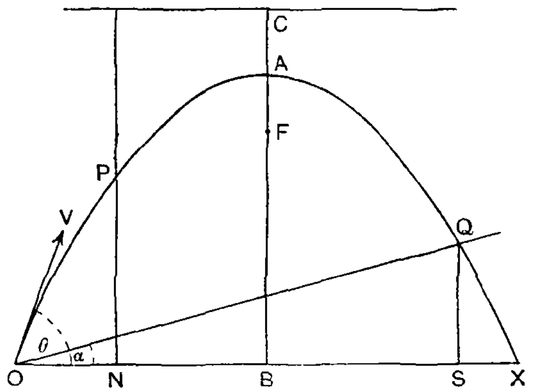

FIG. 20. Path of a projectile.

$NP=y$. Let u, v, be the horizontal and vertical components of the initial velocity V, then $u=V\cos\theta$ and $v=V\sin\theta$.

Now the only acceleration of the particle is directed vertically downwards, and is therefore $-g$; hence, the equations of motion are

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-g.$$

Integrating each of these with respect to t,

$$\frac{dx}{dt}=A_1, \quad \frac{dy}{dt}=-gt+A_2.$$

When $t=0$, $\dfrac{dx}{dt}=u=V\cos\theta$ and $\dfrac{dy}{dt}=v=V\sin\theta$.

Hence, $A_1=V\cos\theta$ and $A_2=V\sin\theta$, so that

$$\frac{dx}{dt}=V\cos\theta, \quad \frac{dy}{dt}=V\sin\theta-gt.$$

Integrating again,

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \tfrac{1}{2}gt^2,$$

the constants of integration being zero, since $x=0$ and $y=0$ when $t=0$.

The equation of the path, *i.e.* the relation between x and y, may now be found by eliminating t from these equations ; this gives

$$y = x \,.\, \tan\theta - \frac{1}{2}\frac{g}{V^2} \,.\, x^2\sec^2\theta,$$

or
$$\left(x - \frac{V^2}{2g} \,.\, \sin 2\theta\right)^2 = -\frac{2V^2}{g} \,.\, \cos^2\theta\left(y - \frac{V^2}{2g}\sin^2\theta\right).$$

Changing the origin to the vertex A, *i.e.* the point

$$(V^2\sin 2\theta/2g, \ V^2\sin^2\theta/2g),$$

and writing l for $-V^2\cos^2\theta/g$, the equation reduces to

$$\mathbf{x}^2 = 2\mathbf{l}\mathbf{y},$$

which represents a parabola whose vertex is the highest point in the flight and whose latus rectum is equal in magnitude to

$$2V^2\cos^2\theta/g.$$

Its focus is at a point F distant $\frac{1}{2}l$, or $V^2\cos^2\theta/(2g)$ below the vertex, the directrix being the perpendicular at C to BA produced, where $AC = FA = V^2\cos^2\theta/(2g)$.

Let $h = BA$ = greatest height attained by the particle during its flight, then it is evident from the equation to the parabola that

$$h = V^2\sin^2\theta/2g.$$

If T be the line of flight, then, since the curve is symmetrical about its axis AB, the particle reaches A in time $T/2$; and at this point the vertical velocity is zero ;

$$\therefore \ V\sin\theta - \tfrac{1}{2}gT = 0,$$

or
$$T = 2V\sin\theta/g.$$

Further, let R = the horizontal range OX, then $OX = 2OB$ and $OB = \frac{1}{2}VT\cos\theta = V^2\sin\theta \,.\, \cos\theta/g$, on putting in the value of T just found.

Hence, $$R = 2V^2\sin\theta\cos\theta/g = V^2\sin 2\theta/g.$$

This will be a maximum when $\sin 2\theta$ is a maximum, V being constant, *i.e.* R is greatest when $\sin 2\theta = 1$, or $\theta = \pi/4$.

To find the velocity at any point P in the path,

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = V^2 \cos^2\theta + (V \sin\theta - gt)^2$$

$$= V^2 - 2g(Vt \sin\theta - \tfrac{1}{2}gt^2)$$

$$= V^2 - 2gy;$$

$$\therefore \frac{ds}{dt} = \sqrt{V^2 - 2gy}.$$

Had the particle fallen vertically from the directrix to P, its velocity, v_1, would be given by

$$v_1^2 = 2g \,.\, PH = 2g(BA + AC - y)$$

$$= V^2(\sin^2\theta + \cos^2\theta) - 2gy$$

$$= V^2 - 2gy = \left(\frac{ds}{dt}\right)^2.$$

Hence the velocity at any point P on the parabola is equal in magnitude to that acquired by a particle falling vertically to that point from the directrix.

Ex. 9. *A particle is projected up a plane inclined to the horizontal at an angle α, with a velocity V whose direction makes an angle θ with the horizontal; shew that the range on the plane is*

$$\frac{2V^2 \cos\theta \,.\, \sin(\theta - \alpha)}{g \cos^2\alpha},$$

and that this will be a maximum when the direction of projection bisects the angle between the vertical and the inclined plane.

Let $OQ = r$ be the range on the inclined plane (Fig. 20), and let (x, y) be the coordinates of Q, referred to AB as y-axis and A as origin.

Then $\quad x = BS = OS - OB = r \cos\alpha - V^2 \sin\theta \,.\, \cos\theta/g,$

$$y = h - SQ = V^2 \sin^2\theta/(2g) - r \sin\alpha.$$

Substituting in the equation

$$x^2 = 2V^2 \cos^2\theta \,.\, y/g,$$

taking y downwards as positive,

$$(r \cos\alpha - V^2 \sin\theta \,.\, \cos\theta/g)^2 = 2V^2 \cos^2\theta\{V^2 \sin^2\theta/(2g) - r \sin\alpha\}/g.$$

Expanding and dividing throughout by r, which is not zero,

$$r\cos^2\alpha = 2V^2\cos\theta(\cos\alpha\,.\,\sin\theta - \cos\theta\sin\alpha)/g$$
$$= 2V^2\cos\theta\,.\,\sin(\theta-\alpha)/g\,;$$
$$\therefore\ r = \frac{2V^2\cos\theta\,.\,\sin(\theta-\alpha)}{g\cos^2\alpha}.$$

Since V and α are constants, r will be a maximum when

$$2\cos\theta\,.\,\sin(\theta-\alpha)$$

is greatest; but, by (14b),

$$2\cos\theta\,.\,\sin(\theta-\alpha) = \sin(2\theta-\alpha) - \sin\alpha,$$

and this will be greatest when $\sin(2\theta-\alpha)=1$, or $\theta = \frac{1}{2}\left(\frac{\pi}{2}+\alpha\right)$.

Let β = angle between the vertical at O and the direction OV, of V, then

$$\beta = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \frac{\pi}{4} - \frac{\alpha}{2} = \frac{\pi}{4} - \frac{\alpha}{2}.$$

But $$\angle QOV = \theta - \alpha = \frac{\pi}{4} + \frac{\alpha}{2} - \alpha = \frac{\pi}{4} - \frac{\alpha}{2} = \beta.$$

Hence, for maximum range, the direction of projection bisects the angle between the vertical and the inclined plane.

73. The Central Conics. The respective conditions necessary for the general equation, $F(x, y)=0$, to represent the various conics have already been briefly investigated in Ex. 5. These will, however, now be deduced in a more convenient form for practical application by considering the coordinates of the centre of each conic, *i.e.* the point which is symmetrically situated with respect to a curve. It will be evident from this fact, together with the analysis of Ex. 5, that **the equation of a conic, referred to its centre as origin, will contain no terms of the first degree in x and y.**

Ex. 10. *Write down the conditions that the general equation*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should represent (1) *two straight lines*, (2) *a circle*, (3) *a parabola*, (4) *an ellipse*, (5) *a hyperbola*. (L.U.)

Although the question asks that the respective conditions should be written down, a complete investigation will here be made.

Change the origin to the point (ξ, η), then replacing x, y by $x+\xi$, $y+\eta$ respectively, the equation becomes

$$ax^2+2hxy+by^2+2x(a\xi+h\eta+g)+2y(h\xi+b\eta+f)+F(\xi, \eta)=0.$$

Now, if (ξ, η) be the centre of the curve, there will be no terms of the first degree in the equation. Hence, the conditions for (ξ, η) to be the centre are

$$a\xi+h\eta+g=0, \quad h\xi+b\eta+f=0.$$

Solving these equations,

$$\boldsymbol{\xi=(hf-bg)/(ab-h^2), \quad \eta=(gh-af)/(ab-h^2),} \quad \ldots\ldots\ldots\ldots\ldots\mathbf{(71)}$$

which are the coordinates of the centre.

The values of ξ and η will remain finite as long as ab is not equal to h^2, but when $ab=h^2$, there is no finite centre, and the terms of the second degree in $F(x, y)=0$ form a perfect square, so that the general equation may be written

$$(lx+my)^2+2gx+2fy+c=0,$$

or $$(lx+my+\lambda)^2=2x(\lambda l-g)+2y(\lambda m-f)+\lambda^2-c,$$

i.e. $$\left(\frac{lx+my+\lambda}{\sqrt{l^2+m^2}}\right)^2=2k\,.\,\frac{2x(\lambda l-g)+2y(\lambda m-f)+\lambda^2-c}{2\sqrt{(\lambda l-g)^2+(\lambda m-f)^2}},$$

where $$k^2=\{(\lambda l-g)^2+(\lambda m-f)^2\}/(l^2+m^2)$$

and λ is an arbitrary constant; hence, by (66), this equation may be interpreted as

$$\{\text{length of perpendicular from } (x, y) \text{ to line } lx+my+\lambda=0\}^2$$
$$=2k\{\text{length of perpendicular from } (x, y) \text{ to line}$$
$$2x(\lambda l-g)+2y(\lambda m-f)+\lambda^2-c=0\}.$$

If now λ be chosen so that the lines are perpendicular to each other, *i.e.* if $-m/l=(\lambda l-g)/(\lambda m-g)$, by (65*b*), or

$$\lambda=(lg+mf)/(l^2+m^2),$$

and, in this case, if Y and X denote the respective perpendicular lengths, the equation becomes

$$Y^2=2kX,$$

which represents a parabola.

Hence, **the general equation $\mathbf{F(x, y)=0}$ represents a parabola when**

$$\mathbf{ab=h^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(72a)}$$

It should be noted that, if in addition, $fh = gb$, the **parabola** degenerates into two parallel straight lines, by (67*b*).

The **parabola** is obviously a non-central conic, since the coordinates of its centre are infinite.

When ab is not equal to h^2, then the equation of the curve referred to $(-\xi, -\eta)$ as origin becomes

$$ax^2 + 2hxy + by^2 + f(\xi, \eta) = 0.$$

Now .
$$\begin{aligned} F(\xi, \eta) &= a\xi^2 + 2h\xi\eta + b\eta^2 + 2g\xi + 2f\eta + c \\ &= \xi(a\xi + h\eta + g) + \eta(h\xi + b\eta + f) + g\xi + f\eta + c \\ &= g\xi + f\eta + c, \text{ since } a\xi + h\eta + g = h\xi + b\eta + f = 0, \\ &= \{g(hf - bg) + f(gh - af) + c(ab - h^2)\}/(ab - h^2) \\ &= (abc + 2fgh - af^2 - bg^2 - ch^2)/(ab - h^2) \\ &= \Delta/(ab - h^2), \end{aligned}$$

so that the equation becomes

$$ax^2 + 2hxy + by^2 = \Delta/(ab - h^2).$$

Hence, when $\Delta = 0$,

$$ax^2 + 2hxy + by^2 = 0 \quad \text{or} \quad (\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y) = 0,$$

which represents two straight lines through the origin, so that **the equation $\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$ represents two straight lines when $\Delta = 0$.**

This is condition (67*a*) found another way in Ex. 4.

When Δ is not zero, the equation may be written

$$Ax^2 + 2Hxy + By^2 = 1,$$

where
$$A/a = H/h = B/b = (h^2 - ab)/\Delta.$$

This equation thus represents the **Central Conics.** To distinguish them, turn the axes through an angle ϕ to remove the term in xy; thus, from (70),

$$A(x' \cos\phi - y' \sin\phi)^2 + 2H(x' \cos\phi - y' \sin\phi)(x' \sin\phi + y' \cos\phi) + B(x' \sin\phi + y' \cos\phi)^2 = 1.$$

Choosing ϕ so that the coefficient of xy vanishes, *i.e.*

$$\tan 2\phi = 2H/(A - B),$$

then
$$\sin 2\phi = 2H/R \quad \text{and} \quad \cos 2\phi = (A - B)/R,$$

where $R^2 = 4H^2 + (A - B)^2$, and the equation, after simplification, becomes, neglecting dashes,

$$\tfrac{1}{2}(A + B + R)x^2 + \tfrac{1}{2}(A + B - R)y^2 = 1.$$

From (69*b*), this equation will be an ellipse if

$$\alpha^2 = 2/(A+B+R) \quad \text{and} \quad \beta^2 = 2/(A+B-R).$$

and these values are real.

But from (69*c*), the eccentricity is given by

$$e^2 = 1 - \beta^2/\alpha^2 = (\alpha^2 - \beta^2)/\alpha^2 = 2R/(R-A-B),$$

and since $e < 1$, $\quad 2R < R - A - B,$

or $\quad R < -(A+B),$

i.e. $\quad R^2 < (A+B)^2,$

or $\quad 4H^2 - (A-B)^2 < (A+B)^2,$

i.e. $\quad H^2 < AB.$

Now when $\alpha = \beta$, the ellipse degenerates into a circle, so that in this case

$$A + B + R = A + B - R,$$

or

$$R = 0;$$

$$\therefore \; 4H^2 + (A-B)^2 = 0,$$

and since both squares are positive, their sum can only be zero when each separately vanishes, *i.e.*

$$H = 0 \quad \text{and} \quad A = B.$$

Hence the equation for a central conic will only represent a circle when the coefficients of x^2 and y^2 are equal and there is no term in xy.

Finally, for a hyperbola, comparing with (69*d*),

$$\alpha^2 = 2/(A+B+R), \quad \beta^2 = -2/(A+B-R),$$

so that by (69*c*) the eccentricity is given by

$$e^2 = 1 + \beta^2/\alpha^2 = 2R/(R-A-B),$$

and for a hyperbola, $e > 1$;

$$\therefore \; 2R > R - A - B,$$

from which $\quad H^2 > AB.$

Hence, summing up these results,

The equation $Ax^2 + 2Hxy + By^2 = 1$ represents a Central Conic, whose centre is the origin, and the curve is

$$\left.\begin{array}{ll} \textbf{(i) an Ellipse if} & H^2 < AB, \\ \textbf{(ii) a Circle if} & H = 0 \text{ and } A = B, \\ \textbf{(iii) a Hyperbola if} & H^2 > AB. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(72b)}$$

Since $A/a = H/h = B/b = (h^2 - ab)/\Delta$, and neither $h^2 - ab$ nor Δ is zero, these results, applied to the general equation $F(x, y) = 0$, lead to the following analytical tests for the respective curves :

The general equation

$$\mathbf{F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$$

represents

$$\left.\begin{array}{ll} \textbf{(i) an Ellipse if} & \mathbf{h^2 < ab,} \\ \textbf{(ii) a Circle if} & \mathbf{h = 0 \text{ and } a = b,} \\ \textbf{(iii) a Hyperbola if} & \mathbf{h^2 > ab.} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(72c)}$$

Ex. 11. *The equation of motion of a particle, whose polar coordinates at any time t are* (r, θ), *moving under a central acceleration inversely proportional to r, is*

$$c^2u^3\left(u + \frac{d^2u}{d\theta^2}\right) = \mu,$$

where $u = 1/r$ *and* c, μ *are constants. Shew that its path is an ellipse or a hyperbola according as* μ *is positive or negative.*

From the given equation,

$$\frac{d^2u}{d\theta^2} = \frac{\mu}{c^2u^3} - u.$$

Multiplying by $2 \cdot \dfrac{du}{d\theta}$, $\quad 2 \cdot \dfrac{du}{d\theta} \cdot \dfrac{d^2u}{d\theta^2} = 2\left(\dfrac{\mu}{c^2u^3} - u\right)\dfrac{du}{d\theta}.$

Integrating, $\quad \left(\dfrac{du}{d\theta}\right)^2 = -\dfrac{\mu}{c^2u^2} - u^2 + A = (Au^2 - u^4 - \mu/c^2)/u^2$

$$= \{(A^2/4 - \mu/c^2) - (u^2 - A/2)^2\}/u^2$$
$$= \{k^2 - (u^2 - A/2)^2\}/u^2,$$

where $\qquad k^2 = A^2/4 - \mu/c^2.$

Separating the variables,

$$d\theta = \frac{u \,.\, du}{\sqrt{k^2 - (u^2 - A/2)^2}}.$$

Put $u^2 - A/2 = v$, then

$$2d\theta = \frac{dv}{\sqrt{k^2 - v^2}},$$

which, on integration, gives

$$2\theta = \sin^{-1}\frac{v}{k} + \beta \quad \text{or} \quad v = k \sin(2\theta - \beta).$$

Putting in the value of v, and writing p for $\sin\beta$ and q for $\cos\beta$,

$$u^2 - A/2 = 2kq \sin\theta \cos\theta - kp\,(\cos^2\theta - \sin^2\theta).$$

Transferring to Cartesian coordinates by (62), and putting $1/r$ for u,

$$1/r^2 - A/2 = 2kqxy/r^2 - kp(x^2 - y^2)/r^2\,;$$

$\therefore$ the equation of the curve becomes

$$x^2(A/2 - kp) + 2kqxy + y^2(A/2 + kp) = 1.$$

Hence, by (72b), this will represent an ellipse or hyperbola according as

$$k^2q^2 - (A/2 - kp)(A/2 + kp) \text{ is } < \text{ or } > 0.$$

Now the expression $= k^2q^2 - A^2/4 + k^2p$

$= k^2 - A^2/4$, since $p^2 + q^2 = 1$,

$= A^2/4 - \mu/c^2 - A/^24$, on putting in the value of k.

$= -\mu/c^2$,

so that, if μ is positive, $-\mu/c^2 < 0$, and if μ is negative, $-\mu/c^2 > 0$.

$\therefore$ the path of the particle will be an ellipse if μ is positive and a hyperbola if μ is negative.

74. Some Properties of the Ellipse. The chief tangent properties of the conics are dealt with in the next chapter, but the ellipse is so important in practice that its equation will now be derived independently from the locus definition, and some of the chief characteristics of the curve developed which do not depend upon the properties of tangents.

Ex. 12. *Find the equation to an ellipse referred to its centre as origin and its diameters* $2a$, $2b$, *as axes.*

Prove that

(*a*) *the focal distances of a point* (x', y') *on the curve are* $a \pm ex'$; (L.U.)

(*b*) *the sum of the focal distances of any point on the curve is constant ;* (L.U.)

(*c*) *the sum of the eccentric angles at the extremities of any chord of a system of parallel chords is constant, and that the locus of the mid-points of these chords is a straight line ;* (L.U.)

(*d*) *the difference of the eccentric angles of two points which are the extremities of two conjugate diameters is a right angle ;*
(*e*) *the sum of the squares of a pair of conjugate diameters is constant, and that their product varies as the cosecant of the angle between them.* (L.U.)

Let AA_1 (Fig. 21) be the x-axis, KH the directrix, F, F_1 the foci, and A, A_1 the points where the curve crosses the axis of x. Bisect AA_1 at O, then $A_1O = OA = a$.

Now, since A is a point on the curve, $FA = e \,.\, AK$.

Similarly, $$A_1F = e \,.\, A_1K,$$

where e is the eccentricity, then

$$FA + A_1F = e(AK + A_1K) ;$$

$$\therefore\ 2a = e(OK - a + a + OK) = 2e \,.\, OK,$$

so that $$OK = a/e.$$

Also $$A_1F - FA = e(A_1K - AK) ;$$

$$\therefore\ a + OF - a + OF = e(a + OK - OK + a) = 2ae ;$$

$$\therefore\ OF = ae.$$

Hence, and

$$\left.\begin{array}{l}\textbf{Distance of directrix from centre} = \mathbf{a/e}, \\ \quad\text{,,} \qquad \text{,,}\ \textbf{a focus} \quad \text{,,} \qquad \text{,,} \quad = \mathbf{ae}.\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots(\mathbf{73a})$$

Suppose $P(x, y)$ be any point on the curve, then $ON = x$, $NP = y$, and by definition

$$FP = e \,.\, PH = e \,.\, NK = e(OK - ON) = a - ex ;$$

$$\therefore\ FP^2 = (a - ex)^2.$$

But $$FP^2 = y^2 + NF^2 = y^2 + (OF - x)^2 = y^2 + (ae - x)^2 ;$$

$$\therefore\ y^2 = (a - ex)^2 - (ae - x)^2 = (1 - e^2)(a^2 - x^2).$$

Let $y = b$ when $x = 0$, then $b^2 = a^2(1 - e^2)$, giving $e^2 = 1 - b^2/a^2$, the relation (69*c*) already found by another method in Ex. 5.

Substituting b^2/a^2 for $1 - e^2$, the equation finally becomes

$$x^2/a^2 + y^2/b^2 = 1,$$

which corresponds with (69*b*).

(*a*) The focal distances of P are FP and F_1P ; and from the above analysis, $$FP = a - ex.$$

Similarly, since the curve is symmetrical about OB,

$$F_1P = e \,.\, NK_1 = e \,.\, (OK_1 + x) = a + ex.$$

Hence, the focal distances of any point (x', y') on the curve are

$$\mathbf{a \pm ex'}. \quad \text{.....................................(73b)}$$

(*b*) The sum of these focal distances

$$= a - ex + a + ex = 2a = \text{length of major axis.}$$

This property gives a simple mechanical construction of the curve. If a loop of thread whose perimeter is $F_1F + FP + PF_1 = 2a(e+1)$, be threaded over two pins firmly fixed at F, F_1, then by moving a pencil vertically at P, keeping the thread always taut, the curve will be traced, since $F_1P + FP$ remains constant.

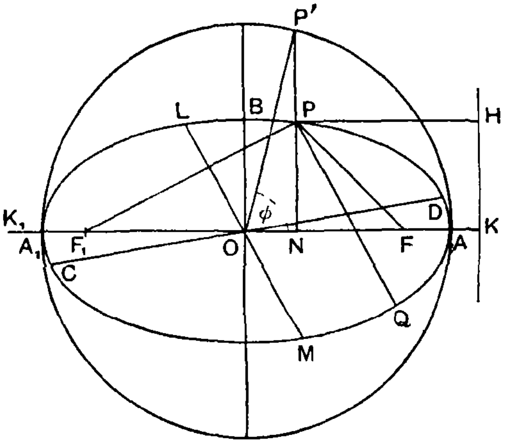

FIG. 21. Properties of the ellipse.

(*c*) Let a circle be described on A_1A as diameter, then this circle is known as the **Auxiliary Circle,** and its equation is clearly $x^2 + y^2 = a^2$.

Produce NP to meet the circle in P'; join OP', and let the angle $NOP' = \phi$. This angle is called the **Eccentric Angle of the point P.** Now $x = ON = OP' \cos\phi = a\cos\phi$, and since P lies on the ellipse,

$$x^2/a^2 + y^2/b^2 = 1.$$

By substitution for x,

$$\cos^2\phi + y^2/b^2 = 1, \quad \text{or} \quad y^2/b^2 = 1 - \cos^2\phi = \sin^2\phi\,;$$

$$\therefore\ y = NP = b\sin\phi.$$

Hence the coordinates of any point P on the ellipse may be taken as

$$\mathbf{a\cos\phi \quad and \quad b\sin\phi}, \quad \text{...........................(73c)}$$

where ϕ is the eccentric angle of P.

Let $y=mx+c$ be any chord PQ, and let the coordinates and eccentric angles of P, Q be (x_1, y_1) ϕ_1 and (x_2, y_2) ϕ_2 respectively; then

$$x_1=a\cos\phi_1, \quad y_1=b\sin\phi_1, \quad x_2=a\cos\phi_2, \quad y_2=b\sin\phi_2,$$

and the points P, Q are on both the chord and the ellipse;

$$\therefore\ b\sin\phi_1=ma\cos\phi_1+c,$$
$$b\sin\phi_2=ma\cos\phi_2+c.$$

Hence, by subtraction to eliminate c,

$$b(\sin\phi_1-\sin\phi_2)=ma(\cos\phi_1-\cos\phi_2);$$

$\therefore$ by (14),

$$b\cos\tfrac{1}{2}(\phi_1+\phi_2)\sin\tfrac{1}{2}(\phi_1-\phi_2)=-ma\sin\tfrac{1}{2}(\phi_1+\phi_2)\sin\tfrac{1}{2}(\phi_1-\phi_2).$$

Divide out by $\sin\frac{1}{2}(\phi_1-\phi_2)$, which is not zero, since P and Q are not coincident points; then

$$b\cos\tfrac{1}{2}(\phi_1+\phi_2)=-ma\sin\tfrac{1}{2}(\phi_1+\phi_2);$$
$$\therefore\ \tan\tfrac{1}{2}(\phi_1+\phi_2)=-b/(ma),$$

or

$$\boldsymbol{\phi_1+\phi_2=2\tan^{-1}\{-b/(ma)\}}, \quad \ldots\ldots\ldots\ldots(73d)$$

which is constant as long as m is constant, *i.e.* as long as the chords are parallel.

Let (x, y) be the coordinates of the middle point of PQ, then

$$2x=x_1+x_2=a(\cos\phi_1+\cos\phi_2)=2a\cos\tfrac{1}{2}(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_1-\phi_2),$$
$$2y=y_1+y_2=b(\sin\phi_1+\sin\phi_2)=2b\sin\tfrac{1}{2}(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_1-\phi_2).$$

$\therefore$ By division,

$$y/x=b/a\,.\tan\tfrac{1}{2}(\phi_1+\phi_2)=-b^2/(a^2m), \text{ by } (73d);$$
$$\therefore\ y=-b^2x/(a^2m),$$

which is a straight line; hence, the locus of the mid-points of a system of parallel chords of an ellipse is a straight line.

Writing the equation in the form $y=m'x$, where $m'm=-b^2/a^2$, and proceeding as before, it is obvious that the locus of the mid-points of the system of chords parallel to $y=m'x$ is

$$y/x=-b^2/(a^2m'),$$

i.e.

$$y=mx.$$

$\therefore$ Chords parallel to $y=mx$ are bisected by $y=m'x$, and chords parallel to $y=m'x$ are bisected by $y=mx$. Pairs of straight lines passing through the centre which possess this property are called **Conjugate Diameters.**

Hence, the equation of conjugate diameters of an ellipse may be written

$$\left.\begin{aligned}&\mathbf{y=mx,\quad y=m'x,}\\ \text{where}\quad &\mathbf{mm'=-b^2/a^2.}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(73e)$$

(*d*) Let LM, CD be two conjugate diameters whose equations are $y=mx$, $y=m'x$, and let θ, ϕ be the eccentric angles of the extremities L and D; then, since

$$y=mx, \quad \therefore \; b\sin\theta = ma\cos\theta,$$

and $$y=m'x, \quad \therefore \; b\sin\phi = m'a\cos\phi.$$

Hence, by multiplication,

$$b^2\sin\theta\sin\phi = mm'a^2\cos\theta\cos\phi$$
$$= -b^2\cos\theta\cos\phi, \text{ from } (73e);$$
$$\therefore \; \sin\theta\sin\phi + \cos\theta\cos\phi = 0,$$

or $$\cos(\theta-\phi)=0;$$

$$\therefore \; \boldsymbol{\theta - \phi = \frac{\pi}{2}}. \quad \text{......(73f)}$$

Thus if L', D' are points on the auxiliary circle corresponding to L, D, the angle $L'OD' = \theta - \phi =$ a right angle.

(*e*) $OD^2 = a^2\cos^2\phi + b^2\sin^2\phi,$

$$OL^2 = a^2\cos^2\theta + b^2\sin^2\theta$$
$$= a^2\cos^2\left(\frac{\pi}{2}+\phi\right) + b^2\sin^2\left(\frac{\pi}{2}+\phi\right), \text{ by } (73f),$$
$$= a^2\sin^2\phi + b^2\cos^2\phi;$$

$$\therefore \; OD^2 + OL^2 = a^2(\cos^2\phi + \sin^2\phi) + b^2(\sin^2\phi + \cos^2\phi) = a^2 + b^2.$$

$\therefore$ **Sum of squares on semi-conjugate diameters**

$= a^2 + b^2$

= sum of squares on semi-axes. ......(73g)

Let the coordinates of D and L be (x_1, y_1) and (x_2, y_2); then since area of triangle $LDO = \frac{1}{2} \cdot LO \cdot OD \cdot \sin\psi$, where $\psi = \angle DOL$, by (63),

$$LO \cdot OD \sin\psi = \begin{vmatrix} x_1 & x_2 & 0 \\ y_1 & y_2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a\cos\phi & a\cos\theta & 0 \\ b\sin\phi & b\sin\theta & 0 \\ 1 & 1 & 1 \end{vmatrix}$$
$$= ab(\sin\theta\cos\phi - \cos\theta\sin\phi)$$
$$= ab\sin(\theta-\phi) = ab, \text{ by } (73f);$$
$$\therefore \; \mathbf{LO \cdot OD = ab \operatorname{cosec} \psi}, \quad \text{......(73h)}$$

i.e. the product of the conjugate diameters is directly proportional to the cosecant of the angle between them.

75. Roulettes and Glisettes. Another group of loci which is of practical importance is that known as Roulettes and Glisettes.

A **Roulette** may be defined as the locus of a point carried by a curve which rolls upon another fixed curve, whilst a **Glisette** is the locus of a point carried by a lamina which is constrained to move so that a curve drawn upon it always remains in contact with two given fixed curves.

Only a few practical examples of these loci will be considered.

Ex. 13. (i) *Obtain the equations of the cycloid in the form*

$$x=a(\theta-\sin\theta), \quad y=a(1-\cos\theta). \qquad \text{(L.U.)}$$

(ii) *A circle rolls on and touches the outside of a fixed circle of radius a ; shew that the path of a point on the circumference of the rolling circle will be given by*

$$\begin{cases} x=a(1+n)\cos\theta-na\cos\dfrac{1+n}{n}\theta, \\ y=a(1+n)\sin\theta-na\sin\dfrac{1+n}{n}\theta, \end{cases}$$

or by

$$\begin{cases} x=a(1-n')\cos\theta'+n'a\cos\dfrac{1-n'}{n'}\theta', \\ y=a(1-n')\sin\theta'-n'a\sin\dfrac{1-n'}{n'}\theta', \end{cases}$$

according as the fixed circle is outside or inside the rolling one, the radii of the latter being na and n'a respectively.

If $n=3$, find n' so that the curves may be identical and express θ' in terms of θ. (L.U.)

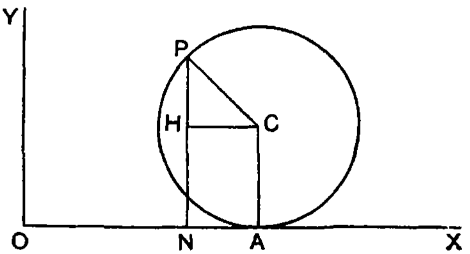

FIG. 22. The cycloid.

(i) The cycloid is the locus of a point carried by a circle which rolls on a fixed straight line. Let P (Fig. 22) be the point situated

on the circumference of the circle with centre C, which rolls without slipping along the straight line OX.

Let O be the initial position of P, so that arc $AP = OA$, then if the coordinates of P are (x, y), and $\theta = \angle ACP$, *i.e.* the angle made by the radius to P with the vertical,

$$x = ON = OA - HC = \text{arc } AP - a \cos PCH = a\theta - a \cos\left(\theta - \frac{\pi}{2}\right)$$
$$= a(\theta - \sin\theta), \text{ where } a = \text{radius of circle},$$

and $y = NP = AC + HP = a + a \sin\left(\theta - \frac{\pi}{2}\right) = a(1 - \cos\theta).$

$\therefore$ **The equations of the cycloid are**

$$\mathbf{x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).} \qquad \text{.......................(74)}$$

(ii) When the circle carrying the point P on its circumference rolls on the outside of a fixed circle, as in Fig. 23, the locus is known as the **epicycloid**.

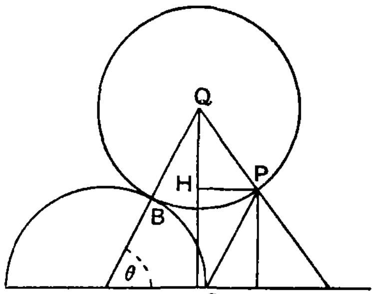

FIG. 23. The epicycloid.

Let O, the centre of the fixed circle, be taken as origin, and suppose that P was initially at C, then if $\angle COQ = \theta$,

$$\angle PQH = \angle OQP - \angle OQH$$
$$= (\text{arc } BP)/(na) - \pi/2 + \theta$$
$$= (\text{arc } BC)/(na) - \pi/2 + \theta$$
$$= \theta/n - \pi/2 + \theta$$
$$= (1+n)\theta/n - \pi/2.$$

Taking OC as x-axis, and (x, y) as the coordinates of P,

$$x = ON = OA + HP = a(1+n)\cos\theta + na \sin PQH$$
$$= a(1+n)\cos\theta + na \sin\{(1+n)\theta/n - \pi/2\}$$
$$= a(1+n)\cos\theta - na \cos\frac{1+n}{n} \,.\, \theta,$$

and $$y = NP = AQ - HQ = a(1+n)\sin\theta - na \cos PQH$$
$$= a(1+n)\sin\theta - na \sin\frac{1+n}{n} \,.\, \theta.$$

When the rolling circle encloses the fixed circle, or rolls inside the fixed circle, the path of the point carried on the circumference is called a **hypocycloid**. The coordinates of the curve may be determined exactly as in the case of the epicycloid, and therefore

may be derived from those found above by replacing n with $-n'$ and θ with θ'; hence

$$x = a(1-n')\cos\theta' - a(-n')\cos\frac{1-n'}{-n'}\theta'$$

$$= a(1-n')\cos\theta' + an'\cos\frac{1-n'}{n'}\theta',$$

and $$y = a(1-n')\sin\theta' - an'\sin\frac{1-n'}{n'}\theta'.$$

When the curves are identical for $n=3$,

$$4\cos\theta - 3\cos\tfrac{4}{3}\theta = (1-n')\cos\theta' + n'\cos\frac{1-n'}{n'}\theta',$$

$$4\sin\theta - 3\sin\tfrac{4}{3}\theta = (1-n')\sin\theta' - n'\sin\frac{1-n'}{n'}.\theta'.$$

Squaring each and adding,

$$25 - 24\cos\tfrac{1}{3}\theta = 1 - 2n' + 2n'^2 + 2n'(1-n')\cos\frac{\theta'}{n'}.$$

This must be an identity, so that

$$2n'^2 - 2n' + 1 = 25,\quad 2n'(1-n') = -24,\quad \frac{\theta'}{n'} = \tfrac{1}{3}(2n\pi \pm \theta);$$

$$\therefore\ n' = 4 \text{ or } -3,\quad \text{giving}\quad \theta' = \tfrac{4}{3}(2n\pi \pm \theta)\quad \text{or}\quad \theta - 2n\pi.$$

76. General Equations for the Cycloids. If b be written for the radius of the rolling circle, the equations of the **Epicycloid** become

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b\cos\frac{a+b}{b}.\theta, \\ y &= (a+b)\sin\theta - b\sin\frac{a+b}{b}.\theta, \end{aligned}\right\} \quad \text{......................(75a)}$$

and those of the **Hypocycloid,**

$$\left.\begin{aligned} x &= (a-b)\cos\theta + b\cos\frac{a-b}{b}\theta, \\ y &= (a-b)\sin\theta - b\sin\frac{a-b}{b}\theta. \end{aligned}\right\} \quad \text{......................(75b)}$$

From these equations, those of two other important curves may be derived as particular cases.

(a) Let $a=b$, then ($75a$) becomes

$$x = a(2\cos\theta - \cos 2\theta),\quad y = a(2\sin\theta - \sin 2\theta).\ \text{............(76a)}$$

These are the equations of the **Cardioid**, but they may be compressed into one simple equation, by changing the origin, and transforming into polar coordinates.

In Fig. 23, let QP produced meet OC produced in K; then since the radii of the circles are equal, and arc BC = arc BP,

$$\angle KQO = \angle QOK = \theta;$$

$$\therefore\ KQ = KO.$$

But $PQ = CO$, so that $KP = KC$;

$$\therefore\ CP \text{ is parallel to } OQ$$

and $$\angle KCP = \angle KOQ = \theta.$$

If, therefore, C be taken as the origin, and (r, θ) the polar coordinates of P, $$CP = r.$$

$$\begin{aligned}\therefore\ r\cos\theta = CN &= ON - OC\\ &= a(2\cos\theta - \cos 2\theta) - a, \text{ from } (76a),\\ &= 2a(\cos - \cos^2\theta).\end{aligned}$$

Dividing out by $\cos\theta$, the polar equation becomes

$$\mathbf{r = 2a(1 - \cos\theta)}. \qquad \textbf{(76b)}$$

(β) Let $a = 4b$, then the equations of the hypocycloid become

$$4x = a(3\cos\theta + \cos 3\theta) = 4a\cos^3\theta, \text{ from } (13),$$

giving $$x^{\frac{1}{3}} = a^{\frac{1}{3}}\cos\theta.$$

Similarly, $$y^{\frac{1}{3}} = a^{\frac{1}{3}}\sin\theta.$$

Hence, by squaring and adding,

$$\mathbf{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}. \qquad \textbf{(77)}$$

The curve represented by this equation is called the **Four-cusped hypocycloid.**

Ex. 14. *A bar of given length moves with its extremities on two fixed straight lines at right angles. Prove that a marked point on the bar describes an ellipse.*

If the length of the bar be 20 *in., and the marked point be* 4 *in. from one end, determine the eccentricity of the ellipse and the position of its foci.*

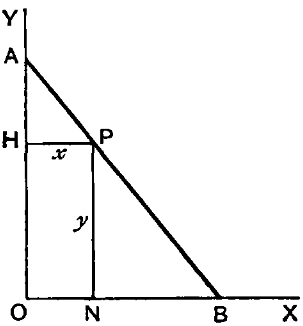

FIG. 24. Mechanical tracing of an ellipse.

Let AB (Fig. 24) be the bar, OX, OY the fixed straight lines, and P the marked point.

Take OX, OY as axes, (x, y) as the coordinates of P, and suppose $AB=l$, and $AP=k$, then

$$HP/PA=NB/BP\,;$$

$$\therefore\ HP^2/PA^2=NB^2/BP^2=(BP^2-NP^2)/BP^2,$$

i.e. $$x^2/k^2=\{(l-k)^2-y^2\}/(l-k)^2,$$

giving $$x^2/k^2+y^2/(l-k)^2=1,$$

which, by (69*b*), is an ellipse whose centre is O, and whose semi-axes are k and $l-k$.

In the given case, take $k=20-4=16$ in. and $l-k=4$ in.; then, from (69*c*), the eccentricity e of the ellipse is given by

$$e^2=1-(4/16)^2=15/16,$$

so that $$e=\sqrt{15}/4=0{\cdot}968.$$

Also, from (73*a*), distance of foci from $O=16e=$ **15·5 in. approx.**

Ex. 15. *A is any point on the circle $x^2+y^2=a^2$ whose centre is O, and B is a fixed point on the axis of x such that AB is of constant length l. If OA meets the line through B parallel to the axis of y in C, shew that the locus of C is given by the equation*

$$(x^2+y^2)(x^2+a^2-l^2)^2=4a^2x^4. \qquad \text{(L.U.)}$$

From the equation of the circle, O is the origin; hence take the coordinates of C as (x, y), then if $\angle BOC=\phi$ (Fig. 25),

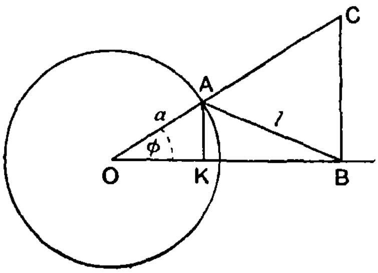

FIG. 25.

$$l^2=KA^2+KB^2=a^2-OK^2+(x-OK)^2=a^2+x^2-2x\,.\,OK$$
$$=a^2+x^2-2ax\,.\,OB/OC.$$

Multiplying out by OC, and squaring,

$$OC^2(a^2+x^2-l^2)^2=4a^2x^2\,.\,OB^2,$$

i.e. $$(x^2+y^2)(a^2+x^2-l^2)^2=4a^2x^4.$$

Exercises 10.

1. Find the area of the triangle ABC, whose vertices are A (2, 3), B (2, 17), C (12, 14); and find the length of the perpendicular from C to AB.

2. Find the area of the triangle whose vertices are $(a+b, 2a)$, $(a, a-b)$, $(2b, b)$. Explain the case when $a=2b$.

3. Shew that the three points (3, 11), (-2, $-1{\cdot}5$), (9, 26) are collinear, and find the equation of the straight line passing through them.

4. Find the value of a such that the three points $(a, 8)$, $(0, a)$, $(2a, 13)$ are collinear, and find the equation of the line passing through them.

5. A straight line is drawn through the point (5, 9) inclined at 45° to the axis of x. This straight line is cut in points P, Q by the lines $x+3y=20$, $7x+y=120$, which pair themselves intersect at T. Shew that the triangle PQT is isosceles, and give the length of each of the equal sides and the tangent of the angle at the vertex.

6. Find the coordinates of the point which divides the line joining the points (x_1, y_1), (x_2, y_2), internally in the ratio $l : m$.

Find the ratio of the segments into which the lines joining (1, 3) to (5, -3) and (4, 5) to (-1, -4) are divided by their point of intersection. (L.U.)

7. Determine the ratio in which the segment of the straight line joining (2, 3) to (-1, 4) is cut by the straight line $x+y+1=0$. (L.U., Sc.)

8. Shew that the polar equation of a straight line may be written in the form $p=r\cos(\theta-\alpha)$, where p is the length of the perpendicular to the line from the origin and α the angle it makes with the x-axis. Deduce the Cartesian form,

$$p=x\cos\alpha+y\sin\alpha.$$

Shew also that the polar equation of the straight line passing through the given points (r_1, θ_1), (r_2, θ_2) is

$$rr_1\sin(\theta-\theta_1)+r_1r_2\sin(\theta_1-\theta_2)+r_2r\sin(\theta_2-\theta)=0.$$

9. Find the coordinates of a point at a distance r from the point (α, β) on a line through (α, β) which makes an angle θ with the positive direction of the axis of x. (L.U.)

10. Shew that the equation $60x^2-103xy-72y^2=0$ represents two straight lines, and find the angles between them.

11. Find the angle between the lines $ax^2+2hxy+by^2=0$, in terms of the coefficients. (L.U.)

12. Shew that the equation $2x^2-13xy-7y^2+x+23y-6=0$ represents two straight lines, and determine the angle between them. Write down the equation of the pair of straight lines parallel to these through the point (1, 2). (L.U.)

13. Find the equation of the straight line which joins the point (3, 5) to the intersection of the two lines

$$4x+y-1=0, \quad 7x-3y-35=0.$$

Shew that this line is equidistant from the origin and the point (8, 34).

14. Find the equation of the straight line which passes through the point (h, k), and is perpendicular to the line $ax+by+c=0$. (L.U., Sc.)

15. Two straight lines meet at an angle θ, and p_1, p_2 are the lengths of the perpendiculars on them from any point. Shew that the lengths of the perpendiculars from this point on the lines bisecting the angles between them are

$$\tfrac{1}{2}(p_1+p_2)\operatorname{cosec}\frac{\theta}{2} \quad \text{and} \quad \tfrac{1}{2}(p_1-p_2)\sec\frac{\theta}{2},$$

where the point lies in the angle denoted by θ. (L.U., Sc.)

16. Perpendiculars are drawn from the point (c, d) upon the lines

$$ax^2+2hxy+by^2=0.$$

Shew that their equation is

$$a(y-d)^2-2h(x-c)(y-d)+b(x-c)^2=0. \qquad \text{(L.U., Sc.)}$$

17. Shew that the equation

$$1/r=a+b\cos\theta$$

represents a straight line if $a=0$, and a parabola if $a=b$.

Prove that the line $c/r=\cos(\theta-\alpha)$ meets the parabola

$$l/r=1+\cos(\theta-\beta)$$

in real points if $2c\cos(\alpha-\beta)<l$. (L.U., Sc.)

18. Prove that the equation of the chord joining two points whose vectorial angles are $\alpha+\beta$, $\alpha-\beta$, on the conic $l/r=1+e\cos\theta$, is

$$l/r=\sec\beta\,.\,\cos(\theta-\alpha)+e\cos\theta.$$

If PSQ be any chord of the above conic which passes through the focus S, shew that $1/SP+1/QS=2/l.$

By reducing each of the following equations to its simplest form, indicate the nature of the locus represented, giving also the coordinates of its centre:

19. $3y(2-3x)=7(3x-2)$.

20. $288x^2-168xy+337y^2=576$. (L.U., Sc.)

21. $12x^2-2xy-2y^2+14x+8y-6=0$.

22. $9x^2+24xy+16y^2+9x+12y+2=0$.

23. $x^2+6xy+9y^2-68x-4y+56=0$.

24. $9x^2+24xy+16y^2-170x-185y+625=0$.

25. $(15y-8x+1)^2-4(23x-7y)=0$. **26.** $(2x+y-3)^2-8x-4y-9=0$.

27. $3x^2-4xy+y^2+12x-4y+48=0$. (L.U., Sc.)

28. $2x^2+5xy+2y^2-11x-7y-4=0$. (L.U.)

29. Interpret geometrically the following equations:

$$(a)\ x^2+4xy+4y^2+2x+4y-3=0,$$

$$(b)\ 5x^2+4xy+8y^2-12x-12y=0. \qquad \text{(Br.U.)}$$

30. Prove that the equation of the chord of an ellipse $Ax^2+By^2=1$, which has (h, k) for its middle point, is

$$Ahx+Bky=Ah^2+Bk^2. \qquad \text{(L.U., Sc.)}$$

31. Find the equation of the conic which passes through the six points $(0, 1)$, $(0, 3)$, $(1, 0)$, $(3, 0)$, $(2, 5)$, $(5, 2)$. (L.U.)

32. Prove that the points of bisection of all parallel chords of a parabola lie on a straight line.

In the parabola $y^2=6x$, chords are drawn through the fixed point $(9, 5)$. Shew that the locus of middle points of these chords is the parabola

$$y^2-5y-3x+27=0.$$

33. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ when referred to new rectangular axes, the origin remaining unchanged, becomes $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$, shew that $a'+b'=a+b$ and $a'b'-h'^2=ab-h^2$.

34. Prove that, if (ξ, η), (ξ', η') are the coordinates of the extremities of two conjugate diameters of the ellipse

$$x^2/a^2+y^2/b^2=1,$$

then $\xi'/a=\pm\eta/b$, $\eta'/b=\pm\xi/a$, both upper and lower signs being taken. (L.U., Sc.)

35. Find that diameter of the ellipse $x^2-xy+y^2=1$, which is conjugate to the axis of x. Hence prove that $a^2+b^2=8/3$, where a and b are the semi-axes. (Br.U.)

36. Prove that the locus of a point which is equidistant from two given circles, one of which lies entirely within the other, is an ellipse. Find the eccentricity of the ellipse in terms of the radii of the circles and the distance between their centres. (L.U.)

37. AB and CD are rods of equal length a_1, joined crosswise by rods AD and BC of equal length a_2; P and Q divide BA and BC in the same ratio r. If the straight line PQ cuts DA and DC in R and S, prove that these rods are also divided in the ratio r, and that

$$PQ\,.\,PR=\frac{r}{(r+1)^2}(a_2{}^2-a_1{}^2).$$

Hence, or otherwise, shew that if Q be constrained to describe a circular arc passing through P, R will describe a certain straight line. (L.U.)

38. A circle of radius a rolls on the outside of another circle of the same radius. Shew that the polar equation of the path traced out by any point on the rolling circle is

$$r = 2a(1 - \cos\theta),$$

the origin being the point of contact of the tracing point with the fixed circle, and the initial line, the radius of this circle through the point. (L.U.)

39. An ellipse has its centre at O; its axes lie on the coordinate axes OX, OY, and it passes through the points P (2, 7) and Q (4, 3). Find the equation of the ellipse and give the position of the foci. Shew that the length of the semi-diameter conjugate to OP is $\sqrt{(841/30)}$, and give its equation.

40. Shew that the length of a focal chord of an ellipse inclined at an angle ϕ to the major axis is $2l/(1 - e^2\cos^2\phi)$, where $2l$ is the length of the latus rectum.

41. The coordinates of a point P are given by

$$x = at^2 + bt + c, \quad y = bt^2 - at + k,$$

where t is a variable. Shew that the locus of P is a parabola whose axis is the line

$$(a^2 - b^2)(ay - bx) = (a^2 + b^2)(bc - ak).$$

42. AB is a fixed diameter of a circle, PN the perpendicular let fall from any point P of the circle to AB. On PN a point Q is taken such that $QN = \frac{1}{3}PN$. Shew that the locus of Q is an ellipse, and give its eccentricity.

***43.** A rod PQ rests with one end Q on the arc of a smooth curve, and the other end P against a smooth vertical wall in the same vertical plane. If the rod remains in equilibrium for all positions when P is higher than Q, shew that the curve is an ellipse and determine its eccentricity.

***44.** The coordinates of a particle in time t are given by

$$x = a\cos pt, \quad y = b\cos(pt + \epsilon),$$

where a, b, p and ϵ are constants. Shew that the path of the particle is, in general, an ellipse, and deduce the nature of the locus in each of the following particular cases:

(a) when $\epsilon = n\pi$,
(b) when $\epsilon = (2n+1)\pi/2$, and $a = b$,

n being a positive integer in each case.

***45.** Find the path of a particle whose coordinates at time t are given by $x = a\cos pt$, $y = a\cos(2pt + \epsilon)$, and shew that in the particular case when $\epsilon = 0$, the path is a parabola whose latus rectum is $\frac{1}{2}a$.

***46.** $BCDE$ is a parallelogram having $BC=ED=a$, $BE=CD=b$, CB is produced to any point A, and the whole figure thus represents a linkage of bars freely hinged at B, C, D, E, whilst the point A is fixed. Shew that if E be constrained to move along a straight line passing through A, then D will describe an ellipse whose eccentricity e is given by

$$e^2(a+2b)=4b(a+b).$$

***47.** Two equal rods AB, CD, each of length b, are joined crosswise by two other equal rods AC, BD, each of length a. The rod BD is fixed and A constrained to move in a circle whose centre is B. Shew that the locus of the instantaneous centre of the rod AC is an ellipse or a hyperbola according as a is less than or greater than b.

***48.** $PQRS$ is a rhombus each of whose sides is a; OQ, OS are two lines each of length b, O being outside the rhombus. Prove that the rectangle $OP \,.\, OR=b^2-a^2$.

The whole figure represents Peaucellier's linkage to transform circular motion into rectilinear motion, P being constrained to describe the arc of a circle passing through O; prove that R traces out a straight line perpendicular to the diameter through O of the circle, and at a distance $(b^2-a^2)/2r$ from it, where r is the radius of the circle.

***49.** A particle whose polar coordinates are (r, θ) describes a curve, under a central acceleration, which is given by the equation

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)=k-\frac{1}{r},$$

where k is constant; shew that the path is a conic, and determine the necessary conditions that it should be an ellipse.

***50.** OA and OB are two lines at right angles; the circles with O as centre and these lines as radii are drawn, and any line through O meets the circles in P and Q; lines are drawn through P and Q parallel to OA and OB meeting in R and S. Prove that R and S move on two ellipses. (L.U.)

51. A body is projected with an initial velocity of V ft. per sec. at θ° to the horizontal. Shew that its least speed is $V\cos\theta$, and that if the horizontal range and greatest height are to be equal, θ must be $75^\circ\ 58'$ approximately. In this case find V for a range of 425 ft.

***52.** A particle is projected at an angle θ with the horizontal up an inclined plane in a vertical plane through a line of greatest slope α. If the particle strike the plane perpendicularly, prove that

$$\cot\alpha=2\tan(\theta-\alpha). \qquad \text{(S.U.)}$$

***53.** AB is a rod, sliding with its ends on two rectangular rods OX, OY. C is a fixed point on the rod, and CP is perpendicular to AB and of fixed length. Find the locus of P for various positions of AB and determine what the locus becomes when $CP^2=CA \,.\, CB$. (Br.U.)

Let the line $y = mx + c$ intersect the curve $ay^2 = x^3$, then th abscissae of the points of intersection are given by

$$a(mx+c)^2 = x^3,$$

or $$x^3 - am^2x^2 - 2macx - ac^2 = 0.$$

If the line is a tangent to the curve, then two of the roots c this cubic must be equal; the condition for this is, by (78),

$$(c + \tfrac{2}{9}m^3a)^2 = \tfrac{4}{9}m^3a(\tfrac{2}{3}c + \tfrac{1}{9}m^3a),$$

or $$c = -\tfrac{4}{27}m^3a.$$

Hence the equation of the tangent becomes

$$y = mx - \tfrac{4}{27}m^3a,$$

or $$27y = m(27x - 4m^2a).$$

78. Tangents and Normals to the Conics. If the straight lin $y = mx + n$ intersects the conic $F(x, y) = 0$, then the quadrati $F(x, mx+n) = 0$, gives the two abscissae of the points of inte section. Hence, when the line becomes a **tangent** these tw abscissae will be equal, and the quadratic must therefore hav equal roots. The condition for this gives the value of n in term of m, and the general equation of the tangent to the conic determined in terms of m.

The **normal** to a conic is defined as the line perpendicular to th tangent at its point of contact with the curve, so that by (65b if m' be the gradient of the normal, $mm' = -1$, giving $m' = -1/n$ From this fact, the general equation of the normal in terms of is also easily derived, as the following example will shew.

Ex. 2. Find the equations of the tangent and normal i terms of the gradient m at any point on each of the following curves the parabola, $y^2 = 4ax$; the circle, $x^2 + y^2 = a^2$; the ellipse and th hyperbola whose equations are $x^2/a^2 \pm y^2/b^2 = 1$ respectively.

Let the straight line $y = mx + n$ intersect the parabola $y^2 = 4a$ then the abscissae of the points of intersection are given by th equation

$$(mx+n)^2 = 4ax,$$

or $$m^2x^2 + 2(mn - 2a)x + n^2 = 0.$$

This quadratic will have equal roots if

$$(mn - 2a)^2 = m^2n^2,$$

i.e. if $$n = a/m.$$

Hence the line $y=mx+a/m$ touches the parabola for all values of m.

Further, since the sum of the roots of the above quadratic is $-2(mn-2a)/m^2$, and, in the case of the tangent, the roots are equal,

$\therefore$ each root $= -(mn-2a)/m^2 = a/m^2$, on putting $n=a/m$.

With this value of x substituted in the equation of the parabola, the corresponding coordinate is $\pm 2a/m$.

Hence **the line $\mathbf{y=mx+a/m}$ touches the parabola $\mathbf{y^2=4ax}$ at the point $\mathbf{(a/m^2, \pm 2a/m)}$.**

If $\mu=1/m$, then any point on the parabola $y^2=4ax$ may be represented by the coordinates $(a\mu^2, 2a\mu)$, and μ is called **a variable parameter.** The importance of this lies in the fact that two variables, x, y, have been replaced by a single variable μ, which is often of great convenience in solving problems. It should be observed that if $\theta=$ slope of the tangent, then $m=\tan\theta$, and $\mu=1/m=\cot\theta$.

From § 78, it is evident that the equation of the normal at the point $(a/m^2, 2a/m)$ is of the form

$$y=-x/m+n',$$

and since the point $(a/m^2, 2a/m)$ is on the normal,

$$\therefore\ 2a/m=-a/m^3+n',$$

from which

$$n'=2a/m+a/m^3;$$

$$\therefore\ y=-x/m+2a/m+a/m^3.$$

If, therefore, m' be the gradient of the normal, $m'=-1/m$, and the equation becomes

$$\mathbf{y=m'x-2am'-am'^3}.$$

Taking now the ellipse $x^2/a^2+y^2/b^2=1$, the abscissae of the points of intersection of the curve and the line $y=mx+n$ are given by

$$b^2x^2+a^2(mx+n)^2=a^2b^2,$$

or

$$x^2(b^2+a^2m^2)+2a^2mnx+a^2(n^2-b^2)=0.$$

The condition for equal roots is

$$a^4m^2n^2=a^2(n^2-b^2)(b^2+a^2m^2),$$

giving

$$n^2=b^2+a^2m^2.$$

Hence, **$\mathbf{y=mx\pm\sqrt{b^2+a^2m^2}}$ is a tangent to the ellipse for all values of m.** The double sign indicates that there are two tangents for every value of m.

Let the line $y=mx+c$ intersect the curve $ay^2=x^3$, then the abscissae of the points of intersection are given by

$$a(mx+c)^2=x^3,$$

or

$$x^3-am^2x^2-2macx-ac^2=0.$$

If the line is a tangent to the curve, then two of the roots of this cubic must be equal; the condition for this is, by (78),

$$(c+\tfrac{2}{9}m^3a)^2=\tfrac{4}{9}m^3a(\tfrac{2}{3}c+\tfrac{1}{9}m^3a),$$

or

$$c=-\tfrac{4}{27}m^3a.$$

Hence the equation of the tangent becomes

$$y=mx-\tfrac{4}{27}m^3a,$$

or

$$27y=m(27x-4m^2a).$$

78. Tangents and Normals to the Conics. If the straight line $y=mx+n$ intersects the conic $F(x, y)=0$, then the quadratic, $F(x, mx+n)=0$, gives the two abscissae of the points of intersection. Hence, when the line becomes a **tangent** these two abscissae will be equal, and the quadratic must therefore have equal roots. The condition for this gives the value of n in terms of m, and the general equation of the tangent to the conic is determined in terms of m.

The **normal** to a conic is defined as the line perpendicular to the tangent at its point of contact with the curve, so that by (65*b*), if m' be the gradient of the normal, $mm'=-1$, giving $m'=-1/m$. From this fact, the general equation of the normal in terms of m is also easily derived, as the following example will shew.

Ex. 2. *Find the equations of the tangent and normal in terms of the gradient m at any point on each of the following curves; the parabola, $y^2=4ax$; the circle, $x^2+y^2=a^2$; the ellipse and the hyperbola whose equations are $x^2/a^2\pm y^2/b^2=1$ respectively.*

Let the straight line $y=mx+n$ intersect the parabola $y^2=4ax$, then the abscissae of the points of intersection are given by the equation

$$(mx+n)^2=4ax,$$

or

$$m^2x^2+2(mn-2a)x+n^2=0.$$

This quadratic will have equal roots if

$$(mn-2a)^2=m^2n^2,$$

i.e. if

$$n=a/m.$$

Hence the line $y = mx + a/m$ touches the parabola for all values of m.

Further, since the sum of the roots of the above quadratic is $-2(mn - 2a)/m^2$, and, in the case of the tangent, the roots are equal,

$\therefore$ each root $= -(mn - 2a)/m^2 = a/m^2$, on putting $n = a/m$.

With this value of x substituted in the equation of the parabola, the corresponding coordinate is $\pm 2a/m$.

Hence **the line $\mathbf{y = mx + a/m}$ touches the parabola $\mathbf{y^2 = 4ax}$ at the point $\mathbf{(a/m^2, \pm 2a/m)}$.**

If $\mu = 1/m$, then any point on the parabola $y^2 = 4ax$ may be represented by the coordinates $(a\mu^2, 2a\mu)$, and μ is called a **variable parameter.** The importance of this lies in the fact that two variables, x, y, have been replaced by a single variable μ, which is often of great convenience in solving problems. It should be observed that if θ = slope of the tangent, then $m = \tan\theta$, and $\mu = 1/m = \cot\theta$.

From § 78, it is evident that the equation of the normal at the point $(a/m^2, 2a/m)$ is of the form

$$y = -x/m + n',$$

and since the point $(a/m^2, 2a/m)$ is on the normal,

$$\therefore\ 2a/m = -a/m^3 + n',$$

from which

$$n' = 2a/m + a/m^3;$$

$$\therefore\ y = -x/m + 2a/m + a/m^3.$$

If, therefore, m' be the gradient of the normal, $m' = -1/m$, and the equation becomes

$$\mathbf{y = m'x - 2am' - am'^3}.$$

Taking now the ellipse $x^2/a^2 + y^2/b^2 = 1$, the abscissae of the points of intersection of the curve and the line $y = mx + n$ are given by

$$b^2x^2 + a^2(mx + n)^2 = a^2b^2,$$

or

$$x^2(b^2 + a^2m^2) + 2a^2mnx + a^2(n^2 - b^2) = 0.$$

The condition for equal roots is

$$a^4m^2n^2 = a^2(n^2 - b^2)(b^2 + a^2m^2),$$

giving

$$n^2 = b^2 + a^2m^2.$$

Hence, **$\mathbf{y = mx \pm \sqrt{b^2 + a^2m^2}}$ is a tangent to the ellipse for all values of m.** The double sign indicates that there are two tangents for every value of m.

Let $k^2 = b^2 + a^2m^2$, then, since the sum of the roots of the above quadratic $= -2a^2mn/k^2$, when the roots are equal, each will be $-2a^2mn/k^2 = \mp a^2m/k$. Substituting this value in the equation of the ellipse, the corresponding ordinates at the point of contact become $\pm b^2/k$.

Now the normal at this point will have an equation of the form $y = -x/m + n'$, and substituting the coordinates of the point, this becomes

$$y = -x/m \pm (b^2 - a^2)/k,$$

so that, if m' = gradient of normal, $m' = -1/m$, and **the equation of the normal to the ellipse is**

$$\mathbf{y = m'x \pm m'(b^2 - a^2)/\sqrt{a^2 + m'^2b^2}}.$$

The corresponding equations in the cases of the circle and hyperbola may now be easily deduced from the above.

For the **circle**, $b = a$, and the **equations of the tangent and normal become**

$$\mathbf{y = mx \pm a\sqrt{1 + m^2}} \quad \text{and} \quad \mathbf{y = m'x},$$

thus shewing that the normal passes through the centre.

For the hyperbola, b^2 must be replaced by $-b^2$, and the tangent is

$$y = mx \pm \sqrt{a^2m^2 - b^2},$$

and the normal, $$y = m'x \mp m'(a^2 + b^2)/\sqrt{a^2 - m'^2b^2}.$$

In the cases of the ellipse and hyperbola, it is usually more convenient to take a parameter other than m in order to obtain rational coefficients in the equations for the tangent and normal respectively.

For the ellipse, the eccentric angle ϕ will readily effect this. From (73*c*), the coordinates of any point on the curve may be taken as $(a \cos \phi, b \sin \phi)$; hence, at the point of contact of the tangent, $$a \cos \phi = \mp a^2m/k, \quad b \sin \phi = \pm b^2/k;$$

so that by division $\quad b \tan \phi/a = -b^2/(a^2m)$,

from which $$m = -(b/a) \cdot \cot \phi.$$

Substituting this value of m in the equation,

$$y = mx \pm \sqrt{b^2 + a^2m^2},$$

the tangent at $(a \cos \phi, b \sin \phi)$ becomes

$$\frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = 1,$$

taking only the positive sign, so that ϕ is always positive.

Similarly, the equation of the normal at the same point,

$$y = -x/m \pm (b^2 - a^2)/k,$$

transforms to
$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2.$$

For the hyperbola, $x^2/a^2 - y^2/b^2 = 1$, it is obvious that if $x = a\sec\phi$, then $y = b\tan\phi$, so that the coordinates of any point on the curve may be taken as $(a\sec\phi,\ b\tan\phi)$.

Replacing b^2 by $-b^2$ in the m-coordinates of the ellipse, the point of contact of the tangent to the hyperbola is

$$(\mp a^2 m/k,\ \mp b^2/k)\,;$$

$$\therefore\ a\sec\phi = \mp a^2 m/k, \quad b\tan\phi \mp b^2/k,$$

which, by division, gives $m = b/(a\sin\phi)$.

Hence, substituting in the equation

$$y = mx \pm \sqrt{a^2m^2 - b^2},$$

the equation of the tangent at the point $(a\sec\phi,\ b\tan\phi)$ assumes the form
$$\frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = 1.$$

Similarly the normal at this point becomes

$$\frac{ax}{\sec\phi} + \frac{by}{\tan\phi} = a^2 + b^2.$$

All the above equations are collected in § 80 for reference.

Ex. 3. (i) *Determine the coordinates of the point of intersection T of the tangents at P $(am_1^2, 2am_1)$ and Q $(am_2^2, 2am_2)$ to the parabola $y^2 = 4ax$, and prove that the area of the triangle TPQ is*

$$\tfrac{1}{2}a^2(m_1 \sim m_2)^3. \qquad \text{(L.U.)}$$

(ii) *From any point P inside the parabola $y^2 = 4ax$, two normals are drawn to the curve which are perpendicular to each other; shew that the locus of P is also a parabola whose vertex is the point* $(3a, 0)$, *and whose latus rectum is a.*

From the above analysis, the equations of the tangents at P and Q are

$$y = x/m_1 + am_1, \quad y = x/m_2 + am_2.$$

Hence, by solution, the point (x', y') of intersection T is

$$x' = am_1m_2, \quad y' = a(m_1 + m_2)\,;$$

and, by (63), the area of the triangle TPQ

$$=\frac{1}{2}\begin{vmatrix} am_1^2 & x' & am_2^2 \\ 2am_1 & y' & 2am_2 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}a^2\begin{vmatrix} m_1^2 & m_1m_2 & m_2^2 \\ 2m_1 & m_1+m_2 & 2m_2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}\mathbf{a}^2(\mathbf{m}_1 - \mathbf{m}_2)^3.$$

(ii) Let the coordinates of P be (x, y), then the equation of the normal to the parabola is

$$y = mx - 2am - am^3,$$

or $$am^3 - (x - 2a)m + y = 0.$$

Since this is a cubic in m, it has three roots; thus three normals can be drawn from (x, y) to the curve.

Let m_1, m_2, m_3 be the roots, then since two of the normals are perpendicular to each other,

$$m_1m_2 = -1.$$

But by § 77, $$m_1 + m_2 + m_3 = 0,$$

so that $$m_1 + m_2 = -m_3,$$

$$m_1m_2 + m_2m_3 + m_3m_1 = -(x - 2a)/a = 2 - x/a;$$

$$\therefore\ m_3(m_1 + m_2) = 2 - x/a - m_1m_2 = 2 - x/a + 1 = 3 - x/a.$$

Substituting the value of $m_1 + m_2$,

$$m_3^2 = x/a - 3.$$

Finally, $$m_1m_2m_3 = -y/a,$$

so that $$m^3 = y/a.$$

$\therefore$ From above relation,

$$y^2/a^2 = x/a - 3,$$

or $$\mathbf{y^2 = a(x - 3a)},$$

which is the equation of a parabola having $(3a, 0)$ as its vertex and a for the length of its latus rectum.

79. Tangent and Normal at a Given Point on a Conic. The foregoing example shews how the general equations of a tangent and normal at any point on a conic may be obtained. It is now necessary to fix these lines by making them pass through a given point on the curve. This will be investigated in Ex. 4.

Ex. 4. *Find the equation of the tangent at the point* (ξ, η) *on the conic* $F(x, y) = 0$. (L.U.)

Deduce (*a*) *the condition that the line* $px + qy + r = 0$ *may be a tangent at the point* (ξ, η); (*b*) *the equation of the normal at* (ξ, η); (*c*) *the equations of the tangent and normal at* (x_1, y_1) *on each of the following curves—the parabola,* $y^2 = 4ax$; *the circle,* $x^2 + y^2 = a^2$; *the ellipse and hyperbola,* $x^2/a^2 \pm y^2/b^2 = 1$.

Let (ξ, η) be any point on the line $y = mx + n$, where as usual m denotes the gradient $\tan\theta$, and θ the slope. Take another point (x, y) on the line, distant r from (ξ, η), then

$$x = \xi + r\cos\theta \quad \text{and} \quad y = \eta + r\sin\theta.$$

Now suppose (x, y) lies on the conic $F(x, y) = 0$, then substituting the values of x and y,

$$a(\xi + r\cos\theta)^2 + 2h(\xi + r\cos\theta)(\eta + r\sin\theta) + b(\eta + r\sin\theta)^2$$
$$+ 2g(\xi + r\cos\theta) + 2f(\eta + r\sin\theta) + c = 0.$$

Arranging this as a quadratic in r, it becomes

$$Ar^2 + 2Br + F(\xi, \eta) = 0,$$

where $\qquad A = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta,$

and $\qquad B = (a\xi + h\eta + g)\cos\theta + (h\xi + b\eta + f)\sin\theta.$

Now if (ξ, η) also lies on the curve, $F(\xi, \eta) = 0$, and

$$Ar^2 + 2Br = 0,$$

from which $r = 0$, or $-2B/A$, and this is the length of the chord when (ξ, η) is not coincident with (x, y).

If, however, (ξ, η) is coincident with (x, y), *i.e.* the line is a tangent to the curve at (ξ, η), then both roots of the above quadratic must be zero, the conditions for which are $F(\xi, \eta) = 0$, and $B = 0$. The former confirms the fact that (ξ, η) lies on the conic, and the latter gives the gradient of the tangent, for replacing B by its value,

$$(a\xi + h\eta + g)\cos\theta = -(h\xi + b\eta + f)\sin\theta,$$

i.e. $\qquad \tan\theta = -(a\xi + h\eta + g)/(h\xi + b\eta + f).$

Hence the equation of the tangent becomes

$$y = -(a\xi + h\eta + g)x/(h\xi + b\eta + f) + n;$$

and since (ξ, η) is on this line,

$$\begin{aligned} n &= \eta + (a\xi + h\eta + g)\xi/(h\xi + b\eta + f) \\ &= (a\xi^2 + 2h\xi\eta + b\eta^2 + g\xi + f\eta)/(h\xi + b\eta + f) \\ &= \{F(\xi, \eta) - (g\xi + f\eta + c)\}/(h\xi + b\eta + f) \\ &= -(g\xi + f\eta + c)/(h\xi + b\eta + f), \text{ since } F(\xi, \eta) = 0. \end{aligned}$$

Hence, the equation of the tangent at (ξ, η) on the conic $F(x, y)=0$ becomes

$$\mathbf{x(a\xi+h\eta+g)+y(h\xi+b\eta+f)+g\xi+f\eta+c=0.} \quad \text{............}\mathbf{(79a)}$$

(*a*) If $px+qy+r=0$ is a tangent to the conic $F(x, y)=0$ at the point (ξ, η), then $p\xi+q\eta+r=0$ Now let λ be a number such that $\lambda p\xi+\lambda q\eta+\lambda r=0$ is identical with (79*a*), then

$$\begin{aligned} a\xi+h\eta-\lambda p+g&=0,\\ h\xi+b\eta-\lambda q+f&=0,\\ g\xi+f\eta-\lambda r+c&=0,\\ p\xi+q\eta\qquad+r&=0. \end{aligned}$$

Eliminating ξ, η and λ from these equations,

$$\begin{vmatrix} a & h & -p & g \\ h & b & -q & f \\ g & f & -r & c \\ p & q & 0 & r \end{vmatrix} = 0.$$

Interchanging the third and fourth columns, **the condition that the line $\mathbf{px+qy+r=0}$ should be a tangent to the conic $\mathbf{F(x, y)=0}$ is**

$$\begin{vmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} & \mathbf{p} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} & \mathbf{q} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} & \mathbf{r} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} & \mathbf{0} \end{vmatrix} = \mathbf{0}. \quad \text{..........................}\mathbf{(79b)}$$

The expanded form of this relation is

$$p^2(bc-f^2)+q^2(ca-g^2)+r^2(ab-h^2)+2pq(fg-ch)+2qr(gh-af) \\ +2pr(hf-bg)=0.$$

(*b*) The gradient of the normal to the conic $= -\cot\theta$

$$=(h\xi+b\eta+f)/(a\xi+h\eta+g);$$

hence the equation to the normal may be written

$$y=(h\xi+b\eta+f)x/(a\xi+h\eta+g)+n.$$

Since it passes through the point (ξ, η),

$$\therefore\ n=\eta-(h\xi+b\eta+f)\xi/(a\xi+h\eta+g),$$

so that the equation in its simplest form becomes

$$\mathbf{(y-\eta)(a\xi+h\eta+g)=(x-\xi)(h\xi+b\eta+f).} \quad \text{..................}\mathbf{(79c)}$$

(*c*) The equation for the tangent (79*a*) may be written

$$ax\xi+h(x\eta+y\xi)+by\eta+g(x+\xi)+f(y+\eta)+c=0,$$

which may be derived from the general equation $F(x, y)=0$ by replacing x^2 by $x\xi$, $2xy$ by $x\eta+y\xi$, y^2 by $y\eta$, $2x$ by $x+\xi$, and $2y$ by $y+\eta$. This is a general rule, and may be applied to any form of equation to the conic; thus, for the parabola, $y^2=4ax$, $yy_1=2a(x+x_1)$ is the tangent at (x, y). Similarly, $xx_1+yy_1=a^2$, $xx_1/a^2 \pm yy_1/b^2=1$, are the equations of the tangents at (x_1, y_1) on the circle $x^2+y^2=a^2$, the ellipse $x^2/a^2+y^2/b^2=1$, and the hyperbola $x^2/a^2-y^2/b^2=1$ respectively.

For the normals, (79*c*) gives for the parabola, on putting $a=h=f=c=0$, $b=1$, $g=-2a$, $\xi=x_1$ and $\eta=y_1$,

$$-2a(y-y_1)=y_1(x-x_1),$$

or

$$2a(y-y_1)+y_1(x-x_1)=0.$$

Similarly for the circle $x^2+y^2=a^2$, the normal is $xy_1=x_1y$.

For the ellipse $x^2/a^2+y^2/b^2=1$, the normal is

$$y_1(x-x_1)/b^2=x_1(y-y_1)/a^2,$$

and for the hyperbola $x^2/a^2-y^2/b^2=1$, the normal is

$$y_1(x-x_1)/b^2+x_1(y-y_1)/a^2=0.$$

Ex. 5. (i) *Any chord of the parabola* $y^2=4ax$ *is drawn through the point* $(0, 2a)$; *shew that the normals at its ends meet on the parabola*

$$(x+y-2a)^2+ay=0. \qquad \text{(L.U., Sc.)}$$

(ii) *Find the equation of the ellipse which passes through the point* $(7, 24)$ *and has its foci at the points* $(25, 0)$, $(-25, 0)$, *and prove that the tangent at the point cuts the minor axis at a distance from the centre equal to the distance of either focus from the centre.* (L.U.)

Let $y=mx+n$ be the chord; since it passes through the point $(0, 2a)$; $\therefore$ $n=2a$, so the chord is $y=mx+2a$.

Suppose this intersects the parabola in $(a\mu^2, 2a\mu)$, then

$$2a\mu=ma\mu^2+2a, \quad \text{or} \quad m\mu^2-2\mu+2=0,$$

thus shewing that there are two values of μ satisfying this equation, *i.e.* the chord intersects the parabola in two points.

Let μ_1, μ_2 be the roots of the quadratic, then $\mu_1+\mu_2=2/m$, and $\mu_1\mu_2=2/m$, so that $\mu_1+\mu_2=\mu_1\mu_2$, or $1/\mu_1+1/\mu_2=1$.

Now the equation of the normal at $(a\mu^2, 2a\mu)$ is, from Ex. 2, p. 240, $y=-\mu(x-2a)+a\mu^3$, or, arranging as a cubic in μ,

$$a\mu^3-\mu(x-2a)-y=0.$$

Let μ_1, μ_2, μ_3 be the roots of this equation; μ_1, μ_2 are obviously roots since the normals are drawn at the extremities of the chord.

Hence, $$\mu_1+\mu_2+\mu_3=0,$$
$$\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1=-(x-2a)/a,$$
$$\mu_1\mu_2\mu_3=y/a.$$

Dividing the second by the third,
$$1/\mu_3+1/\mu_1+1/\mu_2=-(x-2a)/y.$$

Hence, since $1/\mu_1+1/\mu_2=1$,
$$1/\mu_3=-(x+y-2a)/y.$$

Also $\mu_1+\mu_2=\mu_1\mu_2=-\mu_3$, from the first equation;

$\therefore\ \mu_1\mu_2\mu_3=-\mu_3{}^2=y/a$, from the third equation.

Equating the two values of μ_3, the locus of the point of intersection of the normals is
$$y^2/(x+y-2a)^2=-y/a.$$
giving $$(x+y-2a)^2+ay=0,$$
which is a parabola.

Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$; this is justifiable, since the foci are symmetrically placed on the x-axis. If e be the eccentricity of the curve, then, from (73a), $ae=25$, so that $a^2e^2=625$.

But from (69c), $$e^2=1-b^2/a^2;$$
hence $$a^2-b^2=625,$$
or $$a^2=625+b^2.$$

Since (7, 24) lies on the curve, $49/a^2+576/b^2=1$,
or $$576a^2+49b^2=a^2b^2.$$

Substituting the value of a already found,
$$576(625+b^2)+49b^2=b^2(625+b^2),$$
which gives $$b^4=576\times 625=24^2\times 25^2;$$
$$\therefore\ b^2=24\times 25=600,$$
and $$a^2=625+b^2=1225;$$

$\therefore$ Equation of ellipse is $\mathbf{x^2/1225+y^2/600=1}$.

From the first part of the solution, the equation of the tangent at (7, 24) is $7x/1225+24y/600=1$, or $x/175+y/25=1$.

This cuts the minor axis where $y=0$, so that $\mathbf{x=25}=$ distance of either focus from the centre.

80. Summary of Results. The equations established in the last two Examples are so important that they are here tabulated for reference.

Conic.	Parabola.	Circle.	Ellipse.	Hyperbola
Equation.	$y^2=4ax$.	$x^2+y^2=a^2$.	$x^2/a^2+y^2/b^2=1$.	$x^2/a^2-y^2/b^2=1$.
Tangent in terms of gradient m	$y=mx+a/m$	$y=mx\pm a\sqrt{1+m^2}$	$y=mx\pm\sqrt{b^2+a^2m^2}$	$y=mx\pm\sqrt{a^2m^2-b^2}$
Coordinates of Point of Contact	$(a/m^2,\ 2a/m)$	$(\mp am/s,\ \pm a/s)$ $s^2=1+m^2$	$(\mp a^2m/k,\ \pm b^2/k)$ $k^2=b^2+a^2m^2$	$(\mp a^2m/k',\ \mp b^2/k')$ $k'^2=a^2m^2-b^2$
Normal in terms of its gradient m'	$y=m'(x-2a)-am'^3$	$y=m'x$	$y=m'x$ $\pm m'(b^2-a^2)/\sqrt{a^2+m'^2b^2}$	$y=m'x$ $\mp m'(a^2+b^2)/\sqrt{a^2-m'^2b^2}$
Tangent at (x_1, y_1) on Conic	$yy_1=2a(x+x_1)$	$xx_1+yy_1=a^2$	$xx_1/a^2+yy_1/b^2=1$	$xx_1/a^2-yy_1/b^2=1$
Normal at (x_1, y_1)	$2a(y-y_1)+y_1(x-x_1)$ $=0$	$xy_1=x_1y$	$y_1(x-x_1)/b^2$ $=x_1(y-y_1)/a^2$	$y_1(x-x_1)/b^2$ $=-x_1(y-y_1)/a^2$
Tangent in terms of parameter ϕ	—	$x\cos\phi+y\sin\phi=a$	$x\cos\phi/a$ $+y\sin\phi/b=1$	$x\sec\phi/a$ $-y\tan\phi/b=1$
Normal in terms of parameter ϕ	—	$x\sin\phi=y\cos\phi$	$ax/\cos\phi-by/\sin\phi$ $=a^2-b^2$	$ax/\sec\phi+by/\tan\phi$ $=a^2+b^2$
Coordinates of Point of Contact	—	$(a\cos\phi,\ a\sin\phi)$	$(a\cos\phi,\ b\sin\phi)$	$(a\sec\phi,\ b\tan\phi)$

81. General Equations for Tangents and Normals to any Curve. The determination of the equations of a tangent and normal to a conic has so far been dealt with by purely algebraical methods. With the aid of the calculus, however, these equations may be obtained by a much shorter process, which has the further advantage of being applicable to any curve defined by its equation.

Let $F(x, y)=0$ be *any* curve, not necessarily a conic, and $y=mx+n$, a tangent to the curve at any point on the curve. Then, by definition, m is the gradient of the curve and is therefore equal to dy/dx; hence the equation of the tangent may be written, $y=x \cdot dy/dx+n$. If this is a tangent at the point (ξ, η), then $n=\eta-\xi \cdot dy/dx$, where x and y are to be replaced by ξ and η in the value of dy/dx; hence **the equation of the tangent at the point (ξ, η), on the curve $F(x, y)=0$, is**

$$y-\eta=\frac{dy}{dx}(x-\xi), \qquad \text{(81a)}$$

where ξ, η are substituted for x, y in dy/dx *after differentiation.*

Sometimes, when the equation of the curve involves an implicit relation between the variables x and y, it may be more convenient to replace dy/dx by its equivalent

$$-(\partial F/\partial x)/(\partial F/\partial y);$$

then the equation of the tangent becomes

$$\frac{\partial F}{\partial y}(y-\eta)+\frac{\partial F}{\partial x}(x-\xi)=0. \qquad \text{(81b)}$$

For the normal, the gradient is $-1/m=-dx/dy$, and the corresponding equations for the normal at (ξ, η) on the curve become

$$\left.\begin{aligned} &(a)\ (x-\xi)+\frac{dy}{dx}(y-\eta)=0,\\ &(b)\ (x-\xi)\Big/\frac{\partial F}{\partial x}=(y-\eta)\Big/\frac{\partial F}{\partial y}, \end{aligned}\right\} \qquad \text{(82)}$$

where, as before, x, y must be replaced by ξ, η in the values of the derivatives.

Ex. 6. (*a*) ***Find the equation of the tangent at the point*** (ξ, η) ***on the conic,*** $ax^2+2hxy+by^2+2gx+2fy+c=0$. (L.U.)

(*b*) *Prove that the tangent to the curve* $x^3+y^3=3axy$ *at the point* (ξ, η) *is* $x(\xi^2-a\eta)+y(\eta^2-a\xi)=a\xi\eta$.

Write down the equation of the tangent at the point $(6a/7, -12a/7)$, *and verify that it meets the curve again at the point*

$$(-16a/21, 4a/21). \qquad \text{(L.U.)}$$

(*a*) Differentiating the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

with respect to x,

$$ax+hx\frac{dy}{dx}+hy+by\frac{dy}{dx}+g+f\frac{dy}{dx}=0;$$

$$\begin{aligned}\therefore\ \frac{dy}{dx}&=-(ax+hy+g)/(hx+by+f) \text{ at the point } (x, y)\\ &=-(a\xi+h\eta+g)/(h\xi+b\eta+f) \text{ at the point } (\xi, \eta).\end{aligned}$$

$\therefore$ By (81*a*), the equation of the tangent at (ξ, η) on the conic $F(x, y)=0$ is

$$(y-\eta)(h\xi+b\eta+f)+(x-\xi)(a\xi+h\eta+g)=0,$$

which, as in Ex. 3, becomes

$$x(a\xi+h\eta+g)+y(h\xi+b\eta+f)+g\xi+f\eta+c=0.$$

The above method should be compared with the algebraic process of Ex. 3.

(*b*) Since $\qquad x^3+y^3=3axy;$

$$\therefore\ x^2+y^2\frac{dy}{dx}=ay+ax\frac{dy}{dx},$$

giving
$$\begin{aligned}\frac{dy}{dx}&=(ay-x^2)/(y^2-ax) \text{ at the point } (x, y)\\ &=(a\eta-\xi^2)/(\eta^2-a\xi) \text{ at the point } (\xi, \eta).\end{aligned}$$

Hence the equation of the tangent at (ξ, η) is

$$(y-\eta)(\eta^2-a\xi)=(x-\xi)(a\eta-\xi^2),$$

or
$$\begin{aligned}x(\xi^2-a\eta)+y(\eta^2-a\xi)&=\xi^3+\eta^3-2a\xi\eta\\ &=\xi^3+\eta^3-3a\xi\eta+a\xi\eta\\ &=a\xi\eta,\end{aligned}$$

since (ξ, η) lies on the curve.

When $\xi=6a/7$, $\eta=-12a/7$, this equation becomes

$$\mathbf{20x+17y+12a=0.}$$

To verify that this line meets the curve again at

$$(-16a/21, 4a/21),$$

it is really only necessary to shew that these values of x and y satisfy both the equation of the curve and the equation of the tangent. In order, however, to illustrate the use of relations between the roots of an equation and its coefficients, given in § 77. the coordinates of the third point of intersection will be found.

Since the equation of the curve is of the third degree, a straight line will intersect it in three points. Now it has just been shewn that the line $20x + 17y + 12a = 0$ is a tangent to the curve at $(6a/7, -12a/7)$; hence this point is really two coincident points. Let (α, β) be the coordinates of the third point of intersection; then all three values of y are given by the equation

$$-(12a + 17y)^3/8000 + y^3 = -3ay(12a + 17y)/20,$$

on eliminating x between the equation of the curve and its tangent.

On dividing out by the coefficient of y^3, the product of the roots will be equal to $-$(absolute term).

Coefficient of $y^3 = 1 - 17^3/8000 = 1 - 4913/8000 = 3087/8000$.

$\therefore$ Absolute term $= -1728a^3/8000 \div 3087/8000 = -1728a^3/3087$.

But product of roots $= (-12a/7)^2 \,.\, \beta = 144a^2\beta/49$;

$\therefore$ $144a^2\beta/49 = 1728a^3/3087$, or $\beta = (1728 \times 49a)/(3087 \times 144)$,

giving $$\beta = 4a/21.$$

Substituting $(\alpha, 4a/21)$, in the equation of the tangent,

$$20\alpha = -(68a/21 - 12a) = -320a/21,$$

giving $$\alpha = -16a/21;$$

$\therefore$ The tangent intersects the curve again at $(-16a/21, 4a/21)$.

Ex. 7. *Find the conditions that the two curves* $F(x, y) = 0$, $G(x, y) = 0$, *may* (i) *touch each other*, (ii) *cut one another orthogonally. Hence determine the nature of the intersection of the curve* $bx^2 - a^2y = 0$ *and the ellipse* $x^2 + 2y^2 = 2$.

The equations of the tangents at (ξ, η) on the curves

$$F(x, y) = 0 \quad \text{and} \quad G(x, y) = 0,$$

are by (81*b*),

$$\frac{\partial F}{\partial y}(y - \eta) + \frac{\partial F}{\partial x}(x - \xi) = 0.$$

$$\frac{\partial G}{\partial y}(y - \eta) + \frac{\partial G}{\partial x}(x - \xi) = 0.$$

And by (65a),

the angle between these lines is the principal value of

$$\tan^{-1}\left(\frac{\frac{\partial F}{\partial x}\cdot\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial x}}{\frac{\partial F}{\partial x}\cdot\frac{\partial G}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial y}}\right). \quad \text{.......................(83a)}$$

If, therefore,

$$\frac{\partial F}{\partial x}\Big/\frac{\partial G}{\partial x}=\frac{\partial F}{\partial y}\Big/\frac{\partial G}{\partial y}, \quad \text{.......................(83b)}$$

the curves touch each other,

and if
$$\frac{\partial F}{\partial x}\cdot\frac{\partial G}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial y}=0, \quad \text{.......................(83c)}$$

the curves cut each other orthogonally, ***i.e.*** **at right angles.**

Applying these results to the given curves, let

$$F(x, y)\equiv bx^2-a^2y=0\ ; \quad G(x, y)\equiv x^2+2y^2-2=0,$$

then
$$\frac{\partial F}{\partial x}=2bx, \quad \frac{\partial F}{\partial y}=-a^2, \quad \frac{\partial G}{\partial x}=2x, \quad \frac{\partial G}{\partial y}=4y\ ;$$

$$\therefore \frac{\partial F}{\partial x}\cdot\frac{\partial G}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial y}=4bx^2-4a^2y=4(bx^2-a^2y)=0,$$

since
$$F(x, y)=0.$$

$\therefore$ the curves cut orthogonally.

82. Subtangents and Subnormals. Let P be any point (x, y) on the curve $F(x, y)=0$ (Fig. 26), and suppose the tangent and normal at P cut the axis of x in T and G respectively. Then if N be the foot of the ordinate from P, the intercept TN is called the **subtangent**, and the intercept NG, the **subnormal** of P. Let $\theta=$ slope of the tangent PT, then

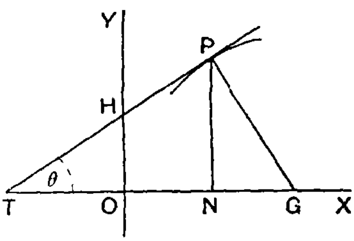

FIG. 26. Subtangent and subnormal.

$$\frac{dy}{dx}=\tan\theta=\frac{NP}{TN}=\frac{y}{TN}\ ;$$

$$\therefore \textbf{Subtangent } \mathbf{TN}=\mathbf{y}\Big/\frac{\mathbf{dy}}{\mathbf{dx}}. \quad \text{.......................(84a)}$$

Also $\angle NPG =$ complement of $\angle NPT = \theta$;

$$\therefore \frac{NG}{NP} = \tan\theta = \frac{dy}{dx}.$$

$$\textbf{Subnormal } \mathbf{NG = y \cdot \frac{dy}{dx}}. \quad \text{..........................(84b)}$$

Again, $TP =$ length of tangent between the x-axis and P

$$= TN \sec\theta = TN\sqrt{1+\tan^2\theta};$$

$$\therefore \mathbf{TP = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\Big/\frac{dy}{dx} = y \cdot \frac{ds}{dy}}, \quad \text{.....................(84c)}$$

since $$ds^2 = dx^2 + dy^2.$$

Similarly, $GP =$ length of normal between the x-axis and P

$$= NP \sec\theta = y\sqrt{1+\tan^2\theta},$$

giving $$\mathbf{GP = y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = y \cdot \frac{ds}{dx}}. \quad \text{.......................(84d)}$$

Ex. 8. (*a*) *If the ordinate and normal at any point P on the curve $y = \phi(x)$ meet the axis of x in N and G, find an expression for the length of NG.*

Shew from this expression that the locus of P is a parabola if NG is constant, and that the locus is a central conic if NG varies as the abscissa of P. (L.U.)

(*b*) *In the case of the parabola $y^2 = 4ax$ shew also that the subtangent is double the abscissa.* (L.U.)

(*c*) *Find the curves in which* (i) *the subtangent is constant, and* (ii) *the length of the tangent between the x-axis and the point of contact is constant.*

(*a*) The first part of this question is answered in § 82, the required expression being given in (84*b*).

For the second part, let $NG = c$, where c is constant, then from (84*b*),

$$y \cdot \frac{dy}{dx} = c,$$

or $$y \cdot dy = c \cdot dx,$$

so that, by integration, $$\tfrac{1}{2}y^2 = cx + A',$$

or $$\mathbf{y^2 = 2cx + A}, \quad \text{where } A = 2A'.$$

This is a parabola whose axis is the axis of x, whose vertex is the point $(0, \sqrt{A})$, and whose latus rectum is $2c$.

When NG varies as x, let $NG = kx$, where k is constant, then

$$y \cdot \frac{dy}{dx} = kx,$$

or
$$y \cdot dy = kx \cdot dx,$$

which gives, on integration,

$$y^2 = kx^2 + B,$$

or
$$\alpha x^2 + \beta y^2 = 1,$$

where
$$\alpha = -k/B \text{ and } \beta = 1/B.$$

This locus is (i) an ellipse if α and β are positive, *i.e.* if k is negative; (ii) a hyperbola if α and β have opposite signs, *i.e.* if B is negative and k positive; or (iii) a circle if $\alpha = \beta$, *i.e.* if $k = -1$. In every case, therefore, the locus represents a central conic.

(*b*) Since $y^2 = 4ax$,

$$\therefore\ y \cdot \frac{dy}{dx} = 2a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}.$$

Hence subtangent $= y \Big/ \frac{dy}{dx} = y^2/2a = 2x$, since $y^2 = 4ax$.

$\therefore$ Subtangent = twice the abscissa.

(*c*) In the first case, the subtangent is constant.

Let
$$y \Big/ \frac{dy}{dx} = c, \text{ where } c \text{ is constant;}$$

then
$$dx = c \cdot dy/y,$$

which, on integration, gives

$$x + A = c \log y;$$

$$\therefore\ y = e^{(x+A)/c} = Be^{x/c}, \text{ where } B = e^{A/c}.$$

In the second case, let a = constant length of tangent between the x-axis and the point of contact, then from (84*c*),

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = a \cdot \frac{dy}{dx}.$$

Square both sides, and solve for $\left(\frac{dy}{dx}\right)^2$, then

$$\left(\frac{dy}{dx}\right)^2 = y^2/(a^2 - y^2);$$

$$\therefore\ \frac{dy}{dx} = y/\sqrt{a^2 - y^2},$$

taking the positive root.

Hence $x+A=\int(\sqrt{a^2-y^2}/y)\,.\,dy$

$$=-\int\frac{u^2du}{a^2-u^2},\text{ on putting } a^2-y^2=u^2$$

$$=\int\left(1-\frac{a^2}{a^2-u^2}\right)du=u-\tfrac{1}{2}a\log\frac{a+u}{a-u}$$

$$=u+a\log\frac{a-u}{y}$$

$$=\sqrt{a^2-y^2}+a\log\frac{a-\sqrt{a^2-y^2}}{y};$$

$$\therefore\quad\frac{a-\sqrt{a^2-y^2}}{y}=e^{(x-\sqrt{a^2-y^2}+A)/a},$$

or $$\mathbf{a}-\sqrt{\mathbf{a}^2-\mathbf{y}^2}=\mathbf{Bye}^{(\mathbf{x}-\sqrt{\mathbf{a}^2-\mathbf{y}^2})/\mathbf{a}},\text{ where } B=e^{A/a}.$$

This curve is sometimes called the **tractory.**

83. Asymptotes. When the point of contact between a tangent to a curve and the curve is at an infinite distance from the origin, the tangent is called an **Asymptote.** Thus, although an asymptote is a real straight line, it never meets the curve in the finite part of the plane, but the curve approaches indefinitely near to it as the distance from the origin increases. A pair of asymptotes, PQ, PQ', to a curve are shewn in Fig. 27.

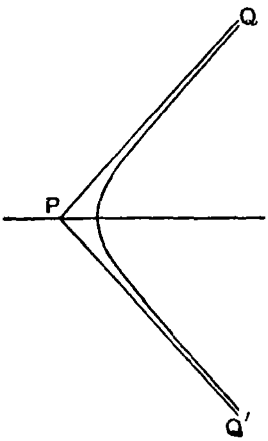

FIG. 27. Asymptotes.

Ex. 9. *Determine the conditions that must be fulfilled for the straight line $y=mx+n$ to be an asymptote to the curve $F(x, y)=0$.*

Find the equations of the asymptotes of the hyperbola

$$2x^2+5xy+2y^2-11x-7y-4=0. \qquad \text{(L.U.)}$$

Determine also the asymptotes to the curve

$$x^4+4y^4+(a^2-5x^2)y^2=b^2(a^2-x^2),$$

and shew that they intersect the curve again in eight points which lie on an ellipse.

(*a*) The line $y=mx+n$ intersects the curve $F(x, y)=0$ in the

points whose abscissae are given by the equation $F(x, mx+n)=0$, formed by substituting $mx+n$ for y in the equation of the curve.

Suppose that $F(x, y)$ be of the rth degree, and that when expanded in powers of x,

$$F(x, mx+n) \equiv A_r x^r + A_{r-1}x^{r-1} + A_{r-2}x^{r-2} + \ldots + A_1 x + A_0 = 0.$$

In this equation, put $x = 1/\xi$, then after multiplying out by ξ^r, the equation becomes

$$A_r + A_{r-1}\xi + A_{r-2}\xi^2 + \ldots + A_1\xi^{r-1} + A_0\xi^r = 0.$$

When the equation in x has two infinite roots, the equation in ξ will have two zero roots, and the conditions for this are $A_r = 0$ and $A_{r-1} = 0$. In this case $y = mx+n$ will meet $F(x, y) = 0$ in two coincident points at infinity, and will therefore be an asymptote to the curve; hence, **$\mathbf{y = mx + n}$ will be an asymptote to the curve $\mathbf{F(x, y) = 0}$, when the coefficients of the two highest powers of $\mathbf{x}$ in the equation $\mathbf{F(x, mx+n) = 0}$ vanish,** *i.e.* if

$$\mathbf{F(x, mx+n) \equiv A_r x^r + A_{r-1}x^{r-1} + A_{r-2}x^{r-2} + \ldots + A_1 x + A_0}$$

when $\mathbf{A_r = 0}$ and $\mathbf{A_{r-1} = 0}$. ....(85)

(*b*) Applying this result to the given equation of the hyperbola, the abscissae of the points of intersection of the curve and the straight line $y = mx+n$ are given by

$$2x^2 + 5x(mx+n) + 2(mx+n)^2 - 11x - 7(mx+n) - 4 = 0,$$

i.e. $(2m^2 + 5m + 2)x^2 + (5n + 4mn - 7m - 11)x + 2n^2 - 7n - 4 = 0.$

Hence, by (85), $y = mx+n$ will be an asymptote if

$$2m^2 + 5m + 2 = 0 \quad \text{and} \quad 5n + 4mn - 7m - 11 = 0.$$

The first equation gives

$$(2m+1)(m+2) = 0 \quad \text{or} \quad m = -2 \text{ or } -\tfrac{1}{2}.$$

The second gives

$$n = (7m+11)/(4m+5),$$

so that, when $m = -2$, $n = 1$,

and when $m = -\frac{1}{2}$, $n = \frac{5}{2}$;

$\therefore$ the equations of the asymptotes are

$$\mathbf{2x + y - 1 = 0, \quad x + 2y - 5 = 0.}$$

(*c*) Similarly, the abscissae of the points of intersection of the line $y = mx+n$ and the given curve are given by

$$x^4 + 4(mx+n)^4 + (a^2 - 5x^2)(mx+n)^2 = b^2(a^2 - x^2).$$

The two highest powers of x are x^4 and x^3; hence, for these to vanish,

$$4m^4 - 4m^2 + 1 = 0 \quad \text{and} \quad 2mn(5 + 8m^2) = 0.$$

The first gives $(4m^2 - 1)(m^2 - 1) = 0$, so that $m = \pm 1$ or $\pm\frac{1}{2}$, and the second gives $n = 0$, since m is not zero;

$\therefore$ The asymptotes are

$$\mathbf{y} + \mathbf{x} = 0, \quad \mathbf{y} - \mathbf{x} = 0, \quad 2\mathbf{y} + \mathbf{x} = 0 \quad \text{and} \quad 2\mathbf{y} - \mathbf{x} = 0.$$

Writing the given equation in the form

$$x^4 - 5x^2y^2 + 4y^2 + b^2x^2 + a^2y^2 = a^2b^2,$$

or
$$(x - y)(x + y)(x + 2y)(x - 2y) + b^2x^2 + a^2y^2 = a^2b^2,$$

it is evident that each of the asymptotes will intersect the curve in two points other than infinity, and these will be given by

$$b^2x^2 + a^2y^2 = a^2b^2,$$

since each of the factors in the first term is zero, as a condition of being an asymptote; hence the other eight points of intersection lie on the curve

$$b^2x^2 + a^2y^2 = a^2b^2,$$

or
$$x^2/a^2 + y^2/b^2 = 1,$$

which is an ellipse whose semi-axes are a and b.

84. Rectangular Hyperbola. When the angle between the asymptotes of a hyperbola is a right angle, the curve is called a **Rectangular Hyperbola,** and it then bears the same relation to the general hyperbola that a circle does to an ellipse. This conic is of much importance in practice.

Ex. 10. (*a*) *Find the asymptotes of the conic*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and deduce the conditions that the curve should be a rectangular hyperbola.

(*b*) *Apply the result to the curve $x^2/a^2 - y^2/b^2 = 1$, and find the equation of this hyperbola referred to its perpendicular asymptotes as axes.*

(*a*) Let $y = mx + n$ intersect the conic $F(x, y) = 0$. then the abscissae of the points of intersection are given by

$$ax^2 + 2hx(mx + n) + b(mx + n)^2 + 2gx + 2f(mx + n) + c = 0.$$

For $y = mx + n$ to be an asymptote, the coefficients of x^2 and x must vanish;

$$\therefore \quad a + 2hm + bm^2 = 0 \quad \text{and} \quad hn + bmn + g + fm = 0.$$

Let m_1, m_2 be the roots of the first of these equations, then

$$m_1 + m_2 = -2h/b \quad \text{and} \quad m_1 m_2 = a/b.$$

Thus there are two asymptotes, and their joint equation is

$$(y - m_1 x - n_1)(y - m_2 x - n_2) = 0,$$

or

$$y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 - (n_1 + n_2)y + (n_1 m_2 + n_2 m_1)x + n_1 n_2 = 0.$$

But from the second of the above equations,

$$n_1 = -(g + fm_1)/(h + bm_1) \quad \text{and} \quad n_2 = -(g + fm_2)/(h + bm_2);$$

$$\begin{aligned} \therefore \quad n_1 + n_2 &= -\{2gh + (hf + bg)(m_1 + m_2) + 2bfm_1 m_2\}/ \\ &\qquad\qquad \{h^2 + hb(m_1 + m_2) + b^2 m_1 m_2\} \\ &= 2f/b, \end{aligned}$$

on substituting the values of $m_1 + m_2$ and $m_1 m_2$, provided h^2 is not equal to ab.

Similarly,

$$\begin{aligned} n_1 m_2 + n_2 m_1 &= -(gm_2 + fm_1 m_2)/(h + bm_1) \\ &\quad - (gm_1 + fm_1 m_2)/(h + bm_2) \\ &= -(gm_2 + af/b)/(h + bm_1) - (gm_1 + af/b)/(h + bm_2) \\ &= -\{(hg + af)(m_1 + m_2) + bg(m_1^2 + m_2^2) \\ &\qquad\qquad + 2afh/b\}/(ab - h^2) \\ &= 2g/b. \end{aligned}$$

Finally, $\quad n_1 n_2 = (g + fm_1)(g + fm_2)/(h + bm_1)(h + bm_2)$

$$= (bg^2 + af^2 + 2fgh)/b(ab - h^2).$$

Substituting these values in the equation for the asymptotes, and multiplying out by b,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0,$$

where $\qquad k = (bg^2 + af^2 - 2fgh)/(ab - h^2).$

Hence **the equations of a hyperbola and its asymptotes differ only by a constant.**

This fact leads to a direct derivation of the equation of the asymptotes, for since this equation differs only by a constant from the equation of the conic, the asymptotes will be given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + \lambda = 0,$$

provided λ be chosen so that the above equation represents two straight lines. The condition for this is, by (67*a*),

$$ab\lambda + 2fgh - af^2 - bg^2 - \lambda h^2 = 0,$$

from which $\lambda = (af^2 + bg^2 - 2fgh)/(ab - h^2)$,

thus agreeing with the value of k above. This is a much shorter method, and is therefore preferable in every way.

For a rectangular hyperbola, the asymptotes must be at right angles to each other, the condition for which is $m_1m_2 = -1$, but from the above analysis, $m_1m_2 = a/b$.

Hence $\mathbf{F(x, y) = 0}$ will represent a rectangular hyperbola when

$$\mathbf{b = -a.} \qquad \ldots\ldots\ldots\ldots\ldots \mathbf{(86a)}$$

(*b*) Applying (86*a*) to $x^2/a^2 - y^2/b = 1$, it readily follows that for the rectangular hyperbola, $a = b$, and the resulting equation becomes $x^2 - y^2 = a^2$. It is also obvious from the above that the asymptotes of this hyperbola are $y = \pm x$, so that each asymptote meets the x-axis at an angle of 45°.

To refer the equation to the asymptotes as axes, it is therefore necessary to turn the axes through 45° in a clockwise direction, *i.e.* through $-45°$; hence x and y must be replaced by

$$x' \cos(-45°) - y' \sin(-45°) = (x' + y')/\surd 2,$$

and $x' \sin(-45°) + y' \cos(-45°) = (y' - x')/\surd 2$, respectively, by (70).

Substituting these values in $x^2 - y^2 = a^2$, and omitting dashes, the equation of a rectangular hyperbola referred to its asymptotes as axes becomes

$$\mathbf{xy = a^2.} \qquad \ldots\ldots\ldots\ldots\ldots \mathbf{(86b)}$$

Ex. 11. *Shew that the locus of the middle point of a chord of constant length* $2l$ *of the curve* $xy = k^2$ *is*

$$(x^2 + y^2)(xy - k^2) = l^2xy. \qquad \text{(L.U.)}$$

Let (x_1, y_1), (x_2, y_2) be the extremities of the chord, and (x, y) its mid-point of the chord, then

$$x = \tfrac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \tfrac{1}{2}(y_1 + y_2).$$

But $x_1 + x_2 = k^2(1/y_1 + 1/y_2) = k^2(y_1 + y_2)/y_1y_2$;

$$\therefore\ y_1y_2 = k^2y/x.$$

Similarly $\quad x_1x_2 = k^2x/y.$

Now
$$4l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (x_1 + x_2)^2 - 4x_1x_2 + (y_1 + y_2)^2 - 4y_1y_2$$
$$= 4x^2 - 4k^2x/y + 4y^2 - 4k^2y/x\,;$$
$$\therefore\ l^2xy = xy(x^2 + y^2) - k^2(x^2 + y^2)\,;$$

i.e.
$$\mathbf{(x^2 + y^2)(xy - k^2) = l^2xy.}$$

Exercises 11.

1. A circle passes through the points (4, 1), (6, 5), and has its centre on the line $4x + y = 16$; find its equation.

Tangents are drawn to this circle from (7, 10); shew that the length of their chord of contact is $2\sqrt{105/13}$.

2. Find the condition that the equation
$$x^2 + y^2 + 2gx + 2fy + c = 0$$
may represent a real circle, and determine the radius and centre of this circle.

Find the equation to and the length of the tangents from the point $(-2, 3)$ to the circle
$$x^2 + y^2 - 2x - 2y + 1 = 0. \qquad \text{(L.U.)}$$

3. AB, AC are tangents to a circle and BC is their chord of contact; prove that the square of the distance of any point on the circumference from BC is equal to the product of its distances from AB and AC.

Prove also the converse, viz. that if ABC is an isosceles triangle and P is a point within the angle BAC such that the square of its distance from the base BC is equal to the product of its distances from the equal sides AB, AC, then the locus of P is the circle which touches AB and AC at B and C. (L.U.)

4. In the parabola $(bx + ay - ab)^2 = 4abxy$, shew that

(i) The axis is the line $(a^2 + b^2)(bx - ay) + ab(a^2 - b^2) = 0$.

(ii) The tangent at the vertex is the line
$$(a^2 + b^2)(ax + by) = a^2b^2.$$

(iii) The directrix is the line $ax + by = 0$.

5. Obtain the equations of the tangent and normal at any point $(a/m^2, 2a/m)$ on the parabola $y^2 = 4ax$.

Shew that the subtangent is double the abscissa, and that the subnormal is constant. (L.U.)

6. Two equal parabolas have the same vertex and their axes are at right angles; prove that their common tangent touches each of them at an extremity of the latus rectum. (Li.U.)

7. Prove that the line $lx+my+n=0$ is normal to the parabola $y^2=4ax$ if $al^3+2alm^2+m^2n=0$, and is a tangent to the parabola if $am^2=ln$.

8. Obtain the equation of the normal to $y^2=4ax$ in the form,

$$y=t(x-2a)-at^3.$$

Shew that this normal touches the curve $27ay^2=4(x-2a)^3$ at the point $(2a+3at^2, 2at^3)$, and intersects it at the point $(2a+\frac{3}{4}at^2, -\frac{1}{4}at^3)$. (Li.U., Sc.)

9. If P be any point on the parabola $y^2=4ax$, and F its focus, shew that the locus of the mid-point of FP is the co-axial parabola

$$2y^2=2a(2x-a).$$

10. Shew that in the parabola $y^2=4ax$, the locus of the point of intersection of two normals which are perpendicular to each other is $y^2=a(x-3a)$. (M.U.)

11. Prove that the locus of the foot of the perpendicular drawn from the vertex to a tangent of the parabola $y^2=4ax$ is

$$(x^2+y^2)x+ay^2=0. \qquad \text{(M.U., Sc.)}$$

12. The straight line $2y=3x+34$ is a tangent to the ellipse

$$b^2x^2+a^2y^2=a^2b^2,$$

whose eccentricity is 0·6. Find the semi-axes a, b, and the coordinates of the point of contact.

13. Shew that if $lx+my+n=0$ is a tangent to the ellipse

$$x^2/a^2+y^2/b^2=1,$$

then $a^2l^2+b^2m^2=n^2$, and if it is normal to the curve, then

$$n^2(a^2m^2+b^2l^2)=l^2m^2(a^2-b^2)^2.$$

14. Shew that $3x-4y-7=0$ is a common tangent to the ellipse $x^2+20y^2-5=0$ and the circle $x^2+y^2+8x+2y+8=0$.

15. Find the equation to the ellipse which has the point $(1, 2)$ as focus and the line $2x-3y+6=0$ as the corresponding directrix, and which is of eccentricity $2/3$.

Determine the value or values of m, so that

$$y=mx+2$$

may be a tangent, and find the point or points of contact. (L.U.)

16. The extremities of a straight rod AB are free to slide one in each of two mutually perpendicular grooves; shew that, as the rod is moved, any marked point P on it describes an ellipse. (See Ex. 14, p. 231.)

If the lengths AP, BP are a, b, shew that, when the rod passes through a focus of the ellipse it makes an angle $\tan^{-1}\sqrt{b/\{2(a-b)\}}$ with the tangent of the ellipse at P. (Li.U.)

17. Prove that the product of the focal distances of a point (x_1, y_1) on the ellipse $b^2x^2+a^2y^2=a^2b^2$ is equal to the square on the semi-diameter of the ellipse which is parallel to the tangent at (x_1, y_1). (L.U.)

18. If ϕ_1, ϕ_2 be the eccentric angles of the extremities of a focal chord of an ellipse of eccentricity e, prove that

$$\cos \tfrac{1}{2}(\phi_1-\phi_2)=e\cos \tfrac{1}{2}(\phi_1+\phi_2).$$

19. Prove that the normals to an ellipse at the extremities of a focal chord intersect on the straight line parallel to the major axis and passing through the mid-point of the chord.

Hence shew that the locus of the intersection of these normals is also an ellipse.

20. Tangents are drawn to the hyperbola $5x^2-4y^2=44$, where it is cut by the line $3x-4y=12$. Shew that they intersect at the point $(11/5, 11/3)$.

21. Prove that the tangents at the points $(a\sec\alpha, b\tan\alpha)$ and $(a\sec\beta, b\tan\beta)$ on the hyperbola $x^2/a^2-y^2/b^2=1$, intersect upon the curve

$$\frac{x^2}{a^2}-\left(\frac{y^2}{b^2}+1\right)\cos^2\tfrac{1}{2}(\alpha-\beta). \qquad \text{(L.U., Sc.)}$$

22. The ordinate of any point P on the hyperbola $x^2/a^2-y^2/b^2=1$ is produced to meet the asymptotes in Q, Q'; prove that

$$PQ\,.\,PQ'=b^2.$$

23. Prove that the product of the perpendicular distances of a point on the rectangular hyperbola $x^2-y^2=a^2$ from its asymptotes is constant. (Li.U.)

24. Shew that the line $x-3y+1=0$ is a tangent to the conic

$$2x^2+3xy-2y^2-3x+4y-1=0,$$

and find the coordinates of the point of contact.

25. Shew that the tangents from the origin to the circle

$$8x^2+8y^2-24x-64y+73=0$$

are at right angles. (L.U.)

26. Obtain the equation of the tangent and of the normal at the point (ξ, η) of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

If b, c be the parts of the axes of x and y intercepted by the tangent, shew that $b^2+c^2=a^2$. (L.U., Sc.)

27. The straight line $x+y=a$ makes an angle θ with the curve $x^3+y^3=c^3$ at a point of intersection (x, y). Prove that

$$\tan\theta=(x^2-y^2)/(x^2+y^2). \qquad \text{(L.U., Sc.)}$$

28. Find the equation of the tangent at the point (am^2, am^3) on the curve $ay^2=x^3$.

The tangent at a point P on this curve cuts the curve again at Q, and is normal to it there; find the value of m at P. (L.U.)

29. A circle of radius 2 in. rolls upon the outside of a circle of radius 4 in. Prove that the curve described by the middle point of a radius of the rolling circle is given by

$$x=6\cos\theta-\cos 3\theta, \quad y=6\sin\theta-\sin 3\theta.$$

Also prove that the tangent at the point $\theta=\frac{1}{4}\pi$ has the equation

$$y+3x=13\sqrt{2}. \qquad \text{(Br.U.)}$$

***30.** Prove that the line $y=mx+c$ is a normal to the curve $x^4+y^4=a^4$ if

$$\frac{c}{a}=\frac{m^{\frac{1}{3}}(1-m^{\frac{2}{3}})}{(1+m^{\frac{4}{3}})^{\frac{1}{4}}}. \qquad \text{(L.U., Sc.)}$$

***31.** Shew that if the line $lx+my=1$ touches the curve

$$(x/a)^n+(y/b)^n=1,$$

then $$(al)^{n/(n-1)}+(bm)^{n/(n-1)}=1.$$

***32.** If $T=0$ be the general equation of the tangent at (ξ, η) on the conic $F(x, y)=ax^2+2hxy+by^2+2gx+2fy+c=0$, shew that the equation of the tangents drawn from any point (x_1, y_1) not on the curve is

$$F(x, y)\,.\,F(x_1, y_1)=T^2,$$

and that the locus of the point (x_1, y_1) when the tangents to the conic are perpendicular to each other is

$$(ab-h^2)(x^2+y^2)+2x(bg-fh)+2y(af-gh)+c(a+b)-f^2-g^2=0,$$

and deduce the nature of the locus for each curve.

***33.** Find the equations of the tangent and normal at any point of the curve given by

$$x=3at/(1+t^3), \quad y=3at^2/(1+t^3). \qquad \text{(B.U.)}$$

34. Find the angle at which the circles

$$x^2+y^2=4 \quad \text{and} \quad x^2+y^2-6x+8y=0$$

intersect; find also the length of their common chord.

Shew that the equation of the circle through the common points of the two given circles and through the point (3, 2) is

$$x^2+y^2+27x-36y-22=0. \qquad \text{(L.U.)}$$

35. Find the coordinates of the points of intersection of the parabolas $y^2=8ax$ and $x^2=ay$, and determine the angles at which the curves intersect at these points.

36. Two curves $x^2y=a$, $xy=b$ pass through the point (3, 4); find the angle at which they intersect at that point.

37. Find the angles at which the curves $y^2=4ax$ and $cy^2=x^3$ cut each other at their points of intersection. (L.U., Sc.)

38. Shew that the circles

$$x^2+y^2+2gx+2fy+c=0, \quad x^2+y^2-a^2=0,$$

cut orthogonally if $c-a^2=0$, and touch if $(c+a^2)^2=4a^2(g^2+f^2)$. (D.U., Sc.)

39. Find the tangents at one of the common points of the circles $x^2+y^2=9$, $x^2+y^2-10x+9=0$, and shew that these circles cut at right angles. (Li.U.)

40. Prove that the conics $(x/a)^2+(y/b^2)=1$ and $(x/\alpha)^2+(y/\beta)^2=1$, intersect at right angles, provided $\alpha^2-a^2=\beta^2-b^2$. (M.U.)

41. Draw roughly on the same diagram, the curves $x^2/100+y^2/25=1$, $xy=24$.

Prove that they intersect at $(\pm 8, \pm 3)$ and $(\pm 6, \pm 4)$, and find the equation of the tangent to each curve at one of these points. What is the angle of intersection of the curves at this point? (D.U.)

42. Prove that the curves $y=x^3$ and $6y=7-x^2$ intersect at right angles. (B.U.)

***43.** Trace the curve $xy^2=(x+y)^2$, and find for what value of c the line $x=c$ cuts the curve at right angles. (Li.U.)

44. A body is projected with an initial velocity of 200 ft. per sec. at 65° to the horizontal. Find (i) when its speed is least, (ii) at what points its direction of motion is inclined 45° to the horizontal, and (iii) its range on a horizontal plane. (B.U.)

***45.** Prove that the line $3y+2x=1$ passes through a point common to both the curves $x^3+8xy-9y^3-1=0$, $x^2+12xy+y^2+19=0$; hence shew that the curves touch each other at this point, and find the equation of their common tangent.

CHAPTER XII

CURVATURE OF PLANE CURVES. PROPERTIES OF THE CONICS

85. Curvature. Let $\delta\psi$ be a small angle subtended at the centre of a circle of radius r by a small arc PQ of length δs, then from the definition of the circular measure of an angle, $\delta\psi = \delta s/r$, or $\delta\psi/\delta s = 1/r$. But each radius is normal to the circle, so that if tangents are drawn at P and Q, the difference of their slopes is $\delta\psi$. This angle is called the **angle of contingence,** and the limit of $\delta\psi/\delta s$ when Q ultimately becomes coincident with P, *i.e.* $d\psi/ds$, is called the **curvature of the circle.** In this case $d\psi/ds = 1/r$, so that the curvature of the circle is constant and equal to the reciprocal of its radius. In precisely the same way, the **curvature of any plane curve** is measured by $d\psi/ds$, where ψ is the slope of the curve and s the length of the arc. It is not constant in general, however, and is denoted by $1/\rho$, where ρ is called the **radius of curvature,** so that, at any point (x, y) on a continuous plane curve, $y = f(x)$, or $F(x, y) = 0$, the radius of curvature is given by the formula

$$\rho = \frac{ds}{d\psi}. \qquad \text{(87a)}$$

This is not always the most convenient formula to use, but another one is established in the following example, and other forms are given on pages 273 and 277.

Ex. 1. (*a*) *Prove that the curvature at any point of the curve $y = f(x)$ is given by*

$$\frac{d^2y}{dx^2}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}}$$

If ρ be the radius of curvature at a point P on a parabola, shew that $\rho^2 \propto SP^3$, where S is the focus. (L.U.)

(*b*) *Prove that the radius of curvature at any point on the cycloid, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, is twice the length of the normal.* (L.U.)

(*a*) Let ψ be the slope at any point on the curve $y = f(x)$, then

$$\frac{dy}{dx} = \tan\psi\,; \quad \therefore \quad \frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{d\psi}{dx},$$

so that

$$\frac{dx}{d\psi} = \sec^2\psi \Big/ \frac{d^2y}{dx^2} = (1 + \tan^2\psi) \Big/ \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \Big/ \frac{d^2y}{dx^2}.$$

If ρ = radius of curvature at the point (x, y), then

$$\rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \Big/ \frac{d^2y}{dx^2}\,;$$

$$\therefore \text{ the curvature } \frac{1}{\rho} = \frac{d^2y}{dx^2}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}}.$$

From this result a more convenient formula for ρ is thus

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \qquad \text{(87b)}$$

Let $P(x, y)$ be any point on the parabola $y^2 = 4ax$, then

$$\frac{dy}{dx} = \frac{2a}{y}, \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{2a}{y^2} \cdot \frac{dy}{dx} = -\frac{4a^2}{y^3}.$$

$\therefore$ From (87*b*),

$$\rho = -\left(1 + \frac{4a^2}{y^2}\right)^{\frac{3}{2}} \Big/ \frac{4a^2}{y^3} = -2(x + a)^{\frac{3}{2}}/a^{\frac{1}{2}}.$$

But since any point on the parabola is equidistant from the focus and from the directrix, and the latter is distant a from the vertex,

$$\therefore \; SP = x + a.$$

Hence $\rho^2/SP^3 = 4/a$, which is constant;

i.e. $\qquad \rho^2 \propto SP^3.$

(*b*) Since $x = a(\theta - \sin\theta)$,

$$\therefore\ \frac{dx}{d\theta} = a(1 - \cos\theta) = 2a\sin^2\frac{\theta}{2};$$

and
$$y = a(1 - \cos\theta) = \frac{dx}{d\theta},$$

$$\therefore\ \frac{dy}{d\theta} = a\sin\theta = 2a\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2},$$

so that
$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \cot\frac{\theta}{2}.$$

If ψ be the slope of the curve at any point, the gradient at this point

$$= \frac{dy}{dx} = \tan\psi = \cot\left(\frac{\pi}{2} - \psi\right);$$

$$\therefore\ \frac{\theta}{2} = \frac{\pi}{2} - \psi \quad\text{and}\quad \frac{d\theta}{d\psi} = -2.$$

But from (87*a*), $\rho = \dfrac{ds}{d\psi} = \dfrac{ds}{dx}\cdot\dfrac{dx}{d\theta}\cdot\dfrac{d\theta}{d\psi} = -2y\cdot\dfrac{ds}{dx}$,

and from (84*d*), if l = length of normal,

$$l = y\cdot\frac{ds}{dx}.$$

Hence, in magnitude, $\rho = 2l$.

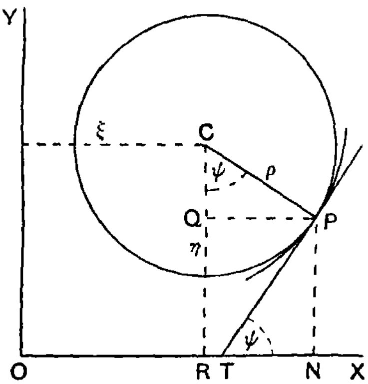

FIG 28. Centre of curvature.

86. Centre of Curvature. Involutes and Evolutes. Let *P* (Fig. 28) be any point on a plane curve, $y = f(x)$, and let ρ be the

radius of curvature at P, then if a circle of centre C and radius ρ be drawn to touch the curve at P, it will have the same tangent and the same curvature at P as the curve. This circle is therefore called the **circle of curvature,** and its centre, C, the **centre of curvature** at P on the curve $y=f(x)$.

Let PT be the common tangent at P which meets the x-axis in T, and ψ its slope; then since CP is perpendicular to TP,

$$\angle PCR = 90^\circ - \angle CPQ = \angle QPT = \angle PTN = \psi.$$

Let (ξ, η), (x, y) be the coordinates of C and P respectively; then since $CP=\rho$,

$$\xi = OR = ON - QP = x - \rho \sin \psi,$$

and
$$\eta = RC = NP + QC = y + \rho \cos \psi.$$

But
$$\tan \psi = \frac{dy}{dx},$$

so that
$$\sin \psi = \frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}} = \frac{dy}{dx}\,\frac{ds}{dx} = \frac{dy}{ds},$$

and
$$\cos \psi = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}} = \frac{dx}{ds}.$$

Hence the coordinates of the centre of curvature at any point (x, y) on a plane curve are given by

$$\left.\begin{aligned} \xi &= x - \rho \sin \psi = x - \frac{\dfrac{dy}{dx}\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}} = x - \frac{\dfrac{dy}{dx}\left(\dfrac{ds}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}, \\ \eta &= y + \rho \cos \psi = y + \frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}} = y + \frac{\left(\dfrac{ds}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}. \end{aligned}\right\} \quad \text{..........(88)}$$

The locus of the point (ξ, η), *i.e.* the centre of curvature for each point on a given curve, is called its **evolute.** If the evolute be regarded as the original curve, then the curve of which it is the evolute is called an **involute,** and it is easily shewn that there is an infinite number of involutes to every evolute.

Ex. 2. *Prove the following properties of the evolute of a plane curve,* $y=f(x)$:

(*a*) *The normal at any point* (x, y) *on* $y=f(x)$ *is a tangent to its evolute at the centre of curvature* (ξ, η) *of* $y=f(x)$ *at* (x, y).

(*b*) *The length of the arc of the evolute is equal to the difference between the radii of curvature of* $y=f(x)$ *at points corresponding to the extremities of the arc.*

(*c*) *The radius of curvature of the evolute at any point is* $\frac{d^2s}{d\psi^2}$, *where* $s=$ *an arc and* $\psi=$ *the slope at the corresponding point on* $y=f(x)$.

(*a*) Let $\psi=$ slope of $y=f(x)$ at any point (x, y), then the gradient of its normal at $(x, y) = -\cot\psi = -\frac{dx}{dy}$, by (65*c*).

But the gradient of the evolute at the point (ξ, η) corresponding to (x, y) is $\frac{d\eta}{d\xi}$.

Now from (88), $\xi = x - \rho\sin\psi$;

$$\therefore \frac{d\xi}{dx} = 1 - \frac{d\rho}{dx}\sin\psi - \rho\cos\psi\cdot\frac{d\psi}{dx} = 1 - \frac{d\rho}{dx}\cdot\frac{dy}{ds} - \frac{ds}{d\psi}\cdot\frac{dx}{ds}\cdot\frac{d\psi}{dx}$$

$$= -\frac{d\rho}{dx}\cdot\frac{dy}{ds}.$$

Similarly, since $\eta = y + \rho\cos\psi$, $\frac{d\eta}{dx} = \frac{d\rho}{dx}\cdot\frac{dx}{ds}$;

$$\therefore \frac{d\eta}{d\xi} = \frac{d\eta}{dx}\Big/\frac{d\xi}{dx} = -\frac{dx}{dy}.$$

$\therefore$ Gradient of tangent to evolute at $(\xi, \eta) =$ gradient of normal to original curve at (x, y).

$\therefore$ The normal to $y=f(x)$ is a tangent to its evolute.

Owing to this property, the evolute is often called the **envelope** of the normals to a curve.

(*b*) Let $d\sigma$ be an element of arc of the evolute, then

$$d\sigma^2 = d\xi^2 + d\eta^2 = \left\{\left(\frac{dy}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2\right\} d\rho^2 \text{ from } (a) \text{ above,}$$

$$= d\rho^2 ;$$

$$\therefore d\sigma = \pm d\rho,$$

the positive or negative sign being taken as σ increases or decreases as ρ increases. Hence if σ be the length of arc between two points

where the corresponding radii of curvature of the original curve are ρ_1, ρ_2, *this equation* on integration gives

$$\boldsymbol{\sigma = \rho_2 - \rho_1}\,; \qquad \text{.......................(89a)}$$

thus **the length of the arc of the evolute = difference between the radii of curvature of the original curve at points corresponding to the extremities of the arc.**

(*c*) Let ρ_e = radius of curvature of the evolute at any point, then the slope ψ' of the tangent to the evolute = slope of normal to original curve, $= \psi + \dfrac{\pi}{2}$;

$$\therefore \ d\psi' = d\psi,$$

so that
$$\rho_e = \frac{d\sigma}{d\psi'}, \text{ by}(87a).$$

$$= \frac{\mathbf{d}\rho}{\mathbf{d}\psi} = \frac{\mathbf{d}}{\mathbf{d}\psi}\cdot\left(\frac{\mathbf{ds}}{\mathbf{d}\psi}\right) = \frac{\mathbf{d^2s}}{\mathbf{d}\psi^2}. \qquad \text{.......................(89b)}$$

Ex. 3. *Find the centre of curvature and the evolute of the parabola* $y^2 = 4ax$, *and shew, by finding the equation of the tangent to the evolute, that it is identical with the equation of the normal to the parabola.*

Determine also the centre of curvature and the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$, *and prove that the length of the evolute corresponding to a quadrant of the ellipse is* $(a^3 - b^3)/ab$.

(i) For the parabola, $y^2 = 4ax$,

$$\frac{dy}{dx} = \frac{2a}{y} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3};$$

$$\therefore \ \xi = x + \frac{2a}{y}\left(1 + \frac{4a^2}{y^2}\right)\frac{y^3}{4a^2} = x + 2x + 2a = \mathbf{3x + 2a},$$

and
$$\eta = y - \left(1 + \frac{4a^2}{y^2}\right)\frac{y^3}{4a^2} = -xy/a = -\mathbf{2x^{\frac{3}{2}}/a^{\frac{1}{2}}}.$$

The locus of (ξ, η) will be given by eliminating x from these two equations. Squaring the second, and replacing x by its value from the first,

$$a\eta^2 = 4x^3 = 4(\xi - 2a)^3/27.$$

Replacing (ξ, η) by (x, y), the equation of the evolute becomes

$$\mathbf{27ay^2 = 4(x - 2a)^3}.$$

This curve is often called the **semi-cubical parabola.**

To find the equation of the tangent to this curve, change the origin to the point $(2a, 0)$ in order to simplify the algebra. The equation then becomes

$$27ay^2 = 4x^3.$$

Let the line $y = mx + n$ intersect this curve, then the abscissae of the common points will be given by

$$27a(mx+n)^2 = 4x^3,$$

or
$$4x^3 - 27am^2x^2 - 54mnax - 27an^2 = 0.$$

Two roots of this cubic will be equal if

$$(\tfrac{27}{8})^2a^2n^2(2n + 3m^3a)^2 = (\tfrac{9}{4})^3a^3m^3n^2(9m^3a + 8n), \text{ by (78)},$$

which reduces to
$$n(n + m^3a) = 0.$$

Assuming n is not, in general, zero,

$$n = -am^3,$$

and the tangent becomes $y = mx - am^3$.

Transferring back to the original origin, this equation becomes

$$\mathbf{y = mx - 2am - am^3},$$

which, by (80), is the equation to the normal of the parabola.

(ii) For the ellipse, $x^2/a^2 + y^2/b^2 = 1$,

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3};$$

$$\therefore \ \xi = x - \frac{b^2x}{a^2y}\left(1 + \frac{b^4x^2}{a^4y^2}\right)\frac{a^2y^3}{b^4} = x(1 - y^2/b^2 - b^2x^2/a^4)$$

$$= x(x^2/a^2 - b^2x^2/a^4) = \mathbf{(a^2 - b^2)x^3/a^4},$$

and
$$\eta = y - \left(1 + \frac{b^4x^2}{a^4y^2}\right)\frac{a^2y^3}{b^4} = y(1 - a^2y^2/b^4 - x^2/a^2)$$

$$= y(y^2/b^2 - a^2y^2/b^4) = \mathbf{(b^2 - a^2)y^3/b^4}.$$

From these values,

$$x^2/a^2 = (\xi a)^{\frac{2}{3}}(a^2 - b^2)^{-\frac{2}{3}} \quad \text{and} \quad y^2/b^2 = (\eta b)^{\frac{2}{3}}(a^2 - b^2)^{-\frac{2}{3}}.$$

$\therefore$ By addition,

$$\{(\xi a)^{\frac{2}{3}} + (\eta b)^{\frac{2}{3}}\}(a^2 - b^2)^{-\frac{2}{3}} = x^2/a^2 + y^2/b^2 = 1.$$

Hence, writing (x, y) for (ξ, η), the equation of the evolute is

$$\mathbf{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}}.$$

If ρ_a, ρ_b be the radii of curvature of the ellipse at the extremities of the major and minor semi-axes respectively, then for any point (x, y)

$$\rho = -\left(1+\frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}} \cdot \frac{a^2y^3}{b^4} = -(a^4y^2+b^4x^2)^{\frac{3}{2}}/(a^4b^4);$$

$$\therefore \ \rho_a = -(a^2b^4)^{\frac{3}{2}}/(a^4b^4) = -b^2/a,$$

and $\quad \rho_b = -(a^4b^2)^{\frac{3}{2}}/(a^4b^4) = -a^2/b.$

$\therefore$ By (89a), the length of the evolute corresponding to a quadrant of the ellipse $=\rho_a - \rho_b = a^2/b - b^2/a = (a^3-b^3)/(ab)$.

87. Curvature at the origin—Newton's Method. When a curve touches either of the axes at the origin, a very simple method of finding ρ may be employed which needs no differentiation. The process is illustrated in the following example.

Ex. 4. *Prove that the radius of curvature at the origin of a curve which touches the axis of x at the origin is*

$$\underset{x\to 0}{Lt}\left(\frac{x^2}{2y}\right).$$

Find the radii of curvature of the curve $y=x^2(x-3)$ at the points where the tangent is parallel to the axis of x. (L.U.)

(a) Let $P(x, y)$ be a point on a curve close to the origin O, (Fig. 29), which will ultimately be coincident with O; then if C be the centre of curvature at O, $OC=\rho_0$, and the diameter OQ of the circle of curvature is perpendicular to the tangent OX to the curve at the origin. If HP be perpendicular to OQ, $HP=x$, $OH=y$, and

$$HP^2 = PC^2 - HC^2 = PC^2 - (OC-OH)^2$$
$$= OH(2OC - OH)$$
$$= OH(OQ - OH) = OH\ .\ HQ,$$

i.e. $\quad x^2 = y\ .\ HQ,$

or $\quad HQ = x^2/y.$

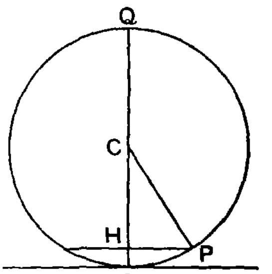

FIG. 29. Curvature at the origin.

When P becomes coincident with O, HQ becomes $OQ=2\rho_0$; hence

$$2\rho_0 = \underset{x\to 0}{\text{Lt}}\,(x^2/y),$$

i.e. $$\rho_0 = \tfrac{1}{2}\underset{x\to 0}{\text{Lt}}\,(x^2/y). \quad \text{.........................(87c)}$$

Similarly, if the curve touches the axis of y at the origin, the radius of curvature is $\frac{1}{2}\operatorname*{Lt}_{y\to 0}(y^2/x)$.

This method is due to Newton, and is therefore known as the Newtonian method.

(b) Since the tangents have to be parallel to the axis of x, $\frac{dy}{dx}=0$, *i.e.* $3x^2-6x=0$, giving $x=0$ or 2; hence the two tangents are $y=0$ and $y+4=0$. The first shews that the curve touches the axis of x at the origin, so that the radius of curvature there is

$$\tfrac{1}{2}\operatorname*{Lt}_{x\to 0}.(x^2/y)=\tfrac{1}{2}\operatorname*{Lt}_{x\to 0}x^2/\{x^2(x-3)\}=\tfrac{1}{2}\operatorname*{Lt}_{x\to 0}1/(x-3)=-1/6.$$

To apply the Newtonian method to the second point, change the origin to $(2, -4)$; the transformed equation is then

$$y-4=(x+2)^2(x-1),$$

or $y=x^2(x+3)$, and the tangent is now the x-axis touching the curve at the origin; hence the radius of curvature there is

$$\tfrac{1}{2}\operatorname*{Lt}_{x\to 0}(x^2/y)=\tfrac{1}{2}\operatorname*{Lt}_{x\to 0}1/(x+3)=1/6.$$

To verify these values, the general value of ρ at any point (x, y) is easily found by (87b) to be

$$\{1+9x^2(x-2^2)\}^{\frac{3}{2}}/\{6(x-1)\},$$

so that, at the points $(0, 0)$ and $(2, -4)$, this value becomes $-1/6$ and $1/6$ respectively, which agrees with the values already found.

88. Polar Formulae. Let P, Q be two points on a curve (Fig. 30) whose polar coordinates are (r, θ), $(r+\delta r, \theta+\delta\theta)$, O being the pole and OX the initial line. Draw PR perpendicular to OQ; then the nearer Q approaches P, the more nearly do OP and OR become equal, and the less the difference becomes between the chord PQ and the arc PQ. Hence when $\delta\theta$ is very small,

$$RP=OP\,.\,\delta\theta=r\,.\,\delta\theta$$

very nearly, and $RQ=OQ-OR=OQ-OP=\delta r$, and if arc $PQ=\delta s$, then $\delta s^2=(\text{chord } PQ)^2=RP^2+RQ^2=r^2\delta\theta^2+\delta r^2$;

$$\therefore \left(\frac{\delta s}{\delta\theta}\right)^2=r^2+\left(\frac{\delta r}{\delta\theta}\right)^2,$$

and proceeding to the limit,

$$\left.\begin{aligned}\left(\frac{ds}{d\theta}\right)^2 &= r^2 + \left(\frac{dr}{d\theta}\right)^2.\\ \text{Similarly,}\qquad \left(\frac{ds}{dr}\right)^2 &= 1 + r^2\left(\frac{d\theta}{dr}\right)^2.\end{aligned}\right\} \quad \text{..........................(90a)}$$

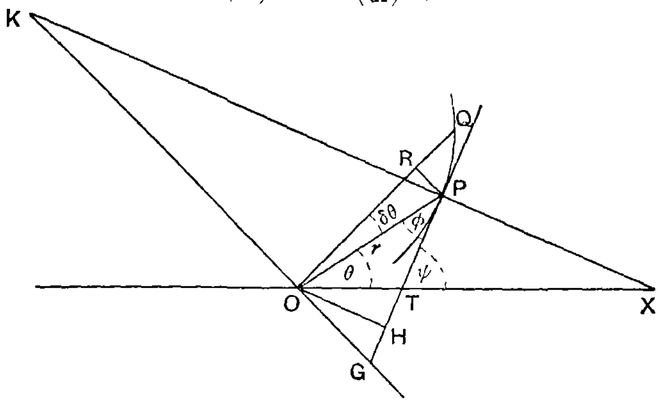

FIG. 30. Polar formulae.

If KOG be drawn perpendicular to OP to meet the tangent and normal at G and K respectively, then OG is called the **Polar Subtangent** and OK the **Polar Subnormal.** Expressions for these are determined in the following example.

Ex. 5. *If ϕ be the angle between the radius vector and the tangent at any point on a plane curve, and p be the perpendicular from the pole to the tangent, shew*

(*a*) *by direct transformation from Cartesian to polar coordinates, that* $\tan \phi = r \,.\, d\theta/dr$, *and find expressions for* $\sin \phi$ *and* $\cos \phi$;

(*b*) *that* $1/p^2 = u^2 + (du/d\theta)^2$, *where* $u = 1/r$.

Hence find expressions for the polar subtangent, subnormal and the radius of curvature.

Prove for the curve $r = a(1 + \cos\theta)$, *that* ρ^4 *varies as the product of the subtangent and normal, and find the value of* p.

(*a*) Let ψ = slope of tangent at any point P (Fig. 30) on the curve, and (x, y), (r, θ) the corresponding Cartesian and polar coordinates of P, then $x = r\cos\theta$, $y = r\sin\theta$;

$\therefore\ dx = dr \,.\, \cos\theta - r\sin\theta \,.\, d\theta$, and $dy = dr \,.\, \sin\theta + r\cos\theta \,.\, d\theta$;

$$\begin{aligned}\therefore\ \tan\psi = dy/dx &= (dr\sin\theta + r\cos\theta \,.\, d\theta)/(dr\cos\theta - r\sin\theta \,.\, d\theta)\\ &= (dr \,.\, \tan\theta + r \,.\, d\theta)/(dr - r\tan\theta \,.\, d\theta).\end{aligned}$$

But from Fig. 30, $\psi = \theta + \phi$;

$\therefore \tan\psi = \tan(\theta+\phi) = (\tan\theta + \tan\phi)/(1 - \tan\theta \,.\, \tan\phi)$.

Equating these two expressions for $\tan\psi$,

$$(dr\tan\theta + r\,.\,d\theta)(1 - \tan\theta\,.\,\tan\phi) = (dr - r\tan\theta\,.\,d\theta)(\tan\theta + \tan\phi),$$

which on multiplying out gives

$$\tan\phi = r\,.\,d\theta/dr.$$

Squaring, $\tan^2\phi = \sec^2\phi - 1 = r^2(d\theta/dr)^2$, so that

$$\sec^2\phi = 1 + r^2(d\theta/dr)^2 = (ds/dr)^2, \quad \text{from } (90a);$$

$$\therefore \cos\phi = dr/ds.$$

Finally, $\qquad \sin\phi = \cos\phi\,.\,\tan\phi = r\,.\,d\theta/ds.$

Hence

$$\sin\phi = r\frac{d\theta}{ds}, \quad \cos\phi = \frac{dr}{ds}, \quad \tan\phi = r\frac{d\theta}{dr}. \quad \ldots\ldots\ldots(90b)$$

(*b*) Let the perpendicular from O to the tangent meet it at H, then

$$p = OH = r\sin\phi = r^2\,.\,\frac{d\theta}{ds};$$

$$\therefore \; 1/p^2 = u^4(ds/d\theta)^2 = u^4\left\{\frac{1}{u^2} + (dr/d\theta)^2\right\}$$

$$= u^2 + u^4\left(\frac{d}{d\theta}\cdot\frac{1}{u}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2;$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^4}\left(\frac{ds}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2. \quad \ldots\ldots\ldots(90c)$$

Again, from Fig. 30,

$$\left.\begin{aligned} OG &= \text{polar subtangent} = r\tan\phi = r^2\,.\,\frac{d\theta}{dr}, \\ OK &= \text{polar subnormal} = r\cot\phi = \frac{dr}{d\theta}. \end{aligned}\right\} \quad \ldots\ldots\ldots(90d)$$

To find the value of ρ in terms of polar coordinates, from $(87a)$,

$$1/\rho = d\psi/ds.$$

But $\psi = \theta + \phi$; $\;\therefore\; d\psi = d\theta + d\phi$;

$$\therefore \; 1/\rho = d\theta/ds + d\phi/ds$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right).$$

From (90*b*), $\cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}$;

$$\therefore\ -\operatorname{cosec}^2\phi\cdot\frac{d\phi}{d\theta} = -\frac{1}{r^2}\cdot\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r}\cdot\frac{d^2r}{d\theta^2};$$

$$\therefore\ \frac{d\phi}{d\theta} = \left\{\left(\frac{dr}{d\theta}\right)^2 - r\cdot\frac{d^2r}{d\theta^2}\right\}\Big/\left(\frac{ds}{d\theta}\right)^2$$

on putting in the value of $\operatorname{cosec}^2\phi$.

Hence, by substitution,

$$\frac{1}{\rho} = \left\{\left(\frac{ds}{d\theta}\right)^2 + \left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}\Big/\left(\frac{ds}{d\theta}\right)^3$$

$$= \left\{r^2 + 2\cdot\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}\Big/\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}.$$

A simpler relation than this may easily be found involving p. Thus, since $p = r\sin\phi$,

$$dp/dr = \sin\phi + r\cos\phi\,.\,d\phi/dr = r(d\theta/ds + d\phi/ds), \text{ from (90b)},$$
$$= r\,.\,d\psi/ds, \text{ since } \psi = \theta + \phi,$$
$$= r/\rho.$$

Hence the two polar values of ρ are given by

$$\boldsymbol{\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} = r\cdot\frac{dr}{dp}.} \qquad \textbf{(87d)}$$

For the curve $r = a(1 + \cos\theta)$,

$$\frac{dr}{d\theta} = -a\sin\theta, \quad\text{and}\quad \frac{d^2r}{d\theta^2} = -a\cos\theta;$$

$$\therefore\ \rho = (r^2 + a^2\sin^2\theta)^{\frac{3}{2}}/(r^2 + 2a^2\sin^2\theta + ar\cos\theta)$$
$$= (2ar)^{\frac{3}{2}}/(3ar) = (8ar)^{\frac{1}{2}}/3.$$

But product of subtangent and subnormal for any curve

$$= r^2\cdot\frac{d\theta}{dr}\cdot\frac{dr}{d\theta} = r^2;$$

$$\therefore\ \rho^4/(\text{this product}) = (64a^2r^2)/(81r^2) = 64a^2/81,$$

which is constant.

$\therefore$ ρ^4 **varies as the product of the polar subtangent and subnormal.**

Finally, from (90c),

$$1/p^2 = 1/r^2 + (1/r^4)(dr/d\theta)^2 = 1/r^2 + a^2 \sin^2\theta/r^4$$
$$= 1/r^2 + (2ar - r^2)/r^4 = 2a/r^3;$$
$$\therefore \ \mathbf{p} = \pm\sqrt{\mathbf{r}^3/(2\mathbf{a})}.$$

89. Some General Properties of the Conic. Some important geometrical properties of the conic will now be discussed, but before dealing with the separate curves, those characteristics common to all conics will first be considered.

Ex. 6. *Find the polar equation of the straight line passing through two given points. Deduce the equations of the tangent and asymptote to the general conic $l/r = 1 - e \cos\theta$.*

Hence shew that

(*a*) *The line joining the point of intersection of two tangents to the focus bisects the angle formed by the radii vectores drawn from the points of contact;*

(*b*) *The tangents drawn at the extremities of a focal chord intersect on the directrix;*

(*c*) *The segment of a tangent intercepted between the point of contact and the directrix subtends a right angle at the focus.*

Let PT (Fig. 31) be any straight line referred to O as pole and OX as initial line, the coordinates of any point on it being (r, θ). Let OQ be the perpendicular from O on the line, then its length will be constant. Denote this by p, and suppose $\angle XOQ = \alpha$, which is also constant, then

$$\mathbf{p = r \cos(\alpha - \theta)}. \qquad \text{(91a)}$$

This is, therefore, the polar equation of the straight line.

Let (r_1, θ_1), (r_2, θ_2) be two given points on the line, then

$$p = r_1 \cos(\alpha - \theta_1) = r_2 \cos(\alpha - \theta_2) = r \cos(\alpha - \theta);$$
$$\therefore \ (r_1 \sin\theta_1 - r_2 \sin\theta_2) \tan\alpha = r_2 \cos\theta_2 - r_1 \cos\theta_1,$$

and $$(r_1 \sin\theta_1 - r \sin\theta) \tan\alpha = r \cos\theta - r_1 \cos\theta_1.$$

Hence, by division, to eliminate α, and simplification, the polar equation of the straight line passing through the given points becomes

$$\mathbf{rr_1 \sin(\theta - \theta_1) - rr_2 \sin(\theta - \theta_2) + r_1r_2 \sin(\theta_1 - \theta_2) = 0}. \qquad \text{(91b)}$$

Suppose now that (r_1, θ_1), (r_2, θ_2) lie on the conic

$$l/r = 1 - e\cos\theta,$$

then $$l/r_1 = 1 - e\cos\theta_1, \quad \text{and} \quad l/r_2 = 1 - e\cos\theta_2.$$

Substituting in (91*b*) these values of r_1 and r_2,

$$r(1-e\cos\theta_2)\sin(\theta-\theta_1)-r(1-e\cos\theta_1)\sin(\theta-\theta_2)$$
$$+l\sin(\theta_1-\theta_2)=0\ ;$$

$$\therefore\ 2r\sin\tfrac{1}{2}(\theta_2-\theta_1)\cos\{\theta-\tfrac{1}{2}(\theta_1+\theta_2)\}+er\cos\theta\sin(\theta_1-\theta_2)$$
$$+l\sin(\theta_1-\theta_2)=0.$$

If θ_1 is not equal to θ_2, this equation may be divided out by $\sin\frac{1}{2}(\theta_1-\theta_2)$, then

$$er\cos\theta\ .\ \cos\tfrac{1}{2}(\theta_1-\theta_2)-r\cos\{\theta-\tfrac{1}{2}(\theta_1+\theta_2)\}=-l\cos\tfrac{1}{2}(\theta_1-\theta_2),$$

which, on division by $\cos\frac{1}{2}(\theta_1-\theta_2)$, gives

$$l/r=\sec\tfrac{1}{2}(\theta_1-\theta_2)\cos\{\theta-\tfrac{1}{2}(\theta_1+\theta_2)\}-\cos\theta.\ \ldots\ldots\ldots\ldots(92a)$$

This is, therefore, the **polar equation of a chord of the conic.**

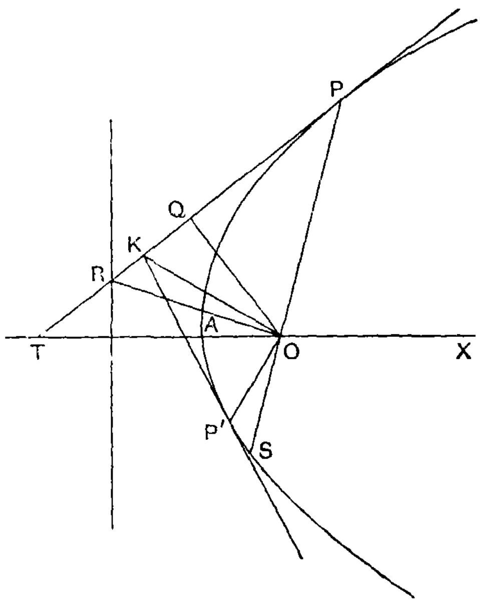

FIG. 31. General properties of the conic.

For this chord to become a tangent at a point whose vectorial angle is α, $\theta_1=\theta_2=\alpha$, and the equation becomes

$$l/r=\cos(\theta-\alpha)-e\cos\theta,\ \ldots\ldots\ldots\ldots(92b)$$

which is thus the **polar equation of a tangent to the conic.**

For an asymptote, the point a must be at infinity; this will be the case when $r=\infty$ in the equation of the conic, *i.e.* when

$$e \cos a = 1.$$

Hence the equation of the asymptote will be the eliminant of a between this equation and that of the tangent, viz.

$$l/r = \cos(\theta - a) - e\cos\theta = \cos\theta\cos a + \sin\theta\sin a - e\cos\theta;$$

$$\therefore\ el/r = (1-e^2)\cos\theta + e\sin\theta\,.\,\sin a, \text{ on putting } \cos a = 1/e;$$

$$\therefore\ \{el/r - (1-e^2)\cos\theta\}^2 = e^2\sin^2\theta\sin^2 a$$
$$= e^2\sin^2\theta(1-\cos^2 a)$$
$$= e^2\sin^2\theta(1-1/e^2)$$
$$= (e^2-1)\sin^2\theta.$$

i.e. $$el/r - (1-e^2)\cos\theta = \pm\sqrt{e^2-1}\,.\,\sin\theta. \quad \text{.................(92c)}$$

These are, therefore, the **polar equations of the asymptotes,** and it is evident that they are real only when $e>1$, *i.e.* the only conic having asymptotes is the hyperbola, as has already been seen in § 83.

As the directrix will be needed later, its equation will now be found.

From (91a), the equation of any straight line is

$$p = r\cos(a-\theta).$$

In the case of the directrix, $p = OB = l/e$, and $a=\pi$;

$\therefore$ the **equation of the directrix** becomes

$$l/r + e\cos\theta = 0. \quad \text{..........................(92d)}$$

The required properties of the conic may now be established.

(a) Let a, β be the vectorial angles of any two points, P, P', on the conic, then by (92b) the equations of the tangents at these points are

$$l/r = \cos(\theta - a) - e\cos\theta,$$

$$l/r = \cos(\theta - \beta) - e\cos\theta.$$

These will intersect when

$$\cos(\theta - a) - \cos(\theta-\beta) = 0,$$

i.e. $$\sin\tfrac{1}{2}(a-\beta)\,.\,\sin\tfrac{1}{2}(2\theta - a - \beta) = 0, \text{ by (14).}$$

Since α is not equal to β,

$$\therefore\ \sin \tfrac{1}{2}(2\theta - \alpha - \beta) = 0,$$

so that $$\theta = \tfrac{1}{2}(\alpha + \beta),$$

i.e. $\angle XOK = \tfrac{1}{2}(\angle POX + \text{reflex} \angle P'OX) = \tfrac{1}{2}\angle P'OP + \angle POX.$

Take away the common angle POX, then

$$\angle POK = \tfrac{1}{2}\angle P'OP\,;$$

$$\therefore\ KO \text{ bisects } \angle P'OP,$$

so that **intersecting tangents to a conic subtend equal angles at the focus.**

(*b*) Let POS be a focal chord, and α the vectorial angle of P, then the vectorial angle of S is $\pi + \alpha$. Let (r_1, θ_1) be the coordinates of R, the point of intersection of the tangents at P and S, then by (92*b*),

$$l/r_1 = \cos(\theta_1 - \alpha) - e\cos\theta_1,$$

$$l/r_1 = \cos(\theta_1 - \pi - \alpha) - e\cos\theta_1$$

$$= -\cos(\theta_1 - \alpha) - e\cos\theta_1.$$

Hence, by addition, $\quad l/r_1 = -e\cos\theta_1,$

which shews, by (92*d*), that (r_1, θ_1) lies on the directrix; hence: **Tangents to a conic at the extremities of a focal chord intersect on the directrix.**

(*c*) Since the tangent at P intersects the directrix at (r_1, θ_1),

$$\therefore\ l/r_1 = \cos(\theta_1 - \alpha) - e\cos\theta_1,$$

and $$l/r_1 = -e\cos\theta_1.$$

$\therefore$ By subtraction, $\quad \cos(\theta_1 - \alpha) = 0,$

$$\therefore\ \theta_1 - \alpha = \frac{\pi}{2},$$

i.e. $$\angle ROX - \angle POX = \pi/2,$$

or $$\angle ROP = \text{a right angle}.$$

Hence, **the segment of a tangent intercepted between the point of contact and the directrix subtends a right angle at the focus.**

90. Properties of the Parabola. The more important characteristic properties of the parabola will now be dealt with.

Ex. 7. *In the parabola, shew that*

(*a*) *The subtangent is bisected at the vertex.*

(*b*) *The tangent bisects the angle between the focal distance and the perpendicular to the directrix from the point of contact.*

(*c*) *Tangents at the extremities of a focal chord intersect at right angles on the directrix.*

(*d*) *The diameter through the point of intersection of two tangents bisects the chord joining their points of contact.* (L.U.)

(*a*) Let PT be any tangent to the curve at P (Fig. 32), PN the ordinate, O the vertex, F the focus and AH the directrix.

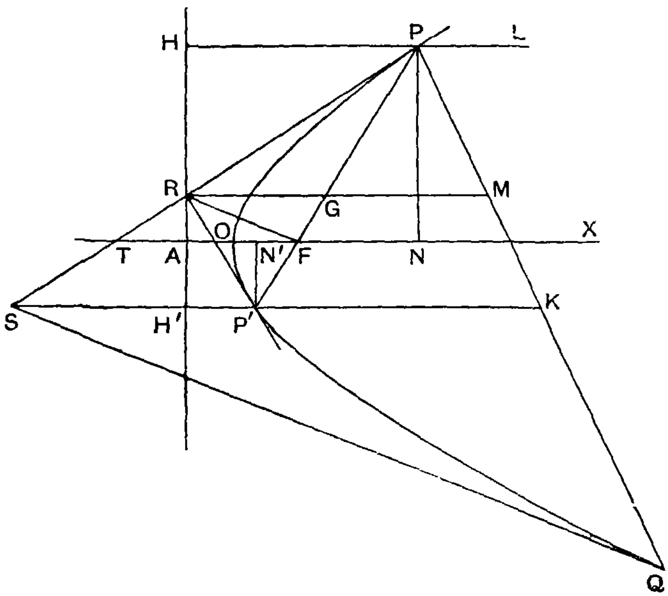

FIG. 32. Properties of the parabola.

If the coordinates of P are (ξ, η), then the equation of PT referred to O as origin and OX as x-axis, is, by (80),

$$y\eta = 2a(x+\xi), \text{ where } AO = OF = a.$$

At T, $y=0$, $\therefore\ x = -\xi$,

so that $TO = ON$;

$$\therefore\ O \text{ is the mid-point of } TN,$$

i.e. **the subtangent is bisected by the vertex of the parabola.**

(*b*) From Fig. 32, $FP = PH$, where PH is perpendicular to AH,

$$= AN = AO + ON = OF + TO = TF;$$

$$\therefore \angle FPT = \angle PTF = \angle TPH;$$

$$\therefore TP \text{ bisects the angle } FPH,$$

i.e. **the tangent bisects the angle between the distances of the point of contact from the focus and the directrix.**

It follows from this fact that the normal at P bisects the angle between PH and FP produced.

(*c*) Let the tangents at P, P' (Fig. 32), the extremities of a focal chord, intersect on the directrix at R. That these tangents do intersect on the directrix has already been proved in (*b*) of Example 6. Also from (*c*) of that example, RF is at right angles to PP'. Let $P'H'$ be drawn perpendicular to $H'H$, then

$$\angle HRP = \pi/2 - \angle HPR = \pi/2 - \angle RPF = \angle PRF.$$

Similarly, $\qquad \angle H'RP = \angle P'RF;$

$$\therefore \angle PRP' = \angle PRF + \angle P'RF = \tfrac{1}{2}(\angle H'RF + \angle HRF) = 90^\circ,$$

i.e. **tangents at the end of a focal chord intersect at right angles on the directrix.**

(*d*) Let (x_1, y_1), (x_2, y_2) be the extremities of any chord, then the equations of the tangents at these points are

$$yy_1 = 2a(x + x_1), \quad yy_2 = 2a(x + x_2).$$

Hence the ordinate at their point of intersection is given by

$$y = 2a(x_1 - x_2)/(y_1 - y_2),$$

and this will be the equation of the diameter, *i.e.* the line parallel to the axis through the point of intersection.

Since (x_1, y_1), (x_2, y_2) lie on the parabola, $x_1 = y_1^2/4a$, $x_2 = y_2^2/4a$.

Hence, by substitution in the above equation,

$$y = \tfrac{1}{2}(y_1^2 - y_2^2)/(y_1 - y_2) = \tfrac{1}{2}(y_1 + y_2),$$

since y_1 is not equal to y_2.

This result shews that the diameter passes through the mid-point of the chord; hence,

The diameter through the point of intersection of two tangents bisects the chord joining their points of contact.

Ex. 8. (*a*) *From any point* $(a\mu^2, 2a\mu)$ *on a parabola a focal chord and a normal chord are drawn. Find the coordinates of the points where these chords meet the curve again, and shew that the diameter through the point of intersection of the tangents at the extremities of the normal chord passes through the other extremity of the focal chord.*

(*b*) *If two tangents at the ends of a focal chord PP' meet in R, and PQ is a chord which is normal to the curve at P, then $PQ = 4RP'$.* (L.U.)

(*a*) Let PFP' be a focal chord and let the coordinates of P' be $(a\mu_1^2, 2a\mu_1)$, then the equation of PP' is, after division by $a(\mu_1 - \mu)$,

$$2(x - a\mu^2) = (y - 2a\mu)(\mu_1 + \mu).$$

But PP' passes through F, the coordinates of which are $(a, 0)$;

$$\therefore\ 2a(1 - \mu^2) = -2a\mu(\mu_1 + \mu),$$

from which $$\mu_1 = 1/\mu.$$

Hence **the coordinates of the extremities of a focal chord of a parabola are**

$$(\mathbf{a\mu^2, 2a\mu}),\quad (\mathbf{a/\mu^2, -2a/\mu}). \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{93a})$$

Again, let the normal at P cut the curve again at Q, then, by § 80, the equation of PQ is, after division by $2a$,

$$y - 2a\mu + \mu(x - a\mu^2) = 0.$$

Let $(a\mu_2^2, 2a\mu_2)$ be the coordinates of Q; then

$$2a(\mu_2 - \mu) + a\mu(\mu_2^2 - \mu^2) = 0,$$

from which, since μ_2 is not equal to μ,

$$\mu_2 = -(\mu^2 + 2)/\mu.$$

Hence **the normal to a parabola at the point $(\mathbf{a\mu^2, 2a\mu})$ meets the curve again at the point whose coordinates are**

$$\{\mathbf{a(\mu^2+2)^2/\mu^2, \quad -2a(\mu^2+2)/\mu}\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{93b})$$

Let the tangents at P and Q intersect at S, and let SK be drawn parallel to OX meeting PQ in K. Then, by (*d*) of Ex. 7, p. 282, K is the mid-point of PQ.

Let y be the ordinate of K, then

$$\begin{aligned} y &= \tfrac{1}{2}(2a\mu + 2a\mu_2) = a(\mu + \mu_2) \\ &= a\{\mu - (\mu^2+2)/\mu\}, \text{ by } (93b), \\ &= -2a/\mu, \end{aligned}$$

which is equal to the ordinate at P, by (93*a*); hence

If a normal chord and a focal chord be drawn from any point on a parabola, the diameter through the point of intersection of the tangents at the extremities of the normal chord not only bisects that chord, but also passes through the other extremity of the focal chord.

From Ex. 6 (*b*), p. 278, R lies on the directrix, and if the diameter through R cuts PP' in G, and PQ in M, then $P'G = GP$.

Further, since GM and $P'K$ are parallel, and G is the mid-point of PP',

$\therefore$ M is the mid-point of PK, *i.e.* $PQ = 4 \,.\, MP$.

But RP' is parallel to PK, each being perpendicular to RP; $\therefore$ $P'RMK$ is a parallelogram, and

$$P'R = MK = \tfrac{1}{4} \,.\, PQ;$$

$$\therefore\ PQ = 4 \,.\, P'R.$$

It should be observed that $PMP'R$ is a rectangle, so that PM is perpendicular to the normal $P'Q$.

91. Properties of the Ellipse. Some important geometrical properties of the ellipse will now be considered. It should be noted that from the fact that the sum of the focal distances of any point on the curve is constant (see Ex. 12, p. 223), this is often used as a convenient definition of an ellipse in practice.

Ex. 9. *Discuss the chief properties of the ellipse*

$$x^2/a^2 + y^2/b^2 = 1.$$

(i) Let AA', BB' (Fig. 33) be the axes of the ellipse, O its centre, F, F' its foci, then from (73*a*) $OF = OF' = ae$, where e is the eccentricity of the curve.

$$\therefore\ A'F \,.\, FA = (a + ae)(a - ae) = a^2(1 - e^2) = b^2.$$

$\therefore$ **Product of distances of extremities of major axis from a focus = square on semi-minor axis.**

Also $$FB^2 = b^2 + OF^2 = b^2 + a^2e^2 = a^2;$$

$$\therefore\ BF = a,$$

so that **the distance of either extremity of the minor axis from a focus = semi-major axis.**

(ii) Let $H'PT$ be a tangent at any point $P(x_1, y_1)$, then its equation is

$$xx_1/a^2 + yy_1/b^2 = 1,$$

and if it intersects the x-axis at T, and N be the ordinate of P, then at T, $y=0$, and $xx_1=a^2$,

i.e. $$ON \,.\, OT = a^2.$$

Let $F'H'$, FH be the perpendiculars on the tangent from the foci; then, taking the equation of the tangent in the form

$$y = mx + \sqrt{a^2m^2 + b^2},$$

the equation of FH will be of the form $y = -x/m + n$.

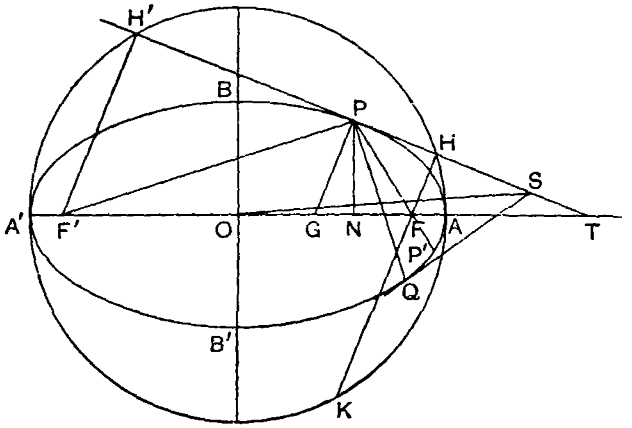

FIG. 33. Properties of the ellipse.

Since F is the point $(ae, 0)$, $\therefore\ n = ae/m$, so that

$$my = ae - x$$

is the equation of FH.

Let (x, y) be the point of intersection H, then, if m be eliminated from the equations

$$y - mx = \sqrt{m^2a^2 + b^2} \quad \text{and} \quad my + x = ae,$$

the resulting equation will represent the locus of H.

To effect this elimination, square each equation and add, then

$$(my + x)^2 + (y - mx)^2 = a^2e^2 + m^2a^2 + b^2,$$

i.e. $$(m^2 + 1)(x^2 + y^2) = a^2(m^2 + 1);$$

$$\therefore\ x^2 + y^2 = a^2,$$

since m is real,

i.e. **the locus of the feet of the perpendiculars from the foci to a tangent is the auxiliary circle.**

Let HF be produced to meet the auxiliary circle again at K, then since FH, $F'H'$ are parallel and equidistant from the centre O, and A, A' is a diameter of the circle,

$$FK = F'H'$$

and $$KF \cdot FH = A'F \cdot FA = b^2, \text{ as proved above,}$$

i.e. $$FH \cdot F'H' = b^2,$$

or, **the product of the perpendiculars from the foci to any tangent is constant, being equal to the square on the semi-minor axis.**

(iii) Let PQ be any chord of the ellipse joining the points $P(x_1, y_1)$, $Q(x_2, y_2)$, then the equations of the tangents at these points are

$$xx_1/a^2 + yy_1/b^2 = 1, \quad xx_2/a^2 + yy_2/b^2 = 1\ ;$$

and if λ be a constant, the equation

$$xx_1/a^2 + yy_1/b^2 - 1 + \lambda(xx_2/a^2 + yy_2/b^2 - 1) = 0$$

will represent any straight line passing through their point of intersection. For the line to be a diameter, it must pass through the origin, and the condition for this is

$$-1 - \lambda = 0, \text{ giving } \lambda = -1\ ;$$

$\therefore$ the equation of the diameter through the point of intersection of the tangents at P and Q is

$$\frac{x_1 - x_2}{a^2} \cdot x + \frac{y_1 - y_2}{b^2} \cdot y = 0.$$

Now let (ξ, η) be the mid-point of PQ, then $2\xi = x_1 + x_2$ and $2\eta = y_1 + y_2$. From the above equation,

$$\begin{aligned} y &= -\{b^2(x_1 - x_2)x\}/\{a^2(y_1 - y_2)\} \\ &= -\{b^2(x_1^2 - x_2^2)(y_1 + y_2)x\}/\{a^2(y_1^2 - y_2^2)(x_1 + x_2)\}. \end{aligned}$$

But since (x_1, y_1), (x_2, y_2) lie on the curve,

$$b^2(x_1^2 - x_2^2) = a^2b^2 - a^2y_1^2 - a^2b^2 + a^2y_2^2 = a^2(y_2^2 - y_1^2)\ ;$$

$$\therefore\ y = (y_1 + y_2)x/(x_1 + x_2) = \eta x/\xi,$$

which shews that (ξ, η) lies on the diameter ; hence,

Tangents at the extremities of a chord of an ellipse intersect on the diameter bisecting that chord.

(iv) The straight line, $y = mx + \sqrt{a^2m^2 + b^2}$, is a tangent to the ellipse for all real values of m ; rearranging this equation and squaring,

$$(y - mx)^2 = a^2m^2 + b^2,$$

or $$(x^2 - a^2)m^2 - 2xym + y^2 - b^2 = 0.$$

This, being a quadratic in m, shews that two tangents can be drawn from any point outside the ellipse.

Let m_1, m_2 be the roots of the quadratic, then

$$m_1 m_2 = (y^2 - b^2)/(x^2 - a^2).$$

If the two tangents be at right angles to each other, then

$$m_1 m_2 = -1;$$

$$\therefore \ (y^2 - b^2)/(x^2 - a^2) = -1,$$

or

$$x^2 + y^2 = a^2 + b^2,$$

which represents a circle of radius $\sqrt{a^2 + b^2}$. This circle is called the **Director Circle** of the ellipse, and it may be defined thus:

The locus of the point of intersection of two tangents to an ellipse which are perpendicular to each other is the director circle.

(v) Let the normal at P intersect the major axis at G, then the subnormal

$$GN = y_1 \tan NPG = y_1 \tan NTP = y_1 \,.\, NP/NT$$

$$= y_1{}^2/(OT - ON) = y_1{}^2/(a^2/x_1 - x_1) = x_1 y_1{}^2/(a^2 - x_1{}^2) = \mathbf{b^2x_1/a^2}.$$

Also $\qquad OG = ON - GN = x_1(1 - b^2/a^2) = \mathbf{e^2x_1},$

and $\qquad FG = OF - OG = ae - e^2x_1 = e(a - ex_1) = \mathbf{e \,.\, FP}.$

Again, $\quad F'G = F'O + OG = ae + e^2x_1 = e \,.\, F'P;$

$$\therefore \ F'G/GF = (e \,.\, F'P)/(e \,.\, FP) = F'P/FP,$$

so that PG bisects the angle $F'PF$; hence

The normal at any point on an ellipse bisects the angle between the focal distances of that point.

The distance PG is called the length of the normal, and it is readily obtained, for

$$PG^2 = GN^2 + NP^2 = b^4x_1{}^2/a^4 + y_1{}^2 = b^4x_1{}^2/a^4 + b^2(a^2 - x_1{}^2)/a^2$$

$$= (b^2/a^2)\{b^2x_1{}^2/a^2 - x_1{}^2 + a^2\} = (b^2/a^2)(a^2 - e^2x_1{}^2);$$

$$\therefore \ \mathbf{PG = b\sqrt{a^2 - e^2x_1{}^2}/a^2}. \quad \ldots\ldots\ldots\ldots(94)$$

92. Properties of the Hyperbola. The properties of the hyperbola are very similar to those of the ellipse, and may, in many cases, be derived from them by changing b^2 into $-b^2$. The chief characteristic of this curve, however, lies in the fact that it is the only conic which has asymptotes, and in this connection

it is important to remember, as has already been shewn in Ex. 10 of § 84, that the equations of a hyperbola and its asymptotes differ only in the constant terms. Probably the most important form of hyperbola used in practice is the rectangular hyperbola, the equation of which is generally referred to its asymptotes as axes. The equation $pv=c$ connecting the pressure and volume of a gas in an isothermal expansion is a good example of this. It is the rectangular hyperbola, too, that gives rise to the hyperbolic functions which are so useful in integration.

Ex. 10. (*a*) *Find the equation of the normal at any point* $(c/t, ct)$ *of the rectangular hyperbola,* $xy=c^2$, *and shew that it meets the curve again at the point* $(-ct^3, -c/t^3)$. (L.U.)

(*b*) *Determine the point of intersection of two normals to the rectangular hyperbola,* $x=2t$, $y=3/t$, *and deduce the coordinates of the centre of curvature in terms of* t. (L.U.)

(*c*) *Shew that four normals can be drawn to a rectangular hyperbola from a point* (ξ, η), *and if these normals meet the curve in* (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), *prove that*

$$x_1+x_2+x_3+x_4=\xi, \quad y_1+y_2+y_3+y_4=\eta,$$

$$x_1x_2x_3x_4=y_1y_2y_3y_4=-c^4 \quad \textit{and} \quad y_1^2+y_2^2+y_3^2+y_4^2=\eta^2.$$

(*a*) From (82*a*) the equation of the normal at the point (x_1, y_1) on a conic is

$$x-x_1+\frac{dy}{dx}(y-y_1)=0.$$

For the rectangular hyperbola, $xy=c^2$, $\dfrac{dy}{dx}=-y/x$.

$\therefore$ The equation of the normal at (x_1, y_1) on the curve is

$$x_1(x-x_1)=y_1(y-y_1).$$

Putting $x=c/t$, $y=ct$, this becomes

$$xt-c=t^3(y-ct).$$

This will meet the curve again where

$$c^2t/y-c=t^3(y-ct,$$

i.e.
$$y^2t^3+c(1-t^4)y-c^2t=0,$$

which is a quadratic in y, thus shewing that the normal meets the curve in two points.

Let y_1, y_2 be the roots of this quadratic, then $y_1y_2 = -c^2/t^2$; but $y_1 = ct$, since the line is normal to the curve at this point;

$$\therefore \quad y_2 = -c^2/(y_1t^2) = -c/t^3,$$

and
$$x_2 = c^2/y_2 = -ct^3;$$

$\therefore$ the normal meets the curve again at the point $(-ct^3, -c/t^3)$.

(*b*) Let t_1, t_2 be the parameters of the points where the normals are drawn, then from (*a*) the equation of the normal at (x_1, y_1) is

$$x_1(x - x_1) = y_1(y - y_1),$$

so that, putting $x_1 = 2t_1, y_1 = 3/t_1, x_2 = 2t_2, y_2 = 3/t_2$, the equations of the normals become

$$2t_1^3(x - 2t_1) = 3(yt_1 - 3),$$

$$2t_2^3(x - 2t_2) = 3(yt_2 - 3).$$

Solving these, the coordinates of their point of intersection are

$$x = \frac{9 + 4t_1t_2(t_1^2 + t_1t_2 + t_2^2)}{2t_1t_2(t_1 + t_2)}, \quad y = \frac{9(t_1^2 + t_1t_2 + t_2^2) + 4t_1^3t_2^3}{3t_1t_2(t_1 + t_2)}.$$

As t_2 approaches the value of t_1, these coordinates tend to those of the **centre of curvature**; hence in the limit, when $t_1 = t_2$, the centre of curvature becomes

$$x = \frac{3(3 + 4t^4)}{4t^3}, \quad y = \frac{27 + 4t^4}{6t}.$$

These values should be checked from the formula.

(*c*) From (*a*) the normal at the point $(c/t, ct)$ on the hyperbola $xy = c^2$ is

$$xt - c = t^3(y - ct);$$

or, since this passes through the point (ξ, η),

$$ct^4 - \eta t^3 + \xi t - c = 0.$$

This is a quartic in t, and therefore has four roots; hence **four normals can be drawn from (ξ, η) to the curve.**

Let t_1, t_2, t_3, t_4 be the four roots of the quartic, then

$$t_1 + t_2 + t_3 + t_4 = \eta/c,$$

$$t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = 0,$$

$$t_1t_2t_3 + t_1t_2t_4 + t_2t_3t_4 + t_1t_3t_4 = -\xi/c,$$

$$t_1t_2t_3t_4 = -1.$$

If $x_r = c/t_r$, $y_r = ct_r$, then

$$x_1 + x_2 + x_3 + x_4 = c(1/t_1 + 1/t_2 + 1/t_3 + 1/t_4)$$
$$= c(t_2t_3t_4 + t_1t_3t_4 + t_1t_2t_4 + t_1t_2t_3)/(t_1t_2t_3t_4)$$
$$= c(-\xi/c)/(-1) = \xi.$$
$$y_1 + y_2 + y_3 + y_4 = c(t_1 + t_2 + t_3 + t_4) = \eta,$$
$$x_1x_2x_3x_4 = c^4/(t_1t_2t_3t_4) = -c^4,$$
$$y_1y_2y_3y_4 = c^4(t_1t_2t_3t_4) = -c^4;$$
$$\therefore\ x_1x_2x_3x_4 = y_1y_2y_3y_4 = -c^4.$$

Finally, $y_1^2 + y_2^2 + y_3^2 + y_4^2 = c^2(t_1^2 + t_2^2 + t_3^2 + t_4^2)^2$

$$= c^2(t_1 + t_2 + \dots)^2 - 2(t_1t_2 + t_1t_3 + \dots)$$
$$= c^2\eta^2/c^2 = \eta^2.$$

Ex. 11. (*a*) *Shew that if a straight line cuts a hyperbola in P and P′, and its asymptotes in Q and Q′, then PP′ and QQ′ have the same middle point.* (L.U.)

(*b*) *Prove that the portion of a tangent to the hyperbola,*

$$x^2/a^2 - y^2/b^2 = 1,$$

intercepted between the asymptotes is bisected at the point of contact. Is this true of the rectangular hyperbola?

(*a*) Let $y = mx + n$ be the line $QPP'Q'$ (Fig. 34), then the ordinates of P, P' are given by

$$(y - n^2)/(m^2a^2) - y^2/b^2 = 1,$$
$$(b^2 - m^2a^2)y^2 - 2b^2ny + b^2n^2 - m^2a^2b^2 = 0.$$

If y_1, y_2 are the roots of this quadratic, *i.e.* the ordinates of P, P',

$$y_1 + y_2 = 2b^2n/(b^2 - m^2a^2).$$

Hence the ordinate of the mid-point of PP' is $b^2n/(b^2 - m^2a^2)$.

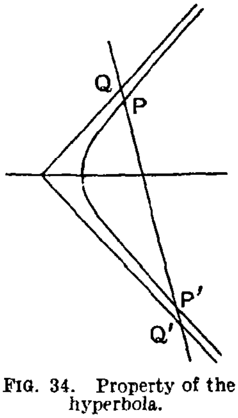

FIG. 34. Property of the hyperbola.

Now the equation of the asymptotes differs only from the equation of the hyperbola in the constant term, so that the quadratic giving the ordinates of Q, Q' will only differ from the above quadratic in the constant term, so that the sum of its roots, *i.e.* the ordinates of Q, Q', will still be $b^2n/(b^2 - m^2a^2)$; hence PP' and QQ' have the same middle point.

(b) It is easy to see from the above result that when the line QQ' moves parallel to itself until P' and P become coincident, then QQ' is a tangent to the curve, which is bisected at P, the point of contact. The proposition may, however, be proved independently as follows.

The equation of the tangent at (ξ, η) on the hyperbola,

$$x^2/a^2 - y^2/b^2 = 1, \text{ is, by (80),}$$

$$x\xi/a^2 - y\eta/b^2 = 1.$$

And from Ex. 7 (a), § 84, the equation of the asymptotes of the hyperbola is

$$x^2/a^2 - y^2/b^2 = 0,$$

so that the ordinates at the points of intersection of the tangent and the asymptotes are given by

$$(1 + y\eta/b^2)^2 a^2/\xi^2 - y^2/b^2 = 0,$$

or

$$y^2(a^2\eta^2 - b^2\xi^2) + 2a^2b^2\eta y + a^2b^4 = 0.$$

But since (ξ, η) lies on the hyperbola,

$$b^2\xi^2 - a^2\eta^2 = a^2b^2.$$

Hence on substituting this, and dividing out by $-a^2b^2$,

$$y^2 - 2\eta y - b^2 = 0.$$

If, therefore, the roots of this quadratic are y_1, y_2,

$$y_1 + y_2 = 2\eta,$$

so that η is the ordinate of the mid-point of the tangent, *i.e.*

The tangent intercepted by the asymptotes of a hyperbola is bisected at the point of contact.

In the case of the rectangular hyperbola, $a = b$, but the result

$$y_1 + y_2 = 2\eta$$

is independent of this condition, so that the proposition still holds.

When the equation of this hyperbola is given in the form $xy = c^2$, it is better to give an independent proof as follows.

The equation of the tangent at (ξ, η) is

$$\xi y + \eta x = 2c^2,$$

and since the axes are the asymptotes, this cuts the x-axis where $y = 0$, and the y-axis where $x = 0$, *i.e.* where $y = 2c^2/\xi = 2\eta$, since $\xi\eta = c^2$; hence the ordinate of its mid-point is η, *i.e.* the ordinate of the point of contact. Hence,

The portion of the tangent to the rectangular hyperbola, $xy = c^2$, intercepted between the axes is bisected at the point of contact.

93. The Common Catenary. The curve in which a heavy, uniform and inelastic string or chain freely hangs when suspended from two fixed points is known as the common **Catenary**. It was originally called a "Chainette," and the problem of finding the form of the curve was first proposed by James Bernoulli, who, together with his brother, Leibnitz, and Huyghens, published solutions in 1691. The curve also attracted the attention of Galileo, who noticed the similarity between the form of the catenary and that of the parabola.

Ex. 12. *Find the Cartesian equation of the common catenary, and investigate its chief properties.*

A steel wire of specific gravity 8·4, 576 *yards long, has a tension of ten tons applied at its ends, which are supported at equal heights. Shew that the sag in the middle is* 6·72 *feet approximately, and that the slope at either end is nearly* 2° 42′.

Let ACB (Fig. 35) be the curve in which a uniform, inelastic string hangs when suspended from two points A, B in the same horizontal line.

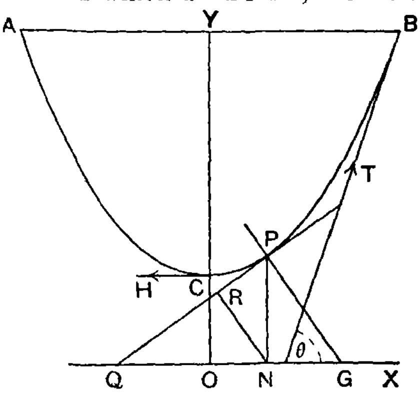

FIG. 35. The catenary.

Let C be the lowest point, then from the symmetry of the curve the vertical CY through C will intersect AB in its mid-point Y. The horizontal distance AB is called the **span**, and the vertical distance CY the **sag** of the catenary.

Denote the semi-length CB of the string by s, and let w = weight of unit length, T = the tension at B or A, and H = the tension at C; then the string CB is in equilibrium under the action of the tensions T and H, and its weight ws.

If θ be the slope of the curve at B, then resolving the forces horizontally and vertically,

$$T \cos \theta = H, \quad T \sin \theta = ws;$$

hence, on division to eliminate T,

$$\tan \theta = ws/H.$$

Now, H is constant, and so is w, since the string is uniform; hence H/w is constant; denote it by c, then

$$s = c \tan\theta = c \,.\, dy/dx.$$

But s is a function of x, and to express it in terms of x, this equation must be differentiated with respect to x.

Writing v for dy/dx, and differentiating,

$$c \,.\, \frac{dv}{dx} = \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+v^2}\,;$$

$$\therefore\ dx = c \,.\, dv/\sqrt{1+v^2},$$

which on integration gives

$$x + A = c \sinh^{-1} v,$$

or $$v = \sinh\{(x+A)/c\}.$$

If CY be taken as the y-axis, then $v=0$ when $x=0$;

$$\therefore\ A = 0,$$

so that $$v = \sinh(x/c),$$

i.e. $$dy = \sinh(x/c) \,.\, dx.$$

Another integration gives

$$y + B = c\cosh(x/c),$$

and if the x-axis be taken at a distance below C, equal to c, then $y=c$, when $x=0$, *i.e.* $$c + B = c,$$

or $$B = 0;$$

$\therefore$ **the equation of the catenary becomes**

$$\mathbf{y = c\cosh\frac{x}{c} = \frac{c}{2}(e^{x/c} + e^{-x/c}),} \qquad \textbf{(95a)}$$

and the x-axis is called the **directrix** of the curve.

Again, from above,

$$s = c \,.\, dy/dx = c\sinh(x/c).$$

$\therefore$ The **semi-length of the curve** is given by

$$\mathbf{s = c\sinh\frac{x}{c} = \frac{c}{2}(e^{x/c} - e^{-x/c}).} \qquad \textbf{(95b)}$$

Now $$\cosh^2(x/c) - \sinh^2(x/c) = 1\,;$$

$$\therefore\ y^2/c^2 - s^2/c^2 = 1,$$

so that $$\mathbf{y^2 = s^2 + c^2.} \qquad \textbf{(95c)}$$

Further, since $$T = ws \,.\, \operatorname{cosec}\theta,$$

$$\therefore\ T = ws \,.\, ds/dy = ws \,.\, y/s, \text{ from (95}c\text{)};$$

$$\therefore\ \mathbf{T = wy}. \qquad \text{...}\mathbf{(95d)}$$

$\therefore$ **The tension at any point = weight of string equal in length to the ordinate at that point.**

Let the tangent and normal at any point P intersect the x-axis at Q and G respectively, then if N be the foot of the ordinate, by (84a) :

$$\text{Subtangent} = QN = y \Big/ \frac{dy}{dx} = c \cosh\frac{x}{c} \Big/ \sinh\frac{x}{c} = c \coth\frac{x}{c}.$$

$$\text{Subnormal} = NG = y \,.\, \frac{dy}{dx} = y \sinh\frac{x}{c} = \tfrac{1}{2}\, c \sinh\frac{2x}{c}.$$

$$\text{Normal} = PG = \sqrt{NP^2 + NG^2} = \sqrt{y^2(1 + \sinh^2 x/c)} = y^2/c.$$

Now the radius of curvature ρ at P

$$= \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \Big/ \frac{d^2y}{dx^2} = c(\cosh^3 x/c)/\cosh x/c = y^2/c\,;$$

$$\therefore\ \boldsymbol{\rho = \mathbf{PG} = \mathbf{y^2/c}}. \qquad \text{..............................}\mathbf{(95e)}$$

Again, let the perpendicular from N to PQ meet it in R, then

$$NR/NP = NP/PG$$

$$= cy/y^2, \text{ by (95}c\text{)};$$

$$\therefore\ NR = c.$$

Hence, **the perpendicular from the foot of any ordinate at any point, to the tangent at that point, is of constant length.**

Turning now to the numerical example, let a = cross section of the wire in square inches, then taking y as the ordinate at one end, from (95d),

$$y = \frac{10 \times 2240 \times a \times 144}{a \times 62{\cdot}5 \times 8{\cdot}4} = 64 \times 96 \text{ feet},$$

since the weight of a cubic foot of water is 62·5 lb.

And from (95c),

$$c^2 = y^2 - s^2 = (64 \times 96)^2 - 288^2 = 96^2(64^2 - 9)$$

$$= 96^2 \times 4087\,;$$

$$\therefore\ c = 96\sqrt{4087}\,;$$

hence, if $h =$ sag, then

$$h = y - c = 64 \times 96 - 96\sqrt{4087} = 96(64 - \sqrt{4087})$$
$$= 96(64 - 63{\cdot}93) = 96 \times 0{\cdot}07$$
$$= 6{\cdot}72 \text{ feet.}$$

If θ be the slope at either end of the wire, then since

$$s = c \tan \theta,$$

$$\therefore \cot \theta = c/s = \sqrt{4087} \times 96/288 = 63{\cdot}93/3 = 21{\cdot}31.$$

From the tables, $\theta = 2^\circ\ 42'$ approximately.

Exercises 12.

Find the radius of curvature ρ for each of the following curves :

1. The circle, $x^2 + y^2 = a^2$.

2. The parabola, $y^2 = px$ at the point (p, p). (L.U

3. The curve, $y = \log \sin x$.

4. The semi-cubical parabola, $y^3 = ax^2$.

5. The four-cusped hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

6. $a^2 y = x^3$.

7. $ay^2 = x^3$.

8. $9ay^2 = x(x - 3a)^2$. **9.** The cissoid, $y^2(2a - x) = x^3$.

10. If the coordinates (x, y) of a point on a curve are given as functions of θ, the inclination of the tangent to the axis of x, prove that the radius of curvature is

$$\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{1}{2}}.$$

Deduce the radius of curvature at any point on the parabola

$$x = a/m^2, \quad y = a/m. \qquad \text{(M.U.)}$$

11. Find the radius of curvature of the tractory,

$$x = a \sin \theta, \quad y = a\left(\log \tan \frac{\theta}{2} + \cos \theta\right).$$

12. The rectangular coordinates of a point on a curve are

$$x = a \sin pt, \quad y = a \cos 2pt,$$

where p is constant and t variable. Find the direction of the tangent at the point where $y = 0$ and the radius of curvature at the point where $x = 0$. (L.U.)

***13.** Shew that if a curve be defined by the equations

$$x = a(nt - \sin t), \quad y = a(n - \cos t),$$

then
$$\rho = \frac{a^{\frac{1}{2}}\{a(1-n^2)+2ny\}^{\frac{3}{2}}}{a(1-n^2)+ny}.$$

(Br.U. and L.U., Sc.)

14. In the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, prove that the inclination of the tangent to the axis of x is $\theta/2$, and that the radius of curvature at any point is twice the intercept on the normal between the curve and the line $y = 2a$. (L.U., Sc.)

***15.** How is the curvature of a plane curve measured? Find the relations between p, a, b when at an intersection of the curves

$$b^2x^2 + a^2y^2 = a^2b^2 \quad \text{and} \quad y^2 = 2px,$$

(i) the tangents form an isosceles triangle with the axis of x, (ii) the curvatures are equal. (L.U.)

16. If the conics $3x^2 + 8y^2 = 24$, $x^2 - 4y^2 = 4$, have P as one of their points of intersection, find the radius of curvature of each of the curves at P, and shew that the distance between the two centres of curvature is the square root of the sum of the squares of the radii of curvature.

17. Find the condition that must hold in order that the hyperbolae $x^2/a^2 - y^2/b^2 = 1$ and $xy = c^2$ may intersect orthogonally, and in this case determine the radii of curvature at the points of intersection. What further condition must be fulfilled if these are equal in length? (L.U.)

18. If the normal at any point P on a parabola intersect the directrix at K, shew that the radius of curvature at P is equal to $2PK$. Find the coordinates of the centre of curvature at one extremity of the latus rectum of the parabola $y^2 = 4ax$.

19. Find ρ at the points where the axis of x meets the curve

$$3y(x + y) = x(3c - x).$$

20. Shew that on the curve $y = ax/(a + x)$,

$$(2\rho/a)^{\frac{2}{3}} = (y/x)^2 + (x/y)^2. \qquad \text{(L.U., Sc.)}$$

***21.** Find the curve in which the radius of curvature at each point P is equal to the length of the normal between P and the axis of x. (L.U., Sc.)

22. Determine the radius of curvature of the curve

$$2x^2 + y^2 = 2(x - y)$$

at the origin. (L.U., Sc.)

23. Trace the curve $y^2(a^2 - x^2) = a^3x$, and find the radius of curvature at the origin. (S.U., Sc.)

24. Prove that the coordinates of the centre of curvature at a point on the ellipse

$$(x/a)^2 + (y/b^2) = 1 \quad \text{are} \quad \frac{a^2-b^2}{a}\cos^3\phi, \quad -\frac{a^2-b^2}{a}\sin^3\phi,$$

where ϕ is the eccentric angle of the point. (Li.U., Sc.)

***25.** If ϕ be the eccentric angle of a point P on the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

shew that the circle of curvature at P cuts the ellipse again at the point $(a\cos 3\phi,\ -b\sin 3\phi)$.

Find ρ for each of the following curves :

26. The cardioid, $r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{1}{2}\theta$.

27. The equiangular spiral, $r = ae^{\theta\cot\alpha}$.

28. The rectangular hyperbola, $r^2\sin 2\theta = 2a^2$.

***29.** If p be the perpendicular from the origin on the tangent at a point on the hypocycloid

$$x = (a-b)\cos\theta + b\cos\frac{a-b}{b}\,.\,\theta, \quad y = (a-b)\sin\theta - b\sin\frac{a-b}{b}\,.\,\theta,$$

distant r from the origin, prove that $r^2 = A + Bp^2$, and determine the constants A and B in terms of a and b. (L.U.)

30. A circle of radius a rolls on the outside of another circle of the same radius. Shew that the polar equation of the path traced out by any point on the rolling circle is

$$r = 2a(1 - \cos\theta),$$

the origin being the point of contact of the tracing point with the first circle and the initial line, the radius of this circle through the point.

Find for this curve the relation between the radius vector and the perpendicular drawn from the origin to the tangent. (L.U.)

31. Shew that if p be the perpendicular from the origin upon the tangent at P to a curve, the radius of curvature at P is given by

$$\rho = p + d^2p/d\psi^2,$$

where ψ is the inclination of the tangent at P to the initial line.

Prove that in the epicycloid ρ is proportional to p. (Li.U.)

***32.** Shew that the curves

$$r^2\cos(2\theta - \alpha) = a^2\sin 2\alpha \quad \text{and} \quad r^2 = 2a^2\sin(2\theta + \alpha)$$

cut at right angles at their points of intersection. (L.U., Sc.)

33. A symmetrically tapered propeller S (Fig. 36) is fixed near the open end of a cylindrical chimney T, of radius a, so that its vertex lies in the plane of the end right section, and with its axis collinear with that of the cylinder. If C is any point on the curve of taper, shew that, for the area swept out by AC to be equal to the internal area of the tubes, the polar equation of C, referred to A as pole and AQ as initial line, is

$$r\sqrt{2}\,.\,\sin\tfrac{1}{2}\theta = a - r.$$

Shew also that the radius of curvature is given by

$$4a^4\rho^2(3a+r)^2 = r^2(r^2+2ar+3a^2)^3.$$

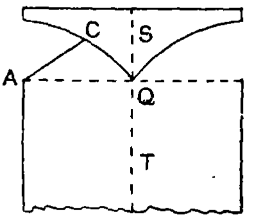

FIG. 36.

34. Prove that tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.

Prove that the length of the chord which is the normal at an extremity of the latus rectum is $2l\sqrt{2}$, where l is the length of the latus rectum. (L.U.)

35. TP, TQ are two tangents to a parabola, and a line through T parallel to the axis of the parabola meets the curve at A and PQ in B; shew that $TA = AB$, and $PB = QB$.

A telegraph wire may be assumed to hang in the form of a parabola between two posts a known distance apart. Shew how to estimate the sag in the centre of the wire by observing the angle that the wire makes with the horizontal chord joining its ends. (L.U.)

36. Prove that the tangent at any point on an ellipse cuts the major axis at the same point as the tangent to the auxiliary circle at the corresponding point.

Shew also that the locus of the foot of the perpendicular from a focus to a tangent is the auxiliary circle. (L.U.)

37. PHQ is a focal chord of an ellipse whose foci are S and H. Prove that the escribed circle of the triangle PSQ which touches PQ externally, touches PQ in H and has its centre at the point where the tangents to the ellipse at P and Q meet. (L.U.)

38. Prove that the normal at any point on an ellipse meets the axis at a distance from the focus which is in a constant ratio to the focal distances of the point.

If PG is the normal at P, prove that the circle on PG as diameter intercepts on the focal distances of P, chords equal in length to the semi-latus rectum. (L.U.)

***39.** Tangents to an ellipse are drawn at the extremities of perpendicular diameters. Prove that the locus of their intersection is another ellipse, and shew that the eccentricities of the two ellipses are connected by the equation

$$1 - e'^2 = (1 - e^2)^2. \qquad \text{(L.U., Sc.)}$$

*40. An elliptic disc, whose semi-axes are a and b, slides in two grooves which are perpendicular to each other, the whole being in a vertical plane. Shew that the locus of its foci is

$$(x^2+y^2)(x^2y^2+b^4)=4a^2x^2y^2,$$

taking the grooves as axes.

41. CP, CD are two conjugate semi-diameters of an ellipse. On the normal at P two points K, K' are taken so that $PK=PK'=CD$. Prove that the sum and difference of the lengths CK, CK' are the lengths of the axes of the ellipse, and that the directions of the axes are the internal and external bisectors of the angle KCK'. (L.U.)

42. Deduce the Cartesian equations of the catenary from the equation

$$s=c\tan\psi.$$

Prove that in this curve the length of the radius of curvature is equal to the length of the normal terminated by the directrix. (B.U.)

43. In the catenary $y=\frac{c}{2}(e^{x/c}+e^{-x/c})$, shew that the length of the perpendicular let fall from N, the foot of the ordinate PN, upon the tangent at the point P, is of constant length.

Also if the normal at P meet the axis of x in G, shew that PG varies as PN^2.

44. Along the tangent at each point P on the catenary $y=c\cosh(x/c)$, a length PQ is measured equal to the ordinate of P; shew that the locus of Q is such that at each point its subtangent is constant. (D.U., Sc.)

45. Find an expression for the radius of curvature at any point of the catenary $y=c\cosh(x/c)$.

Prove that the radius of curvature and the normal are both equal to y^2/c. (L.U.)

46. A uniform chain of length l is to be suspended from two points A and B in the same horizontal line so that the terminal tension is n times the tension at the lowest point. Shew that the span AB must be made equal to

$$\frac{l\log(n+\sqrt{n^2-1})}{\sqrt{n^2-1}}. \qquad \text{(L.U., Sc.)}$$

47. A uniform chain 100 feet long is to be suspended from two points in the same horizontal line with such a span that the tension at the ends is to be three times that at the middle. Find the required span to the nearest inch. (L.U., Sc.)

48. In the catenary prove that the tension T at any point P, the tension T_0 at the lowest point and the weight W of the chain from the lowest point up to P are connected by

$$T^2=T_0^2+W^2.$$

If the total length of the chain be 100 ft., the total weight 40 lb., and the sag 10 ft., shew that the greatest tension is 52 lb. and the distance apart of the supports is X where $\cosh(X/240) = 13/12$.

(L.U.)

***49.** A uniform chain is suspended from two fixed points, the difference of whose heights is h; shew that if an arc s is measured along the chain from its lowest point, $(s^2 - h^2)/h$ is constant.

A uniform chain of length l lies in a straight line on the ground. One end is raised vertically through a height h, and the chain is in limiting equilibrium. Prove that the length s of the curved portion is given by

$$s^2 + 2\mu hs = h^2 + 2\mu hl,$$

where μ is the coefficient of friction. (L.U., Sc.)

***50.** A chain BCD is fixed at the end B and passes over a smooth peg at C which is lower than B, but not in the same vertical, and the portion CD hangs vertically. The heights of C and B above the level of D are h and k respectively, and l is the length of the portion BC. Prove that if the curve BC is continued to its lowest point A, the arc AC is equal to $(k^2 - h^2 - l^2)/2l$. (B.U.)

51. The roadway of a suspension bridge is supported by two chains whose weight is negligible compared with that of the roadway. Assuming that the roadway is uniform, shew that the chains are parabolic in form.

Shew also that the tension at any point of a chain is proportional to the square root of the height above the directrix. (M.U.)

CHAPTER XIII

THREE-DIMENSIONAL GEOMETRY

94. Coordinates. In three-dimensional geometry, or space geometry, as it is sometimes called, three mutually perpendicular planes are taken as the planes of reference, and their lines of intersection are called the coordinate axes of x, y, z respectively. The z-axis is usually taken in the vertical position, as indicated in Fig. 37, and the planes XOY, YOZ, ZOX are known as the xy, yz, zx planes respectively. Only the positive octant is shewn in the figure. The position of any point P in space may be uniquely determined by any one of the following systems of coordinates:

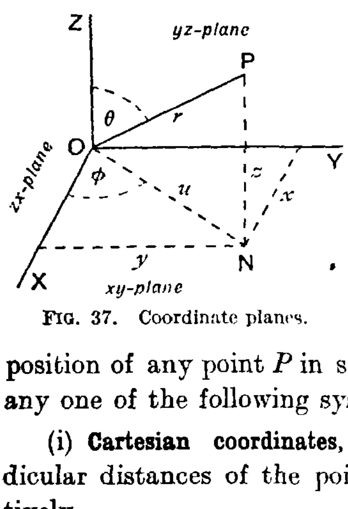

FIG. 37. Coordinate planes.

(i) **Cartesian coordinates,** (x, y, z). These are the perpendicular distances of the point from the yz, zx, xy-planes respectively.

(ii) **Polar coordinates** (r, θ, ϕ). r is the distance of P from the origin, θ the angle between the radius vector r and the z-axis, and ϕ the angle between the zx-plane and the plane containing r and the z-axis. In the figure, $OP = r$, $\angle POZ = \theta$, and $\angle NOX = \phi$, where N is the foot of the perpendicular from P to the xy-plane.

(iii) **Cylindrical coordinates** (u, ϕ, z). u is the projection of r on the xy-plane, ϕ is the angle NOX already defined in (ii), and z is the Cartesian coordinate NP.

From these definitions it is easy to see that (u, ϕ) are the plane polar coordinates of N with reference to OX, and (r, θ) are the plane polar coordinates P in the variable plane POZ with reference to OZ; hence, $x = u\cos\phi$, $y = u\sin\phi$, $z = r\cos\theta$, and $u = r\sin\theta$, so that $x^2 + y^2 = u^2$, and $x^2 + y^2 + z^2 = u^2 + z^2 = r^2$. Hence the relations between the three systems of coordinates may be expressed as follows:

$$\left.\begin{array}{ll} \mathbf{x} = \mathbf{r}\sin\theta\cos\phi, & \mathbf{u} = \mathbf{r}\sin\theta, \\ \mathbf{y} = \mathbf{r}\sin\theta\sin\phi, & \theta = \tan^{-1}(\mathbf{u}/\mathbf{z}) = \tan^{-1}\{\pm\sqrt{\mathbf{x}^2 + \mathbf{y}^2}/\mathbf{z}\}, \\ \mathbf{z} = \mathbf{r}\cos\theta, & \phi = \tan^{-1}(\mathbf{y}/\mathbf{x}), \\ \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 = \mathbf{r}^2, & \mathbf{x}^2 + \mathbf{y}^2 = \mathbf{u}^2, \end{array}\right\} \quad (96)$$

it being understood that the principal values of θ and ϕ are to be taken.

Ex. 1. *Find the distance from the origin, and the polar coordinates of the point $P(0{\cdot}7, 2{\cdot}4, 6)$.*

Shew that the distance from P of the point $Q(1{\cdot}9, 4, 8{\cdot}1)$ is $2{\cdot}9$, and give a general expression for the distance between two points A and B whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Here $x = 0{\cdot}7$, $y = 2{\cdot}4$, $z = 6$, so that

$$r^2 = 0{\cdot}49 + 5{\cdot}76 + 36 = 42{\cdot}25\,;$$

$$\therefore\ \mathbf{r} = 6{\cdot}5.$$

Also $u^2 = r^2 - z^2 = (r+z)(r-z) = 6{\cdot}25$; $\therefore\ u = 2{\cdot}5$,

and $\theta = \tan^{-1}(u/z) = \tan^{-1}(2{\cdot}5/6) = \tan^{-1}0{\cdot}4167 = 22^\circ\ 37'$,

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(2{\cdot}4/0{\cdot}7) = \tan^{-1}3{\cdot}4286 = 73^\circ\ 44' \text{ nearly.}$$

$\therefore$ The polar coordinates of P are $(\mathbf{6{\cdot}5},\ \mathbf{22^\circ\ 37'},\ \mathbf{73^\circ\ 44'})$.

To find the distance PQ, change the origin to P, then

$$\begin{aligned} PQ^2 &= (1{\cdot}9 - 0{\cdot}7)^2 + (4 - 2{\cdot}4)^2 + (8{\cdot}1 - 6)^2 \\ &= 1{\cdot}44 + 2{\cdot}56 + 4{\cdot}41 = 8{\cdot}41 \\ &= \mathbf{PQ} = 2{\cdot}9. \end{aligned}$$

From this analysis it is obvious that by taking A as the origin, the distance AB between the two points is given by

$$\mathbf{AB}^2 = (\mathbf{x}_2 - \mathbf{x}_1)^2 + (\mathbf{y}_2 - \mathbf{y}_1)^2 + (\mathbf{z}_2 - \mathbf{z}_1)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(97)$$

95. The Straight Line. Let the angle POX (Fig. 37) be α, the angle POY be β, and the angle POZ be γ, then if the origin be changed to any other point, these angles will remain constant so long as the axes remain parallel to their original positions, and the line OP remains fixed. α, β, γ will thus determine the direction of OP. It is therefore evident that for any point P on this line

$$x = r\cos\alpha, \quad y = r\cos\beta \quad \text{and} \quad z = r\cos\gamma,$$

and by eliminating r, the equation of any line through the origin becomes

$$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} = r.$$

If the origin be changed to the point (x_1, y_1, z_1), the equation becomes

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma} = r,$$

where r is the distance between the points (x_1, y_1, z_1), (x, y, z), *i.e.*

$$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the **Direction Cosines** of the line, and if they be denoted by l, m, n respectively, **the standard symmetrical form of the equation of a straight line** becomes

$$(\mathbf{x} - \mathbf{x}_1)/\mathbf{l} = (\mathbf{y} - \mathbf{y}_1)/\mathbf{m} = (\mathbf{z} - \mathbf{z}_1)/\mathbf{n} = \mathbf{r}, \quad \text{.................(98a)}$$

and the coordinates of any point on the line may be expressed in the parametric form,

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{lr}, \quad \mathbf{y} = \mathbf{y}_1 + \mathbf{mr}, \quad \mathbf{z} = \mathbf{z}_1 + \mathbf{nr}, \quad \text{.................(98b)}$$

so that $x - x_1 = r\cos\alpha, \quad y - y_1 = r\cos\beta, \quad z - z_1 = r\cos\gamma,$

and $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$;

$$\left.\begin{aligned} \therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma &= 1, \\ \text{or} \qquad l^2 + m^2 + n^2 &= 1, \\ \text{and} \qquad \sin^2\alpha + \sin^2\beta + \sin^2\gamma &= 2. \end{aligned}\right\} \quad \text{.........................(98c)}$$

Ex. 2. (*a*) *Express the equations*

$$Ax + By + Cz + D = ax + by + cz + d = 0,$$

representing a straight line in the standard symmetrical form.

Hence shew that the straight line $3x - 7y + 4z = 4x + 5y - 9z = 0$ *passes through the origin, and is inclined equally to the coordinate axes.*

(*b*) *Prove that the angle between two straight lines whose direction cosines are* $l, m, n,$ *and* l', m', n' *respectively is* $\cos^{-1}(ll' + mm' + nn')$.

Find the angle between the line $3x+2y+z-5=x+y-2z-3=0$, *and the line* $8x-4y-4z=7x+10y-8z=0$. (L.U.)

(*a*) From (98*b*), the coordinates of any point on a straight line may be represented in the parametric form,

$$x=x_1+lr, \quad y=y_1+mr, \quad z=z_1+nr,$$

where l, m, n are its direction cosines, and r the distance between (x, y, z) and (x_1, y_1, z_1).

The given equations may therefore be transformed into the standard symmetrical form of (98*a*), if l, m, n can be found in terms of the coefficients of those equations.

To do this, substitute the parametric values of (x, y, z), then since (x_1, y_1, z_1) is a point on the line, and r is, in general, not zero,

$$Al+Bm+Cn=0, \quad al+bm+cn=0.$$

Solving these for the ratios l/n, m/n,

$$l/n=(Bc-bC)/(Ab-aB), \quad m/n=(Ca-cA)/(Ab-aB).$$

Substituting these in the standard form,

$$(x-x_1)/(Bc-bC)=(y-y_1)/(Ca-cA)=(z-z_1)/(Ab-aB).$$

So far the point (x_1, y_1, z_1) is arbitrary; let it be chosen as the point where the given line intersects the xy-plane, then $z_1=0$, and replacing x, y, z by $x_1, y_1, 0$ in the given equations,

$$Ax_1+By_1+D=ax_1+by_1+d=0;$$

which on solving give

$$x_1=(Bd-bD)/(Ab-aB), \quad y_1=(aD-Ad)/(Ab-aB).$$

Hence with these values, **the symmetrical form of the equations**

$$\mathbf{Ax+By+Cz+D=ax+by+cz+d=0,}$$

is

$$\frac{\mathbf{x-(Bd-bD)/(Ab-aB)}}{\mathbf{Bc-Cb}}=\frac{\mathbf{y-(aD-Ad)/(Ab-aB)}}{\mathbf{Ca-Ac}}=\frac{\mathbf{z}}{\mathbf{Ab-aB}}, \quad \ldots(99a)$$

and if l, m, n are the direction cosines of the line

$$\frac{\mathbf{l}}{\mathbf{Bc-Cb}}=\frac{\mathbf{m}}{\mathbf{Ca-Ac}}=\frac{\mathbf{n}}{\mathbf{Ab-aB}}=\frac{+\mathbf{1}}{\sqrt{\mathbf{(Bc-Cb)^2+(Ca-Ac)^2+(Ab-aB)^2}}}, \quad (99b)$$

since $l^2+m^2+n^2=1$.

Applying these results to the given equations, the standard form becomes
$$x/43 = y/43 = z/43,$$
or
$$x = y = z.$$

Hence the line passes through the origin, and its direction cosines are all equal, *i.e.* it is equally inclined to the axes.

(*b*) Take the point of intersection of the lines as the origin, then the equations of the lines may be written,
$$x/l = y/m = z/n = r, \quad \xi/l' = \eta/m' = \zeta/n' = r'.$$

Let $P(x, y, z)$ and $Q(\xi, \eta, \zeta)$ be points on these lines respectively, then from (97),
$$\begin{aligned} PQ^2 &= (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 \\ &= \xi^2 + \eta^2 + \zeta^2 + x^2 + y^2 + z^2 - 2(\xi x + \eta y + \zeta z) \\ &= r'^2 + r^2 - 2rr'(ll' + mm' + nn'). \end{aligned}$$

But from the plane triangle POQ, if $\theta = \angle POQ$, then
$$PQ^2 = r^2 + r'^2 - 2rr' \cos\theta.$$

Hence, substituting the above value of PQ, and dividing out by $2rr'$,
$$\cos\theta = ll' + mm' + nn'.$$

From (99), by substitution of the numerical values of the coefficients of the given equations, if l, m, n, and l', m', n' are the direction cosines of the respective lines,
$$l/-5 = m/7 = n = 1/\sqrt{25+49+1} = 1/(5\sqrt{3})$$
and
$$l'/2 = m' = n'/3 = 1/\sqrt{4+1+9} = 1/\sqrt{14}.$$

Hence, if θ be the angle between the lines,
$$\begin{aligned} \cos\theta &= ll' + mm' + nn' \\ &= -2/\sqrt{42} + 7/(5\sqrt{42}) + 3/(5\sqrt{42}) = 0\,; \end{aligned}$$
$$\therefore \theta = \pi/2,$$
i.e. the lines are perpendicular to each other.

Ex. 3. (*a*) *Deduce from the first part of Ex.* 2 (*b*) *an expression for the sine of the angle between two lines, and the conditions to be fulfilled for two lines to be either parallel or perpendicular to each other.*

(*b*) *The straight line which passes through the points* (11, 11, 18), (2, −1, 3) *is intersected by a straight line drawn through* (15, 20, 8) *at right angles to the z-axis. Shew that the two lines intersect at* 45°.

(*a*) The value of $\sin\theta$ may be put into a convenient form by the aid of an important algebraic identity.

Let a, b, c, p, q, r be any numbers, then

$$\begin{aligned}(a^2+b^2+c^2)(p^2+q^2+r^2)&-(ap+bq+cr)^2\\ &=a^2q^2+a^2r^2+b^2p^2+b^2r^2+c^2p^2+c^2q^2-2(abpq+acpr+bcqr)\\ &=(a^2q^2-2abpq+b^2p^2)+(b^2r^2-2bcqr+c^2q^2)\\ &\qquad+(c^2p^2-2acpr+a^2r^2)\\ &=(aq-bp)^2+(br-cq)^2+(cp-ar)^2;\end{aligned}$$

$$\therefore\ (\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2)(\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2)=(\mathbf{ap}+\mathbf{bq}+\mathbf{cr})^2+(\mathbf{aq}-\mathbf{bp})^2+(\mathbf{br}-\mathbf{cq})^2+(\mathbf{cp}-\mathbf{ar})^2. \quad\ldots(\mathbf{100})$$

This important result is known as **Lagrange's identity.**

Now
$$\begin{aligned}\sin^2\theta=1-\cos^2\theta&=1-(ll'+mm'+nn')^2\\ &=1-(l^2+m^2+n^2)(l'^2+m'^2+n'^2)+(lm'-l'm)^2\\ &\qquad+(mn'-m'n)^2+(nl'-n'l)^2, \text{ by } (100),\\ &=(lm'-l'm)^2+(mn'-m'n)^2+(nl'-n'l)^2,\end{aligned}$$

which gives a convenient expression for $\sin\theta$.

It is evident that for the lines to be parallel, $\theta=0$, and $\sin\theta=0$, so that
$$(lm'-l'm)^2+(mn'-m'n)^2+(nl'-n'l)^2=0,$$
which can only be satisfied when
$$lm'=l'm,\quad mn'=m'n,\quad nl'=n'l,$$
since the square of every quantity is positive.

Hence $l/l'=m/m'=n/n'=\sqrt{(l^2+m^2+n^2)/(l'^2+m'^2+n'^2)}=1$;
$$\therefore\ l=l',\quad m=m',\quad n=n',$$
as is obvious from the fact that the lines are parallel.

For the lines to be perpendicular, $\theta=\pi/2$, and $\cos\theta=0$;
$$\therefore\ ll'+mm'+nn'=0.$$

Hence the foregoing results may be summarised as follows:

The angle between the lines

$$(\mathbf{x}-\mathbf{x}_1)/\mathbf{l}=(\mathbf{y}-\mathbf{y}_1)/\mathbf{m}=(\mathbf{z}-\mathbf{z}_1)/\mathbf{n},$$
$$(\mathbf{x}-\mathbf{x}'_1)/\mathbf{l}'=(\mathbf{y}-\mathbf{y}'_1)/\mathbf{m}'=(\mathbf{z}-\mathbf{z}'_1)/\mathbf{n}',$$

is given by $\cos\theta=\mathbf{ll}'+\mathbf{mm}'+\mathbf{nn}'$,

and if $\mathbf{l}=\mathbf{l}'$, $\mathbf{m}=\mathbf{m}'$, $\mathbf{n}=\mathbf{n}'$, the lines are parallel, and if $\mathbf{ll}'+\mathbf{mm}'+\mathbf{nn}'=0$, the lines are perpendicular. .... **(101)**

(*b*) Let the equation of the first line be

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n,$$

then, since it passes through the points (11, 11, 18), (2, -1, 3), by substitution, $$9/l=12/m=15/n,$$

or $$l/3=m/4=n/5=1/(5\sqrt{2}), \text{ by } (99b),$$

giving $$l=3/(5\sqrt{2}), \quad m=4/(5\sqrt{2}), \quad n=1/\sqrt{2}.$$

The second line is obviously parallel to the xy-plane, and since it passes through the point (15, 20, 8),

$$x=15+l'r, \quad y=20+m'r, \quad z=8, \text{ by } (98b),$$

or $$m'(x-15)=l'(y-20).$$

Further, since this line is perpendicular to the z-axis, it passes through the point (0, 0, 8), so that $15m'=20l'$;

$$\therefore\ l'/3=m'/4=1/5, \text{ by } (99b), \text{ since } n'=0;$$

$$\therefore\ l'=3/5 \quad \text{and} \quad m'=4/5.$$

Hence the angle between the lines is, by (101),

$$\cos\theta=ll'+mm'$$

$$=9/(25\sqrt{2})+16/(25\sqrt{2})=1/\sqrt{2};$$

$$\therefore\ \theta=45^\circ.$$

96. Straight Line passing through Two Given Points. The line

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n=r$$

passes through the point (x_1, y_1, z_1). Suppose it also passes through another given point (x_2, y_2, z_2), then

$$(x_2-x_1)/l=(y_2-y_1)/m=(z_2-z_1)/n=r_1,$$

where r_1 is the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2); hence, by division,

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=\frac{r}{r_1}, \quad \text{.....................(102a)}$$

which is the required equation.

Let $r/r_1=\lambda$; then each point on the line may be represented in terms of the parameter λ as follows:

$$\left.\begin{aligned} x&=\lambda x_2+(1-\lambda)x_1,\\ y&=\lambda y_2+(1-\lambda)y_1,\\ z&=\lambda z_2+(1-\lambda)z_1.\end{aligned}\right\} \quad \text{..............................(102b)}$$

Ex. 4. *Find an expression for the shortest distance between two non-intersecting straight lines.*

Calculate the shortest distance between the lines,

$$(12-x)/9=(y-1)/4=(z-5)/2,$$
$$(23-x)/6=(19-y)/4=(z-25)/3,$$

and give the equation of the line of shortest distance, and the coordinates of the points of intersection of it and each of the given lines.

(1) Let $(x-x_s)/l_s=(y-y_s)/m_s=(z-z_s)/n_s=k_s$, $(s=1, 2)$ be the equations of the given lines, which for brevity will be referred to as $s=1$, $s=2$ respectively.

Suppose that the line $(x-\xi_1)/l'=(y-\eta_1)/m'=(z-\zeta_1)/n'$ intersects $s=1$ at (ξ_1, η_1, ζ_1) and $s=2$ at (ξ_2, η_2, ζ_2), then

$$\xi_2-\xi_1=rl', \quad \eta_2-\eta_1=rm', \quad \zeta_2-\zeta_1=rn',$$

where r is the distance between (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2).

Now since $s=1$ passes through the former point,

$$\xi_1=k_1l_1+x_1, \quad \eta_1=k_1m_1+y_1, \quad \zeta_1=k_1n_1+z_1.$$

Similarly, $\xi_2=k_2l_2+x_2, \quad \eta_2=k_2m_2+y_2, \quad \zeta_2=k_2n_2+z_2$;

$$\therefore\ rl'=\xi_2-\xi_1=k_2l_2-k_1l_1+x_2-x_1,$$
$$rm'=\eta_2-\eta_1=k_2m_2-k_1m_1+y_2-y_1,$$
$$rn'=\zeta_2-\zeta_1=k_2n_2-k_1n_1+z_2-z_1;$$
$$\therefore\ k_2l_2-k_1l_1+x_2-x_1-rl'=0,$$
$$k_2m_2-k_1m_1+y_2-y_1-rm'=0,$$
$$k_2n_2-k_1n_1+z_2-z_1-rn'=0.$$

Eliminating k_1, k_2 from these three equations,

$$\begin{vmatrix} l_2 & -l_1 & x_2-x_1-rl' \\ m_2 & -m_1 & y_2-y_1-rm' \\ n_2 & -n_1 & z_2-z_1-rn' \end{vmatrix}=0;$$

$$\therefore\ (x_2-x_1-rl')(m_1n_2-m_2n_1)+(y_2-y_1-rm')(n_1l_2-n_2l_1)$$
$$+(z_2-z_1-rn')(l_1m_2-l_2m_1)=0.$$

This gives the distance r between the two points of intersection. But if r has to be the shortest distance between the given lines, the line l', m', n' must be at right angles both to $s=1$ and $s=2$;

$$\therefore\ l_1l'+m_1m'+n_1n'=0,$$
$$l_2l'+m_2m'+n_2n'=0, \text{ by (101)};$$

$$\therefore\ l'/(m_1n_2 - m_2n_1) = m'/(n_1l_2 - n_2l_1) = n'/(l_1m_2 - l_2m_1)$$
$$= 1/\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}^{\frac{1}{2}}, \text{ by (99}b\text{)},$$
$$= 1/\{1 - (l_1l_2 + m_1m_2 + n_1n_2)^2\}^{\frac{1}{2}}, \text{ by (100)},$$
$$= \operatorname{cosec}\theta,$$

where θ is the angle between $s = 1$ and $s = 2$.

This gives the values of l', m', n', and substituting these in the above equation for r, the result becomes, after simple reduction,

$$\mathbf{r} = \{(\mathbf{x}_2 - \mathbf{x}_1)(\mathbf{m}_1\mathbf{n}_2 - \mathbf{m}_2\mathbf{n}_1) + (\mathbf{y}_2 - \mathbf{y}_1)(\mathbf{n}_1\mathbf{l}_2 - \mathbf{n}_2\mathbf{l}_1) + (\mathbf{z}_2 - \mathbf{z}_1)(\mathbf{l}_1\mathbf{m}_2 - \mathbf{l}_2\mathbf{m}_1)\}\operatorname{cosec}\theta, \quad \text{.........(103)}$$

which is the required expression.

(2) Let the line intercepting the shortest distance between the given lines intersect the first at $P(\xi_1, \eta_1, \zeta_1)$, and the second at $Q(\xi_2, \eta_2, \zeta_2)$, then if r_1 is proportional to the distance between P and (12, 1, 5),

$$\xi_1 = 12 - 9r_1, \quad \eta_1 = 1 + 4r_1, \quad \zeta_1 = 5 + 2r_1.$$

Similarly, if r_2 be proportional to the distance between Q and (23, 19, 25),

$$\xi_2 = 23 - 6r_2, \quad \eta_2 = 19 - 4r_2, \quad \zeta_2 = 25 + 3r_2.$$

Now $\xi_2 - \xi_1$, $\eta_2 - \eta_1$, $\zeta_2 - \zeta_1$ are, by (102a), proportional to the direction cosines of PQ, and since PQ is perpendicular to both the given lines, by (101),

$$-9(11 + 9r_1 - 6r_2) + 4(18 - 4r_1 - 4r_2) + 2(20 - 2r_1 + 3r_2) = 0,$$

and

$$-6(11 + 9r_1 - 6r_2) - 4(18 - 4r_1 - 4r_2) + 3(20 - 2r_1 + 3r_2) = 0,$$

i.e. $\quad 101r_1 - 44r_2 - 13 = 44r_1 - 61r_2 + 78 = 0,$

from which $r_1 = 1$ and $r_2 = 2$.

$\therefore$ The coordinates of intersection are:

$$\mathrm{P}(3,\ 5,\ 7) \quad \text{and} \quad \mathrm{Q}(11,\ 11,\ 31).$$

And by (97), $PQ^2 = 8^2 + 6^2 + 24^2 = 676$, so that the shortest distance $= PQ =$ **26**.

The direction cosines of PQ are, from above, proportional to 4, 3, 12, and since PQ passes through (3, 5, 7), its equation is

$$(x - 3)/4 = (y - 5)/3 = (z - 7)/12.$$

97. The Plane. When a straight line moves so that two points on it are always in contact with two fixed intersecting straight lines, a plane is generated provided the motion be always in the same direction and away from the fixed point of intersection. In this way a plane may be looked upon as the locus of a straight line, and its equation will be found according to this definition.

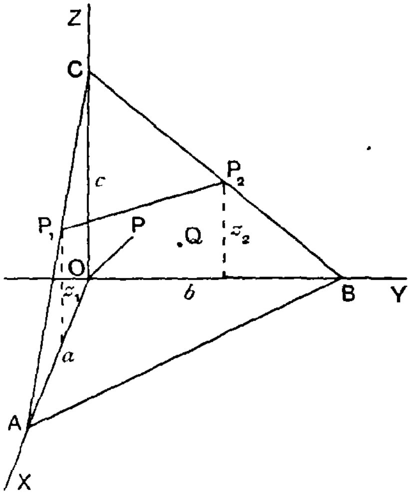

FIG. 38. The plane.

Let AC, BC (Fig. 38) be two fixed lines in the zx and yz planes respectively, and let the intercepts on the coordinate axes be a, b, c, so that $OA=a$, $OB=b$, $OC=c$, then by (64c) the equations of AC and BC are $x/a+z/c=1$ and $y/b+z/c=1$. Take any point $P_1\,(x_1, 0, z_1)$ on AC, and any point $P_2\,(0, y_2, z_2)$ on BC, then the locus of the straight line P_1P_2 is a plane which intersects the coordinate points in the lines AB, AC, BC respectively. Now, by (102b), the coordinates of any point on P_1P_2 are given by the equations :

$$x=(1-\lambda)x_1, \quad y=\lambda y_2, \quad z=\lambda z_2+(1-\lambda)z_1 \,;$$

and as P_1 lies on AC, $\qquad z_1=c(1-x_1/a)$.

Similarly, as P_2 lies on BC,

$$z_2=c(1-y_2/b).$$

Eliminate x_1, y_1, z_1, z_2 from these five equations,

$$\begin{aligned} z &= \lambda c(1-y_2/b)+(1-\lambda)c(1-x_1/a) \\ &= \lambda c\{1-y/(\lambda b)\}+(1-\lambda)c\{1-x/(a-a\lambda)\} \\ &= \lambda c-cy/b+(1-\lambda)c-cx/a \\ &= -cy/b-cx/a+c\,; \end{aligned}$$

$$\therefore\ \frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}+\frac{\mathbf{z}}{\mathbf{c}}=\mathbf{1}, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(104a)}$$

which is the **intercept form of the equation to a plane.**

From this important result it is evident that **any equation of the form**

$$\alpha x + \beta y + \gamma z + \delta = 0 \quad \text{.............................} (104b)$$

represents a plane, and this may, therefore, be taken as the standard equation to a plane.

It follows, also, that when a straight line is defined as in Ex. 2 (*a*), p. 305, by two equations

$$Ax + By + Cz + D = ax + by + cz + d = 0,$$

that, since each of these equations represents a plane, the straight line implied is the line of intersection of these planes.

Ex. 5. *Find the equation of a plane in terms of*

(1) *its perpendicular distance from the origin and the angles which this perpendicular makes with the axes* ;

(2) *the coordinates of three given points lying on it.*

Hence find the equation of the plane which passes through the points (2, 0, 6), (10, 12, 0), (− 2, 3, 6), *and determine the length of the normal to this plane from the origin and the coordinates of the point where it intersects the plane.*

(1) Let the perpendicular from the origin O to the plane ABC (Fig. 38) meet it in $P(\xi, \eta, \zeta)$, and suppose the direction cosines of OP are l, m, n, and its length is p, then $\xi = pl$, $\eta = pm$, and $\zeta = pn$.

Take any point $Q\,(x, y, z)$ on the plane, and let the direction cosines of the line PQ be l', m', n', then

$$x = \xi + l'r, \quad y = \eta + m'r, \quad z = \zeta + n'r,$$

where $r = PQ$.

But PQ lies wholly in the plane, and is therefore perpendicular to OP ; hence, by (101),

$$ll' + mm' + nn' = 0,$$

i.e.
$$(x - \xi)\,l + (y - \eta)\,m + (z - \zeta)\,n = 0,$$

since r is not zero ;

$$\therefore \; xl + ym + zn = \xi l + \eta m + \zeta n = p(l^2 + m^2 + n^2) = p.$$

Hence, **if p be the length of the perpendicular from the origin to a plane and l, m, n, be its direction cosines, the equation of the plane is**

$$\mathbf{lx + my + nz = p},$$

and the point where the perpendicular meets the plane is

$$(\mathbf{pl}, \quad \mathbf{pm}, \quad \mathbf{pn}). \quad \text{..............................} (104c)$$

If the intercept form of the equation be assumed, then the transformation into the above form is simple, for $p = al = bm = cn$;

$$\therefore \quad x/a + y/b + z/c = 1$$

becomes

$$lx + my + nz = p$$

at once, on substitution.

It is evident, therefore, that

$$\mathbf{1/p^2 = 1/a^2 + 1/b^2 + 1/c^2}. \qquad \textbf{(104d)}$$

(2) Let $Ax + By + Cz + D = 0$ be the equation of the plane which passes through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), then

$$Ax + By + Cz + D = 0,$$

$$Ax_1 + By_1 + Cz_1 + D = 0,$$

$$Ax_2 + By_2 + Cz_2 + D = 0,$$

$$Ax_3 + By_3 + Cz_3 + D = 0.$$

Hence, by eliminating A, B, C, D,

$$\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{1} \\ \mathbf{x_1} & \mathbf{y_1} & \mathbf{z_1} & \mathbf{1} \\ \mathbf{x_2} & \mathbf{y_2} & \mathbf{z_2} & \mathbf{1} \\ \mathbf{x_3} & \mathbf{y_3} & \mathbf{z_3} & \mathbf{1} \end{vmatrix} = \mathbf{0}, \qquad \textbf{(104e)}$$

which is the required equation.

Substituting the given coordinates, the equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 0 & 6 & 1 \\ 10 & 12 & 0 & 1 \\ -2 & 3 & 6 & 1 \end{vmatrix} = 0,$$

which, on expansion, becomes

$$18x + 24y + 72z - 168 = 0,$$

or

$$3x + 4y + 12z = 78.$$

Dividing this equation throughout by 78,

$$x/26 + 2y/39 + 2z/13 = 1;$$

the intercepts on the coordinate axes are thus, by (104a), 26, 19·5, 6·5.

Hence, by (104*d*), the length of the normal to the plane from the origin is given by

$$1/p^2 = 1/26^2 + 4/39^2 + 4/13^2 = (1/4 + 4/9 + 4)/13^2 = 1/36 ;$$

$$\therefore \ p = 6,$$

the positive root being taken because the intercepts are all positive, thus shewing that the normal is in the first octant.

If l, m, n be the direction cosines of this normal, the coordinates of the point of intersection with the plane are, by (104*e*), pl, pm, pn.

But $$p = 26l = 19{\cdot}5m = 6{\cdot}5n.$$

$\therefore$ Coordinates are $(p^2/26,\ 2p^2/39,\ 2p^2/13) = (18/13,\ 24/13,\ 72/13)$.

98. The Intersection of Three Planes. Let

$$p_1 = l_1x + m_1y + n_1z,$$
$$p_2 = l_2x + m_2y + n_2z,$$
$$p_3 = l_3x + m_3y + n_3z,$$

be the equations of three planes; then, if the direction cosines, l_s, m_s, n_s, of any one of the planes are different from those of each of the others, no two of the planes are parallel. Assuming this to be the case, the solution of the above equations gives, by (7),

$$-x/D_1 = y/D_2 = -z/D_3 = 1/D,$$

where $$D_1 = \begin{vmatrix} m_1 & n_1 & -p_1 \\ m_2 & n_2 & -p_2 \\ m_3 & n_3 & -p_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} n_1 & -p_1 & l_1 \\ n_2 & -p_2 & l_2 \\ n_3 & -p_3 & l_3 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} -p_1 & l_1 & m_1 \\ -p_2 & l_2 & m_2 \\ -p_3 & l_3 & m_3 \end{vmatrix}, \quad D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

If D does not vanish, the values of x, y, z are finite and the planes pass through a common point.

If $D = 0$, there are two possible cases; either the planes intersect two at a time in parallel lines, or the three planes pass through a single line. The discrimination between the two cases is given in the next example.

Ex. 6. *Find the conditions that* (1) *a straight line, and* (2) *two straight lines should lie on a given plane.*

Hence deduce the necessary conditions for three planes (1) *to intersect in a single line,* (2) *to form a triangular prism.*

The perpendicular from the origin on a given plane meets the plane at the point (3, 5, 2). *Shew that this plane passes through the line of intersection of the planes* $x-2y-3z+22=0$ *and* $13x+7y-6z-26=0$.

(a) Let $ax+by+cz+d=0$ be the given plane, and

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n=r,$$

a line. If this line lies in the plane, the coordinates

$$x=x_1+lr, \quad y=y_1+mr, \quad z=z_1+nr,$$

must satisfy the equation of the plane.

Hence, by substitution,

$$ax_1+by_1+cz_1+d+(al+bm+cn)r=0.$$

Since (x_1, y_1, z_1) is a point on the line, it must also lie in the plane, so that **the line $(x-x_1)/l=(y-y_1)/m=(z-z_1)/n$ will lie in the plane $ax+by+cz+d=0$, when**

$$\left.\begin{aligned} ax_1+by_1+cz_1+d&=0,\\ \text{and} \qquad al+bm+cn&=0.\end{aligned}\right\} \qquad \text{(105a)}$$

(b) From this result it is evident that the two lines

$$(x-x_s)/l_s=(y-y_s)/m_s=(z-z_s)/n_s \quad (s=1, 2)$$

will lie on the plane $ax+by+cz+d=0$, if

$$ax_1+by_1+cz_1+d=0, \quad ax_2+by_2+cz_2+d=0,$$

i.e.

$$a(x_1-x_2)+b(y_1-y_2)+c(z_1-z_2)=0,$$

and

$$al_1+bm_1+cn_1=0,$$

$$al_2+bm_2+cn_2=0.$$

Hence, on eliminating a, b, c.

the two lines

$$(x-x_s)/l_s=(y-y_s)/m_s=(z-z_s)/n_s \quad (s=1, 2)$$

will be coplanar if

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0. \qquad \text{(105b)}$$

(c) Take the equations of the three planes in the form

$$p_s = l_s x + m_s y + n_s z \quad (s = 1, 2, 3),$$

the direction cosines having different values, so that no two of the planes are parallel.

From (99a) the intersection of $s=2$ and $s=3$ is the line

$$(x - x_1)/(m_3 n_2 - m_2 n_3) = (y - y_1)/(l_2 n_3 - l_3 n_2) = z/\lambda,$$

where $x_1 = (m_2 p_3 - m_3 p_2)/\lambda$, $y_1 = (l_3 p_2 - l_2 p_3)/\lambda$, and $\lambda = l_3 m_2 - l_2 m_3$.

This line will by (105a) lie on the plane $s=1$ if $l_1 x_1 + m_1 y_1 = p_1$ and $l_1(m_3 n_2 - m_2 n_3) + m_1(l_2 n_3 - l_3 n_2) + n_1(l_3 m_2 - l_2 m_3) = 0$.

The former gives, on substituting the values of x_1, y_1,

$$l_1(m_2 p_3 - m_3 p_2) + m_1(l_3 p_2 - l_2 p_3) + p_1(l_2 m_3 - l_3 m_2) = 0.$$

or

$$\begin{vmatrix} l_1 & m_1 & p_1 \\ l_2 & m_2 & p_2 \\ l_3 & m_3 & p_3 \end{vmatrix} = 0,$$

i.e., in the notation of § 98, $D_3 = 0$.

The latter condition gives

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0, \quad \text{or } D = 0.$$

The vanishing of these two determinants implies also the vanishing of the remaining two, for if A_1, A_2, A_3, be the minors of n_1, n_2, n_3 in D,

$$D \equiv n_1 A_1 + n_2 A_2 + n_3 A_3 = 0,$$

Similarly $\qquad D_3 \equiv p_1 A_1 + p_2 A_2 + p_3 A_3 = 0;$

$$\therefore \ A_1/(n_2 p_3 - n_3 p_2) + A_2/(n_3 p_1 - n_1 p_3) = A_3/(n_1 p_2 - n_2 p_1).$$

Let each of these ratios be equal to $1/k$, then, from § 98,

$$-D_1 = m_1(n_2 p_3 - n_3 p_2) + m_2(n_3 p_1 - n_1 p_3) + m_3(n_1 p_2 - n_2 p_1)$$
$$= k(m_1 A_1 + m_2 A_2 + m_3 A_3) = 0.$$

Likewise $\qquad -D_2 = k(l_1 A_1 + l_2 A_2 + l_3 A_3) = 0.$

Hence the three planes intersect in a single line when

$$D_1 = D_2 = D_3 = D = 0.$$

(d) When the planes form a triangular prism, they must intersect two at a time in parallel lines; hence the direction cosines $\mu(m_3 n_2 - m_2 n_3)$, $\mu(l_2 n_3 - l_3 n_2)$, $\mu(l_3 m_2 - l_2 m_3)$, where μ is a

constant, must be the same for the three lines. Further, each line will be at right angles to the perpendiculars on the planes from the origin, so that

$$l_1(m_3n_2 - m_2n_3) + m_1(l_2n_3 - l_3n_2) + n_1(l_3m_2 - l_2m_3) = 0,$$

i.e. $$D = 0.$$

The point $(x_1, y_1, 0)$ does not now lie on the plane $s = 1$, so that none of the determinants D_s $(s = 1, 2, 3)$ will vanish in this case.

The various cases of intersection may, therefore, be summarised as follows:

The three planes $\mathbf{p_s = l_s x + m_s y + n_s z}$ (s = 1, 2, 3) will intersect

(i) in a point when D does not vanish, the coordinates of the common point being $(-\mathbf{D_1/D}, \mathbf{D_2/D}, -\mathbf{D_3/D})$;

(ii) in a single straight line when $\mathbf{D_1 = D_2 = D_3 = D = 0}$;

(iii) in three parallel straight lines when D = 0, and when $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$ are each different from zero,

where

$$-\mathbf{D_1} = \begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix}, \quad -\mathbf{D_2} = \begin{vmatrix} n_1 & p_1 & l_1 \\ n_2 & p_2 & l_2 \\ n_3 & p_3 & l_3 \end{vmatrix},$$

$$-\mathbf{D_3} = \begin{vmatrix} p_1 & l_1 & m_1 \\ p_2 & l_2 & m_2 \\ p_3 & l_3 & m_3 \end{vmatrix}, \quad \mathbf{D} = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}. \quad \ldots\ldots\ldots\ldots\ldots (106)$$

(*c*) Let the equation of the plane be

$$p = lx + my + nz,$$

then, from (104*b*), $pl = 3$, $pm = 5$, $pn = 2$;

$$\therefore\ 3x + 5y + 2z = p^2 = 3^2 + 5^2 + 2^2 = 38,$$

so that the equation becomes

$$3x + 5y + 2z - 38 = 0.$$

If this passes through the line of intersection of the two given planes, the determinants of the coefficients of the equations

$$3x + 5y + 2z - 38 = 0,$$

$$x - 2y - 3z + 22 = 0,$$

$$13x + 7y - 6z - 26 = 0,$$

must vanish. The above analysis shows, however, that if two of these determinants vanish, they all vanish.

Now $D = \begin{vmatrix} 3 & 5 & 2 \\ 1 & -2 & -3 \\ 13 & 7 & -6 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 5 \\ 1 & -2 & -2 \\ 13 & 7 & 7 \end{vmatrix} = 0,$

and $D_1 = \begin{vmatrix} 5 & 2 & -38 \\ -2 & -3 & 22 \\ 7 & -6 & -26 \end{vmatrix} = 2\begin{vmatrix} 5 & 2 & -19 \\ -2 & -3 & 11 \\ 7 & -6 & -13 \end{vmatrix}$

$$= 2\begin{vmatrix} 5 & 2 & -19 \\ 3 & -1 & -8 \\ 7 & -6 & -13 \end{vmatrix} = 2\begin{vmatrix} 12 & -4 & -32 \\ 3 & -1 & -8 \\ 7 & -6 & -13 \end{vmatrix} = 0.$$

Hence D_2 and D_3 vanish, and the three planes intersect in a single line.

99. Volume of a Tetrahedron. A tetrahedron is a triangular pyramid, and is thus a solid having four triangular faces. When each of these faces is an equilateral triangle, the solid is known as a regular tetrahedron.

All solids bounded by plane faces are called **Polyhedra,** and Euler has shewn that if F, E, V be the number of faces, edges and vertices respectively in any polyhedron, then

$$\mathbf{E + 2 = F + V}. \qquad \text{(107)}$$

Ex. 7. (*a*) *Shew that the areas of the projections on the xy, yz, and xz-planes of a triangle ABC are* $n\Delta$, $l\Delta$ *and* $m\Delta$ *respectively, where* l, m, n *are the direction cosines of the perpendicular from the origin to the plane of the triangle, and* Δ *is the area of the triangle.*

(*b*) *Prove that the volume of the tetrahedron of which the vertices have coordinates* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) *is*

$$\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

(*a*) To simplify the analysis, take B (Fig. 39) on the y-axis and C on the z-axis. It is obvious that this may be done without loss of generality. Let the coordinates of A, B, C be (x_1, y_1, z_1), $(0, y_2, 0)$ and $(0, 0, z_3)$ respectively, and let the direction cosines of BC be l_2, m_2, n_2; then, since BC lies in the yz-plane, $l_2 = 0$.

Suppose AH to be drawn perpendicular to BC, and let its direction cosines be u, v, w, then $x_1 = u \cdot AH$, and the area of ABC is

$\Delta = \frac{1}{2} \cdot AH \cdot BC = \frac{1}{2} \cdot x_1 \cdot BC/u.$

Let A' be the projection of A on the xy-plane, then $A'BO$ is the projection on that plane of the triangle ABC; hence the area of

$$A'BO = \tfrac{1}{2}x_1y_2 = \tfrac{1}{2}x_1m_2 \cdot BC;$$

$\therefore$ Area of

$$A'BO : \angle ABC = m_2u : 1.$$

But by (101), since AH is perpendicular to BC,

$$\therefore \; m_2v + n_2w = 0.$$

Similarly $m_2m + n_2n = 0$,

so that $\qquad nv - mw = 0;$

also $\qquad mv + nw + lu = 0,$

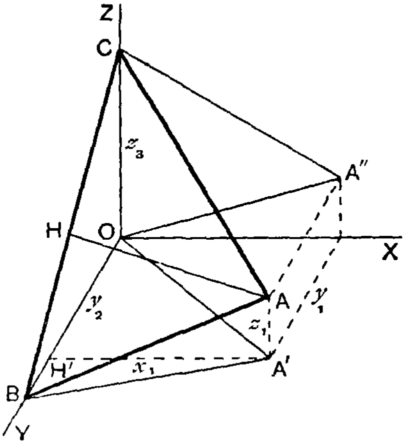

FIG. 39. Areas of the projections of a triangle.

whence, on squaring and adding to eliminate v and w,

$$l^2u^2 = (mv + nw)^2 + (nv - mw)^2 = (m^2 + n^2)(v^2 + w^2)$$
$$= (m^2 + n^2)(1 - u^2), \quad \text{since } u^2 + v^2 + w^2 = 1;$$
$$\therefore \; u^2 = m^2 + n^2.$$
$$\therefore \; m_2{}^2u^2 = m_2{}^2m^2 + m_2{}^2n^2 = n_2{}^2n^2 + m_2{}^2n_2{}^2 = n^2,$$

or
$$m_2u = n;$$
$$\therefore \text{ Area of } AB'O = n \cdot \Delta.$$

Similarly, it may be shewn that area of $BOC = l \cdot \Delta$, and area of $A''OC = m \cdot \Delta$; hence **the areas of the projections on the xy, yz, and zx-planes of any triangle, whose area is Δ, are nΔ, lΔ, and mΔ respectively where l, m, n are the direction cosines of the perpendicular from the origin to the plane of the triangle.** ....(**108**)

(b) Let the vertices of the tetrahedron be $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$, $C\,(x_3, y_3, z_3)$, and $D\,(x_4, y_4, z_4)$. Take D as the origin, then the coordinates of A, B, C will become

$$(x_s - x_4, y_s - y_4, z_s - z_4) \quad (s = 1, 2, 3).$$

Let the equation of the plane ABC be $p = lx + my + nz$.

Since this plane passes through the given points A, B, C, its equation may also, by (104e), be written

$$\begin{vmatrix} x & y & z & 1 \\ x_1-x_4 & y_1-y_4 & z_1-z_4 & 1 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 & 1 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 & 1 \end{vmatrix} = 0,$$

or $$xD_1 - yD_2 + zD_3 - D_4 = 0,$$

where D_1, D_2, D_3, D_4 are the determinant minors of x, y, z, 1 in the first row.

Now suppose $A_1B_1C_1$ to be the projection of ABC on the xy-plane, then the coordinates of A_1, B_1, C_1 are $(x_1-x_4, y_1-y_4, 0)$, $(x_2-x_4, y_2-y_4, 0)$ and $(x_3-x_4, y_3-y_4, 0)$; hence, by (63), the area of $A_1B_1C_1$ is

$$\tfrac{1}{2}\begin{vmatrix} x_1-x_4 & y_1-y_4 & 1 \\ x_2-x_4 & y_2-y_4 & 1 \\ x_3-x_4 & y_3-y_4 & 1 \end{vmatrix} = \tfrac{1}{2}D_3.$$

Similarly, if $A_2B_2C_2$, $A_3B_3C_3$ are the projections of ABC on the yz and zx-planes respectively, area of $A_2B_2C_2 = \frac{1}{2}D_1$, and area of $A_3B_3C_3 = -\frac{1}{2}D_2$, taking regard of the signs. If, therefore, the area of the face ABC be denoted by Δ,

$$\tfrac{1}{2}D_1 = l\,.\,\Delta, \quad -\tfrac{1}{2}D_2 = m\,.\,\Delta, \quad \tfrac{1}{2}D_3 = n\,.\,\Delta, \text{ by (108).}$$

Substituting for l, m, n in the p-equation of the plane ABC,

$$2p\Delta = D_1x - D_2y + D_3z = D_4,$$

from the above coordinate equation.

$\therefore$ **Volume of tetrahedron** $= \frac{1}{3}p\Delta = \frac{1}{6}D_4$

$$= \tfrac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}$$

$$= \tfrac{1}{6}\left\{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} - z_4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - y_4\begin{vmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \end{vmatrix} - x_4\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}\right\}$$

$$= -\tfrac{1}{6}\left\{x_4\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} - y_4\begin{vmatrix} z_1 & 1 & x_1 \\ z_2 & 1 & x_2 \\ z_3 & 1 & x_3 \end{vmatrix} + z_4\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\right\}$$

$$= -\tfrac{1}{6}\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \tfrac{1}{6}\begin{vmatrix} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & 1 \\ \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{z}_2 & 1 \\ \mathbf{x}_3 & \mathbf{y}_3 & \mathbf{z}_3 & 1 \\ \mathbf{x}_4 & \mathbf{y}_4 & \mathbf{z}_4 & 1 \end{vmatrix} \quad \text{.....................(109)}$$

Ex. 8. *Find an expression for the volume of a tetrahedron when the equations of its faces are given.*

Obtain the volume of the tetrahedron formed by the planes

$$\begin{aligned} 4x+4y-5z&=12, \\ 4x-5y+4z&=12, \\ -5x+4y+4z&=12, \\ x+\ y+\ z&=\ 3. \end{aligned} \qquad \text{(L.U.)}$$

To find the volume of a tetrahedron when the equations of its four faces are given, the coordinates of its vertices may be found by solving the equations three at a time. This is a tedious process, and in order to obviate the necessity of solving four sets of simultaneous equations, a general expression may be obtained by the aid of determinants.

Let the given equations of the faces be

$$a_s x+b_s y+c_s z+d_s=0 \quad (s=1, 2, 3, 4),$$

and let
$$\lambda = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Denote the determinant minors of a_s, b_s, c_s, d_s by A_s, B_s, C_s, D_s $(s=1, 2, 3, 4)$ respectively; then

$$\begin{aligned} \lambda &= \ a_1A_1-b_1B_1+c_1C_1-d_1D_1 \\ &= -a_4A_4+b_4B_4-c_4C_4+d_4D_4, \text{ etc.} \end{aligned}$$

If the first row in λ be replaced by any other row, the determinant vanishes; thus any expression of the form

$$a_rA_s-b_rB_s+c_rC_s-d_rD_s,$$

where r and s have different values, is zero.

Now, by (7), the common point of $s=2$, $s=3$, $s=4$, is

$$(-A_1/D_1,\ B_1/D_1,\ -C_1/D_1).$$

Similarly, the coordinates of the remaining three vertices may be written down; hence, by (109), the volume of the tetrahedron is

$$\frac{1}{6}\begin{vmatrix} -A_1/D_1 & B_1/D_1 & -C_1/D_1 & 1 \\ -A_2/D_2 & B_2/D_2 & -C_2/D_2 & 1 \\ -A_3/D_3 & B_3/D_3 & -C_3/D_3 & 1 \\ -A_4/D_4 & B_4/D_4 & -C_4/D_4 & 1 \end{vmatrix} = \frac{1}{6D_1D_2D_3D_4}\begin{vmatrix} -A_1 & B_1 & -C_1 & D_1 \\ -A_2 & B_2 & -C_2 & D_2 \\ -A_3 & B_3 & -C_3 & D_3 \\ -A_4 & B_4 & -C_4 & D_4 \end{vmatrix} = \frac{\Delta}{6D_1D_2D_3D_4},$$

where Δ denotes the determinant of minors.

Let $\mu_r\Omega_s$ denote the expression $-a_rA_s + b_rB_s - c_rC_s + d_rD_s$, then, by the rule for the multiplication of determinants (Ex. 34, p. 33),

$$\lambda\Delta = \begin{vmatrix} \mu_1\Omega_1 & \mu_1\Omega_2 & \mu_1\Omega_3 & \mu_1\Omega_4 \\ \mu_2\Omega_1 & \mu_2\Omega_2 & \mu_2\Omega_3 & \mu_2\Omega_4 \\ \mu_3\Omega_1 & \mu_3\Omega_2 & \mu_3\Omega_3 & \mu_3\Omega_4 \\ \mu_4\Omega_1 & \mu_4\Omega_2 & \mu_4\Omega_3 & \mu_4\Omega_4 \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^4;$$

$\therefore$ $\Delta = \lambda^3$, since λ cannot vanish, as the planes do not all pass through one point.

Hence the volume of the tetrahedron becomes $\lambda^3/(6D_1D_2D_3D_4)$.

$\therefore$ **The magnitude of the volume of a tetrahedron whose faces are the planes** $\mathbf{a_sx + b_sy + c_sz + d_s = 0\ (s = 1, 2, 3, 4)}$ **is**

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}^3 \div \left\{6\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}\begin{vmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \end{vmatrix}\begin{vmatrix} a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right\} \quad \ldots(110)$$

Applying this important result to the given equations, the required volume is

$$\begin{vmatrix} 4 & 4 & -5 & -12 \\ 4 & -5 & 4 & -12 \\ -5 & 4 & 4 & -12 \\ 1 & 1 & 1 & -3 \end{vmatrix}^3 \div \left\{6\begin{vmatrix} 4 & 4 & -5 \\ 4 & -5 & 4 \\ -5 & 4 & 4 \end{vmatrix}\times\begin{vmatrix} 4 & -5 & 4 \\ -5 & 4 & 4 \\ 1 & 1 & 1 \end{vmatrix}\times\begin{vmatrix} -5 & 4 & 4 \\ 1 & 1 & 1 \\ 4 & 4 & -5 \end{vmatrix}\times\begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & -5 \\ 4 & -5 & 4 \end{vmatrix}\right\}$$

$$= -\frac{27 \times 729 \times 729 \times 729}{6 \times 243 \times 81 \times 81 \times 81} = -13\cdot5; \quad \therefore \text{ Volume} = 13\cdot5.$$

As the given equations are in this case quite easy to solve, it will be interesting to check this result.

The solution gives readily (4, 4, 4), (3, 0, 0), (0, 3, 0) and (0, 0, 3) as the coordinates of the vertices; hence, by (109), the volume is

$$\frac{1}{6}\begin{vmatrix} 0 & 0 & 3 & 1 \\ 0 & 3 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 4 & 4 & 4 & 1 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 0 & 3 \\ 1 & 3 & 0 \\ 1 & 4 & 4 \end{vmatrix} + \frac{1}{6}\begin{vmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 4 & 4 & 4 \end{vmatrix} = -7\cdot5-6 = -\mathbf{13\cdot5}$$

as before.

100. Quadric Surfaces. As its name implies, a quadric surface is the locus of a point which moves in three dimensions according to conditions defined by an equation of the second degree in x, y, z. The most general form of such an equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2lx+2my+2nz+d=0.$$

This equation will be referred to as $Q(xyz)=0$. Just as in the case of the two-dimensional equation $F(xy)=0$, this may be reduced (1) by changing the origin where possible to remove the terms of the first degree, and (2) by rotating the axes about the new origin through a suitable angle which will remove the terms in xy, yz, zx. According as the new origin lies in the finite part of space or not, so quadric surfaces are classified into **Central Quadrics and Non-central Quadrics.**

The general equation of a central quadric may accordingly be taken in the form $Ax^2+By^2+Cz^2=1$, the centre being the origin.

If A, B, C are all positive, the surface is an **ellipsoid** whose axes are $2/\sqrt{A}$, $2/\sqrt{B}$, $2/\sqrt{C}$.

If B or C, or both, are negative, the surface is a **hyperboloid.**

When the right-hand side is zero, the surface becomes a **cone.**

The non-central quadrics have no centre in the finite part of space, and the terms of the first degree in the general equation $Q(x, y, z)=0$ cannot therefore all be removed. The surfaces are, in general, **cylinders** or **paraboloids** which under certain conditions may degenerate into **planes.**

Ex. 9. (*a*) *Investigate the nature of the loci represented by the following equations* :

$$\text{(i)}\ 144x^2+225y^2+400z^2+576x+1350y+800z=599.$$

$$\text{(ii)}\ 9x^2+4y^2-144z=12(2y-3x).$$

$$\text{(iii)}\ 9x^2+16y^2-36z^2-20=8(4y+9z).$$

(*b*) *Find the equation of the locus of a point which moves so that its distance from the fixed point* (a, b, c) *is* m *times its distance from the origin of coordinates.*

If a, b, c *be all positive and* $a>b>c$, *shew that the surface will intersect all the planes of reference if*

$$m^2(a^2+b^2+c^2)>a^2. \qquad \text{(L.U.)}$$

(*a*) (i) The given equation on rearrangement becomes

$$144(x^2+4x)+225(y^2+6y)+400(z^2+2z)=599,$$

or $$144(x+2)^2+225(y+3)^2+400(z+1)^2$$
$$=599+576+2025+400$$
$$=3600.$$

Changing the origin to the point $(-2, -3, -1)$, the equation becomes $$144x^2+225y^2+400z^2=3600,$$

or $$x^2/25+y^2/16+z^2/9=1,$$

which represents an ellipsoid generated by the variable ellipse

$$x^2/25+y^2/16=1-\mu^2/c^2, \quad z=\mu,$$

whose centre is the origin.

(ii) The given equation may be written

$$9(x+2)^2+4(y-3)^2-144z=72.$$

Hence, on changing the origin to the point $(-2, 3, -0{\cdot}5)$, the equation takes the form

$$9x^2+4y^2-144z=0,$$

i.e. $$x^2/16+y^2/36-z=0;$$

the surface is therefore generated by the variable ellipse

$$x^2/16+y^2/36=\mu, \quad z=\mu,$$

and the sections made by planes parallel to the yz, zx planes are parabolas. The surface is therefore an elliptic paraboloid.

(iii) Writing the equation in the form

$$9x^2+16(y-1)^2-36(z+1)^2=0,$$

and transferring to $(0, 1, -1)$ as new origin,

$$9x^2+16y^2-36z^2=0,$$

or
$$x^2/16+y^2/9-z^2/4=0,$$

i.e.
$$x^2/16+y^2/9=\mu^2, \quad z=\pm 2\mu ;$$

thus shewing that plane sections parallel to the xy-plane are ellipses which become a point at the origin.

Further, writing the equation as

$$x^2/16-z^2/4=-y^2/9,$$

it is evident that the section on the zx-plane is the pair of lines $z/2=\pm x/4$; similarly the section on the yz-plane is the pair of lines $y/3=\pm z/2$; hence the line

$$x/4=y/3=z/2$$

is said to be a generator of the surface, which is clearly a right elliptic cone, whose vertex is at the origin.

(*b*) Let $P(x, y, x)$ be the coordinates of any point on the locus, then if A be the point (a, b, c),

$$AP^2=(x-a)^2+(y-b)^2+(z-c)^2,$$

and
$$OP^2=x^2+y^2+z^2.$$

But
$$AP=m\,.\,OP;$$

$$\therefore\ (x-a)^2+(y-b)^2+(z-c)^2=m^2(x^2+y^2+z^2),$$

i.e.
$$(m^2-1)(x^2+y^2+z^2)+2ax+2by+2cz=a^2+b^2+c^2,$$

which may be written

$$\{x+a/(m^2-1)\}^2+\{y+b/(m^2-1)\}^2+\{z+c/m^2-1)\}^2$$
$$=m^2(a^2+b^2+c^2)/(m^2-1)^2,$$

which is obviously a sphere whose centre is the point

$$\{-a/(m^2-1),\ -b/(m^2-1),\ -c/(m^2-1)\}$$

and whose radius is

$$m\sqrt{a^2+b^2+c^2}/(m^2-1), \text{ if } m>1,$$

or
$$m\sqrt{a^2+b^2+c^2}/(1-m^2), \text{ if } m<1.$$

The sphere intersects the xy, yz, zx-planes when $z=0$, $x=0$, and $y=0$ respectively, and these sections are

$$\{x+a/(m^2-1)\}^2+\{y+b/(m^2-1)\}^2=\{m^2(a^2+b^2+c^2)-c^2\}/(m^2-1)^2$$
$$\{y+b/(m^2-1)\}^2+\{z+c/(m^2-1)\}^2=\{m^2(a^2+b^2+c^2)-a^2\}/(m^2-1)^2$$
$$\{x+a/(m^2-1)\}^2+\{z+c/(m^2-1)\}^2=\{m^2(a^2+b^2+c^2)-b^2\}/(m^2-1)^2,$$

which are circles of radii

$$\frac{\sqrt{m^2(a^2+b^2+c^2)-c^2}}{m^2-1},\quad \frac{\sqrt{m^2(a^2+b^2+c^2)-a^2}}{m^2-1},\quad \frac{\sqrt{m^2(a^2+b^2+c^2)-b^2}}{m^2-1}$$

respectively. These will be real if

$$m^2(a^2+b^2+c^2) > a^2,$$

since $a > b > c$.

101. Intersection of a Plane and a Quadric. To determine a plane section of a quadric surface, it is convenient to move the axes of the equation representing the surface into a plane through the origin parallel to the intersecting plane, and along its normal. The formulae for effecting this important transformation will now be considered.

Ex. 10. (*a*) *A quadric surface* $ax^2+by^2+cz^2=1$ *is intersected by the plane* $lx+my+nz=0$: *find the formulae to move the axes of the surface into the plane and along its normal, the origin remaining unchanged.*

Hence shew that the section will be an ellipse, parabola or hyperbola according as $l^2/a+m^2/b+n^2/c$ *is greater than, equal to, or less than zero. Find also the conditions to be fulfilled that the section should be a rectangular hyperbola.*

(*b*) *The ellipsoid* $3x^2+8y^2+z^2=c^2$ *is cut by the plane* $x+4y+z=0$. *Shew that the section is an ellipse whose semi-axes are* $c\sqrt{0\cdot3}$, $\frac{1}{2}c\sqrt{3}$, *and eccentricity* $\sqrt{0\cdot6}$.

(*a*) Take the new x-axis as the line of intersection of the xy-plane and the given plane, and the y-axis in the latter plane; the z-axis will then be the normal to this plane, and its direction cosines will therefore be l, m, n.

Let the direction cosines of the new axes of x and y be l_1, m_1, n_1 and l_2, m_2, n_2 respectively, then, since the new OX is perpendicular to OZ, $n_1=0$; hence

$$\text{(i) } l_1^2+m_1^2=1, \qquad \text{(ii) } l_2^2+m_2^2+n_2^2=1,$$
$$\text{(iii) } ll_1+mm_1=0, \qquad \text{(iv) } ll_2+mm_2+nn_2=0,$$
$$\text{(v) } l_1l_2+m_1m_2=0.$$

From (iii) and (v), $l_1/m_1=-m_2/l_2=-m/l$.

Hence, from (i), $\qquad m_1^2=l^2/(l^2+m^2)$,

so that, taking m_1 as positive, $l_1=-m/\sqrt{l^2+m^2}$.

Putting $m_2=ml_2/l$ in (ii) and (iv),

$$l_2^2=l^2(1-n_2^2)/(l^2+m^2) \quad \text{and} \quad (l^2+m^2)\,l_2/l=-nn_2.$$

Hence, on eliminating l_2,

$$n_2^2=l^2+m^2, \quad l_2^2=l^2n^2/(l^2+m^2),$$

and $$m_2^2=m^2n^2/(l^2+m^2).$$

If m_1 be positive, then m_2 will be negative, and since

$$l_1/m_1=-m_2/l_2,$$

l_2 will also be negative, so that

$$m_2=-mn/\sqrt{l^2+m^2}, \quad l_2=-ln/\sqrt{l^2+m^2},$$

and by (iv), $$n=\sqrt{l^2+m^2}.$$

Let $P\,(x_1, y_1, z_1)$ be any point referred to the old axes, (x', y', z') its coordinates referred to the new axes, then taking P as a new origin, the equation of the given plane becomes

$$z'=(x-x_1)\,l+(y-y_1)\,m+(z-z_1)\,n_1$$
$$=-lx_1-my_1-nz_1,$$

since (x, y, z) lies on the plane.

Similar expressions for x', y' may thus be found, and these give the relations between the new and the old coordinates. By solving the three expressions for x_1, y_1, z_1, corresponding expressions giving the old coordinates in terms of the new ones may be found. This is a tedious process, however, and the required relations may be derived directly as follows.

If OX', OY', OZ' be the new axes of x, y, z, then $\cos Z'OX=l$, $\cos X'OX=l_1$, $\cos Y'OX=l_2$, so that

$$\cos XOZ'=-l, \ \cos XOX'=-l_1 \text{ and } \cos XOY'=-l_2.$$

Hence the equation of the plane YOZ referred to the new axes is

$$-l_1x' - l_2y' - lz' = 0,$$

and by the above result,

$$x = l_1x' + l_2y' + lz',$$

and similarly for y and z; hence the following important transformation formulae :

If a surface be cut by a plane $\mathbf{lx + my + nz = 0}$, through the origin, the equation of the surface may be referred to new axes along the line of intersection of the given plane and the xy-plane, the perpendicular to it through the origin lying in the plane and the normal to the plane, by replacing the old coordinates x, y, z, by the following expressions involving the new coordinates, x′, y′, z′,

$$\left.\begin{aligned} &\mathbf{x = l_1x' + l_2y' + lz',} \\ &\mathbf{y = m_1x' + m_2y' + mz',} \\ &\mathbf{z = \qquad\quad n_2y' + nz',} \\ &\textbf{where}\quad \mathbf{l_1 = -m/k,\quad m_1 = l/k,\quad l_2 = -ln/k,\quad m_2 = -mn/k,\quad n_2 = k,} \\ &\textbf{and}\quad \mathbf{k^2 = l^2 + m^2,}\quad \textbf{k being positive.} \end{aligned}\right\} \quad \mathbf{(111)}$$

Applying these formulae to the surface $ax^2 + by^2 + cz^2 = 1$, the section on the plane, $z' = 0$, is

$$a(l_1x' + l_2y')^2 + b(m_1x' + m_2y') + cn_2{}^2y'^2 = 1,$$

which reduces to

$$Ax^2 + 2Hxy + By^2 = 1,$$

on omitting dashes,

where $\quad A = al_1{}^2 + bm_1{}^2 = (am^2 + bl^2)/k^2,$

$$H = al_1l_2 + bm_1m_2 = lmn(a - b)/k^2,$$

$$B = al_2{}^2 + bm_2{}^2 + cn_2{}^2 = \{n^2(al^2 + bm^2) + ck^4\}/k^2.$$

Now

$$\begin{aligned} AB - H^2 &= \{(am^2 + bl^2)(al^2n^2 + bm^2n^2 + ck^4) - l^2m^2n^2(a - b)^2\}/k^4 \\ &= \{abn^2(m^4 + 2l^2m^2 + l^4) + ck^4(am^2 + bl^2)\}/k^4 \\ &= bcl^2 + cam^2 + abn^2 \\ &= (l^2/a + m^2/b + n^2/c)abc. \end{aligned}$$

Hence, from (72*c*), the section will be an ellipse, parabola, or hyperbola according as

$$AB - H^2 >,\ =,\quad \text{or}\quad < 0,$$

i.e. $\qquad \mathbf{l^2/a + m^2/b + n^2/c >,\ =,\ \text{or} < 0,} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(112a)}$

as long as abc is positive.

From (86*a*), the section will be a rectangular hyperbola if

$$A + B = 0,$$

i.e. if $\qquad am^2 + bl^2 + al^2n^2 + bm^2n^2 + ck^4 = 0\ ;$

i.e. if $\qquad a + b - al^2 - bm^2 + ck^2 = 0\ ;$

i.e. if $\qquad (a+b)(l^2+m^2+n^2) - al^2 - bm^2 + c(l^2+m^2) = 0\ ;$

i.e. if $\qquad \mathbf{l^2(b+c) + m^2(c+a) + n^2(a+b) = 0.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(112b)}$

(*b*) From the equation of the plane, its direction cosines l, m, n are given by

$$l = m/4 = n = 1/(3\sqrt{2}),\ \text{from (99}b\text{)}\ ;$$

$$\therefore\ l = n = 1/(3\sqrt{2}),\quad m = 4/(3\sqrt{2}).$$

Replacing x, y, z by the expressions given in (111), the equation of the section of the ellipsoid made by the plane is

$$(3m^2 + 8l^2)x^2 - 10lmnxy + (3l^2n^2 + 8m^2n^2 + k^4)y^2 = c^2k^2,$$

which, on inserting the numerical values, becomes

$$28x^2/9 - 10\sqrt{2}\,.\,xy/27 + 35y^2/27 = 17/18,$$

which represents an ellipse by (72*c*), since $H^2 < AB$.

From Ex. 10, § 73 (p. 221), the reciprocals of the squares of the semi-axes are

$$\frac{9}{17c^2}\left(\frac{28}{9} + \frac{35}{27} \pm \frac{\sqrt{200 + 49^2}}{27}\right)$$

$$= \frac{9}{17c^2}\left(\frac{119 \pm 51}{27}\right) = \frac{10}{3c^2}\ \text{or}\ \frac{4}{3c^2}.$$

Hence the semi-axes are $c\sqrt{0{\cdot}3}$ and $\dfrac{c}{2}\sqrt{3}$.

From (69*c*), the eccentricity $= \sqrt{1 - 0{\cdot}4} = \sqrt{0{\cdot}6}$.

102. Sections of a Right Circular Cone. It will now be shewn that a plane section of a right circular cone is a conic as described

in § 70 (p. 206). Let VAA' (Fig. 40) be a right circular cone whose axis OV coincides with the z-axis, whose vertex is the point $(0, 0, h)$, and whose base radius is r, then the equation of the cone is obviously

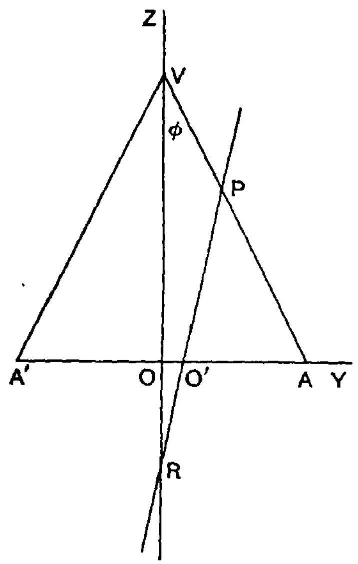

FIG. 40. Sections of a right circular cone.

$$h^2(x^2+y^2)=r^2(h-z)^2.$$

Let the cone be cut by a plane $O'P'$ parallel to the x-axis, having for its line of intersection with the base, $y=a$, $z=0$. Change the origin to the point $(0, a, 0)$, then the equation of the cone becomes

$$h^2x^2+h^2(y-a)^2=r^2(h-z)^2,$$

and the equation of the plane may be written $my+nz=0$, where

$$m=-\cos PRV, \quad n=-\sin PRV,$$

and R is the point of intersection of PO', VO.

Hence, transforming the equation of the cone by (111), the section on the plane is

$$h^2x^2+(h^2n^2-r^2m^2)y^2+2h(hna+r^2m)y+h^2(a^2-r^2)=0.$$

Now if $h^2(h^2n^2-r^2m^2)=0$, the term in y cannot be removed, and from (72a) the curve is, in general, a **parabola.** Hence the condition for this is

$$h^2n^2-r^2m^2=0, \text{ since } h \text{ is not zero};$$

$$h^2n^2=r^2(1-n^2), \text{ since } m^2+n^2=1;$$

$$\therefore\ n^2=r^2/(h^2+r^2)=\sin^2\phi,$$

where ϕ is the semi-vertical angle OVA;

$$\therefore\ \angle PRV=\angle OVA=\angle OVA',$$

so that $O'P$ is parallel to $A'V$; *i.e.* the plane is parallel to a **generator.**

When $h^2n^2 - r^2m^2$ is not zero, the term in y can be removed and the equation transformed into

$$h^2x^2 + (h^2n^2 - r^2m^2)\, y^2 = c^2,$$

where c is the new absolute term.

The curve will therefore be an ellipse or a hyperbola according as $h^2n^2 - r^2m^2$ is positive or negative;

$$i.e. \text{ according as } n^2 > \text{ or } < \sin^2 \phi;$$

$$i.e. \text{ according as } \angle PRV > \text{ or } < \phi.$$

When $O'P$ is perpendicular to OV, $n = -1$, and the equation becomes $x^2 + y^2 = c'$, which represents a circle.

Hence the plane section of a right circular cone is always a conic as defined in § 70.

Ex. 11. *A right circular cone of height* 24 *in. and base radius* 7 *in. is cut by a plane inclined at* $\sin^{-1} 0{\cdot}8$ *to its base, the line of intersection with the base being* $5\frac{25}{32}$ *in. from the centre. Shew that the section is an ellipse of eccentricity* 0·625.

Here $h = 24$, $r = 7$, $m = -3/5$, $n = -4/5$, $a = 185/32$; hence the equation of the section becomes

$$576x^2 + 351y^2 - \tfrac{6}{5}\,.\,234y + c' = 0,$$

where $$c' = -(9 \times 409 \times 39)/16,$$

i.e. $$576x^2 + 351(y - 2/5)^2 = c,$$

where $$c = (10289 \times 351)/400.$$

Hence, changing the origin to $(0, 2/5)$, the final equation is

$$576x^2 + 351y^2 = c,$$

which clearly represents an ellipse.

If a_1, a_2 be its axes, then

$$a_1{}^2 = c/576 = (10289 \times 39)/(400 \times 64),$$

$$a_2{}^2 = c/351 = 10289/400,$$

giving $$a_1 = 3{\cdot}96, \quad a_2 = 5{\cdot}07,$$

and the eccentricity $= \sqrt{1 - 351/576} = 0{\cdot}625$.

103. Circular Sections of a Quadric. To obtain plane circular sections of a quadric, the necessary conditions may be derived by

the methods of the last example. It is, however, simpler to proceed as follows.

Let $$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

be a quadric surface, and $\lambda(x^2 + y^2 + z^2) = 1$, a sphere; then the points of intersection of the quadric and the sphere will lie on the surface

$$F(x, y, z) - \lambda(x^2 + y^2 + z^2) = 0,$$

i.e. $$(a-\lambda)x^2 + (b-\lambda)y^2 + (c-\lambda)z^2 + 2fyz + 2gzx + 2hxy = 0,$$

or, putting $x = \xi z$ and $y = \eta z$,

$$(a-\lambda)\xi^2 + 2h\xi\eta + (b-\lambda)\eta^2 + 2g\xi + 2f\eta + c - \lambda = 0.$$

This will represent a pair of planes if it can be factorised, and the condition for this is, by (67*a*),

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0,$$

which is a cubic in λ. It can be shewn that the roots of this cubic are always real, but that only one of them gives real planes. Hence:

Circular sections of the quadric $\mathbf{F(x, y, z) = 1}$, where

$$\mathbf{F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy}$$

are given by the equation

$$\mathbf{F(x, y, z) - \lambda(x^2 + y^2 + z^2) = 0,}$$

where λ is a root of the cubic

$$\begin{vmatrix} \mathbf{a-\lambda} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b-\lambda} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c-\lambda} \end{vmatrix} = \mathbf{0}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(118)$$

Ex. 12. *Prove that there are two real planes which pass through the centre of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and intersect the surface in circles.*

Find these planes in the case of the ellipsoid

$$12x^2 + 30y^2 + 6z^2 = 4. \qquad \text{(L.U., Sc.)}$$

(*a*) The cubic for λ becomes

$$(1/a^2 - \lambda)(1/b^2 - \lambda)(1/c^2 - \lambda) = 0,$$

giving $\lambda = 1/a^2,\ 1/b^2$, or $1/c^2$.

Hence the six planes are given by

$$y^2(1/b^2 - 1/a^2) = z^2(1/a^2 - 1/c^2),$$
$$x^2(1/a^2 - 1/b^2) = z^2(1/b^2 - 1/c^2),$$
$$x^2(1/a^2 - 1/c^2) = y^2(1/c^2 - 1/b^2).$$

If $a > b > c$, only the second of these gives real planes, and since parallel plane sections are similar and similarly situated conics, the planes

$$x\sqrt{a^2 - b^2}/a + z\sqrt{b^2 - c^2}/c = \mu_1,$$
$$x\sqrt{a^2 - b^2}/a - z\sqrt{b^2 - c^2}/c = \mu_2,$$

will give circular sections for all values of μ_1 and μ_2, consistent with real intersections.

(*b*) For the ellipsoid, $12x^2 + 30y^2 + 6z^2 = 4$,

$$1/a^2 = 3, \quad 1/b^2 = 7{\cdot}5, \quad 1/c^2 = 1{\cdot}5;$$

so that $b < a < c$, and by the foregoing analysis, the planes giving real circular sections are

$$y^2(7{\cdot}5 - 3) = z^2(3 - 1{\cdot}5),$$

or $$\mathbf{y}\sqrt{3} + \mathbf{z} = 0, \quad \text{and} \quad \mathbf{y}\sqrt{3} - \mathbf{z} = 0.$$

104. Tangent Planes. Let the line

$$(x - \xi)/l = (y - \eta)/m = (z - \zeta)/n = r$$

intersect the quadric $Q(x, y, z) = 0$, then

$$Q(\xi + lr, \eta + mr, \zeta + nr) = 0;$$

i.e. $$Q(\xi, \eta, \zeta) + 2r(lA + mB + nC)$$
$$+ r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gln + 2hlm) = 0.$$

where $$A = a\xi + h\eta + g\zeta + u = \frac{1}{2}\frac{\partial Q}{\partial \xi},$$

$$B = b\eta + f\zeta + h\xi + v = \frac{1}{2}\frac{\partial Q}{\partial \eta},$$

$$C = c\zeta + g\xi + f\eta + w = \frac{1}{2}\frac{\partial Q}{\partial \zeta}.$$

Writing L for the coefficient of r^2, the above equation becomes

$$Q(\xi, \eta, \zeta) + r\left(l\frac{\partial Q}{\partial \xi} + m\frac{\partial Q}{\partial \eta} + n\frac{\partial Q}{\partial \zeta}\right) + r^2L = 0,$$

which is a quadratic for r, whose roots give the distances of (ξ, η, ζ) from the two points of intersection.

If (ξ, η, ζ) lies on the quadric also, then $Q(\xi, \eta, \zeta)=0$, and one of the roots is zero ; if, in addition,

$$l\frac{\partial Q}{\partial \xi}+m\frac{\partial Q}{\partial \eta}+n\frac{\partial Q}{\partial \zeta}=0,$$

then both roots are zero, and the two points are coincident. The line is therefore tangent to the surface at (ξ, η, ζ), and the locus of this line gives the tangent plane at the point. This locus may easily be found by eliminating l, m, n, between the equations

$$x=\xi+lr, \quad y=\eta+mr, \quad z=\zeta+nr,$$

and
$$l\frac{\partial Q}{\partial \xi}+m\frac{\partial Q}{\partial \eta}+n\frac{\partial Q}{\partial \zeta}=0,$$

which gives

$$(\mathbf{x}-\xi)\frac{\partial \mathbf{Q}}{\partial \xi}+(\mathbf{y}-\eta)\frac{\partial \mathbf{Q}}{\partial \eta}+(\mathbf{z}-\zeta)\frac{\partial \mathbf{Q}}{\partial \zeta}=\mathbf{0}, \quad \text{.................(114)}$$

as the **tangent plane to the quadric $\mathbf{Q}(\mathbf{x}, \mathbf{y}, \mathbf{z})=\mathbf{0}$ at the point** (ξ, η, ζ).

It should be observed that the direction cosines of the plane are proportional respectively to

$$\frac{\partial Q}{\partial \xi}, \quad \frac{\partial Q}{\partial \eta}, \quad \frac{\partial Q}{\partial \zeta}.$$

Ex. 13. (*a*) *Shew that the condition that the line*

$$(x-p)/l=(y-q)/m=(z-r)/n$$

should touch the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ *is*

$$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}-1\right)=\left(\frac{pl}{a^2}+\frac{qm}{b^2}+\frac{rn}{c^2}\right)^2.$$

(*b*) *Find the equation of the cone with vertex at the origin and generators parallel to tangent lines from* (p, q, r) *to the ellipsoid*

$$x^2/a^2+y^2/b^2+z^2/c^2=1. \qquad \text{(L.U.)}$$

(*c*) *Find also the value of* p_1 *such that the plane* $lx+my+nz=p_1$ *shall touch the ellipsoid : calculate the direction cosines of the tangent plane to the ellipsoid* $x^2/36+y^2/25+9z^2/49=1$ *passing through the point* (12, 25, 49/36).

(*a*) Let $(x-p)/l=(y-q)/m=(z-r)/n=\mu$, then substituting in the equation of the ellipsoid

$$(p+l\mu)/a^2+(q+m\mu)/b^2+(r+n\mu)^2/c^2=1,$$

or
$$\mu^2(l^2/a^2+m^2/b^2+n^2/c^2)+2\mu(lp/a^2+mq/b^2+nr/c^2)$$
$$+p^2/a^2+q^2/b^2+r^2/c^2-1=0.$$

This quadratic will have equal roots when the line is tangent to the surface; the condition for this is

$$(l^2/a^2+m^2/b^2+n^2/c^2)(p^2/a^2+q^2/b^2+r^2/c^2-1)$$
$$=(lp/a^2+qm/b^2+nr/c^2)^2.$$

(*b*) The generators of the cone will be $x/l=y/m=z/n=k$ where l, m, n have the same values as the tangent lines from (p, q, r) to the ellipsoid, *i.e.* l, m, n must satisfy the above condition.

Eliminating l, m, n, by means of the relations $l=x/k$, $m=y/k$, $n=z/k$, the equation of the cone becomes

$$(x^2/a^2+y^2/b^2+z^2/c^2)(p^2/a^2+q^2/b^2+r^2/c^2-1)$$
$$=(px/a^2+qy/b^2+rz/c^2)^2.$$

(*c*) From (114), the equation of the tangent plane at (ξ, η, ζ) on the ellipsoid is

$$(x-\xi)\frac{\partial Q}{\partial \xi}+(y-\eta)\frac{\partial Q}{\partial \eta}+(z-\zeta)\frac{\partial Q}{\partial \zeta}=0,$$

and since in this case, $Q\equiv\xi^2/a^2+\eta^2/b^2+\zeta^2/c^2$, the equation becomes

$$(x-\xi)\xi/a^2+(y-\eta)\eta/b^2+(z-\zeta)\zeta/c^2=0,$$

or

$$x\xi/a^2+y\eta/b^2+z\zeta/c^2=1,$$

since (ξ, η, ζ) lies on the ellipsoid.

If this equation is identical with $lx+my+nz=p_1$, then

$$\xi/a^2=l/p_1, \quad \eta/b^2=m/p_1, \quad \zeta/c^2=n/p_1;$$

$$\therefore\ l^2a^2/p_1^2+m^2b^2/p_1^2+n^2c^2/p_1^2=\xi^2/a^2+\eta^2/b^2+\zeta^2/c^2=1,$$

so that

$$\mathbf{p_1^2=l^2a^2+m^2b^2+n^2c^2}.$$

Taking as the equation of the tangent plane

$$x\xi/a^2+y\eta/b^2+z\zeta/c^2=1,$$

and inserting the numerical values, it becomes

$$\xi/3+\eta+\zeta/4=1.$$

If l, m, n be the direction cosines of the perpendicular from the origin on this plane, then

$$l/(1/3)=m=n/(1/4)=12/13, \quad \text{by (99}b\text{)};$$

$$\therefore\ l=4/13, \quad m=12/13, \quad \text{and} \quad n=3/13.$$

The equation of the plane then becomes

$$4x+12y+3z=13.$$

105. Normal to a Quadric. It is evident that if $lx+my+nz=p$ be a tangent plane to a quadric, l, m, n are the direction cosines of the normal.

Ex. 14. (*a*) *Find the conditions that the plane* $lx+my+nz=p$ *should touch the quadric* $ax^2+by^2+cz^2=1$, *and deduce the equation of the normal.*

(*b*) *Shew that if the line* $l_1x+m_1y+n_1z=1$, $l_2x+m_2y+n_2z=1$ *be a normal to the ellipsoid* $x^2/a^2+y^2/b^2+z^2/c^2=1$, *then*

$$(l_1-l_2)(m_1n_2-m_2n_1)/(b^2-c^2)=(m_1-m_2)(l_2n_1-l_1n_2)/(c^2-a^2)$$
$$=(n_1-n_2)(l_1m_2-l_2m_1)/(a^2-b^2). \quad \text{(L.U., Sc.)}$$

(*a*) Let $lx+my+nz=p$ touch the quadric at the point (ξ, η, ζ), then, from (114), this equation may be written

$$a(x-\xi)\xi+b(y-\eta)\eta+c(z-\zeta)\zeta=0,$$

or
$$ax\xi+by\eta+cz\zeta=1,$$

since (ξ, η, ζ) lies on the quadric; hence

$$l/p=a\xi, \quad m/p=b\eta, \quad n/p=c\zeta;$$

$$\therefore \; a\xi^2+b\eta^2+c\zeta^2=(l^2/a+m^2/b+n^2/c)/p^2=1,$$

i.e.
$$p^2=l^2/a+m^2/b+n^2/c. \quad \ldots\ldots\ldots\ldots(115a)$$

The normal at (ξ, η, ζ) is clearly

$$(x-\xi)/l=(y-\eta)/m=(z-\zeta)/n,$$

or
$$\frac{x-\xi}{a\xi p}=\frac{y-\eta}{b\eta p}=\frac{z-\zeta}{c\zeta p}. \quad \ldots\ldots\ldots\ldots(115b)$$

(*b*) Let $L=l_1m_2-l_2m_1$, $M=m_1n_2-m_2n_1$, $N=n_1l_2-n_2l_1$, then by (99*a*) the planes $l_1x+m_1y+n_1z=1$, $l_2x+m_2y+n_2z=1$, intersect in the line $(x-\xi)/M=(y-\eta)/N=(z-\zeta)/L$.

If this be a normal to the ellipsoid at $P(\xi, \eta, \zeta)$, then, from (115*b*),

$$a^2M/\xi=b^2N/\eta=c^2L/\zeta.$$

Now P lies on both planes, so that

$$a^2l_1M+b^2m_1N+c^2n_1L=1,$$
$$a^2l_2M+b^2m_2N+c^2n_2L=1;$$

$$\therefore \text{ (i) } a^2(l_1-l_2)M+b^2(m_1-m_2)N+c^2(n_1-n_2)L=0.$$

But since the line lies in both planes, by (105*a*),

$$l_1M+m_1N+n_1L=0 \quad \text{and} \quad l_2M+m_2N+n_2L=0;$$

$$\therefore \text{ (ii) } (l_1-l_2)M+(m_1-m_2)N+(n_1-n_2)L=0.$$

Hence, from (i) and (ii),

$$\frac{M}{(m_1-m_2)(n_1-n_2)(b^2-c^2)}=\frac{N}{(n_1-n_2)(l_1-l_2)(c^2-a^2)}$$
$$=\frac{L}{(l_1-l_2)(m_1-m_2)(a^2-b^2)},$$

i.e.
$$\frac{(l_1-l_2)(m_1n_2-m_2n_1)}{b^2-c^2}=\frac{(m_1-m_2)(n_1l_2-n_2l_1)}{c^2-a^2}$$
$$=\frac{(n_1-n_2)(l_1m_2-l_2m_1)}{a^2-b^2}.$$

EXERCISES 13.

1. Determine the equations of the straight line joining the points $(-2, 1, 3)$ and $(3, 1, -2)$. Find also the equation of the plane drawn through the point $(1, 1, 1)$ which contains the straight line. (L.U.)

2. Prove the relation connecting the direction cosines of two perpendicular lines in space.

Find the equation of the plane through the axis of z and parallel to the line joining the points $(3, 2, -1)$, $(1, 3, 4)$.

Calculate the distances of this plane from the given points. (L.U.)

3. Define the direction cosines of a straight line with reference to three mutually perpendicular axes, and find the condition that two lines whose direction cosines are given should be at right angles.

Taking the axis of z as vertical, find the direction cosines of a line of greatest slope in the plane which passes through the points $(0, 0, 0)$, $(3, 5, -2)$ and $(4, 1, 1)$. (L.U.)

4. Define the direction cosines of a line and find the relation between them.

Find the equation of a plane which passes through the origin and is inclined to the horizontal plane XOY at 60° and to the vertical plane XOZ at 45°. (L.U.)

5. If p denotes the perpendicular distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz=0$, shew that $p\sqrt{a^2+b^2+c^2}=ax_1+by_1+cz_1$. (M.U.)

6. A plane is drawn perpendicular to the line
$$x-3=(y+4)/2=(z-1)/4$$
so as to pass through the point $(1, 1, 1)$. If PQ, PR be the lines in which this plane cuts the two planes
$$3x+y-z-2=0 \quad \text{and} \quad 5x-y+2z-25=0,$$
find the coordinates of P and the cosine of the angle QPR.

7. Find an expression for the cosine of the angle between two lines whose direction cosines are given.

A right pyramid stands on a square base, and the vertical angle of each of the isosceles triangular faces is α. Taking the vertex as origin, the axis of z along the axis of the pyramid and the axes of x and y parallel to the diagonals of the base, obtain the equations of the planes of the triangular faces and of their lines of intersection, and find the angle between the lines which join the vertex to a pair of opposite corners of the base. (L.U.)

8. Shew that the line perpendicular to both of the lines whose direction cosines are proportional to l, m, n ; l', m', n', has direction cosines proportional to $mn' - m'n$, $nl' - n'l$, $lm' - l'm$.

A plane parallel to the line $x - 1 = 2y - 5 = 2z$ and to the line

$$3x = 4y - 11 = 3z - 4$$

passes through the point $(2, 3, 3)$. Find its equation. (L.U.)

9. Find the equation of the plane which passes through the origin O, and the points $(1, 6, 4)$, $(6, 15, -4)$. PN is the perpendicular let fall from P $(14, -3, -11/3)$ upon this plane; give the length of PN and shew that $ON = \sqrt{445}/3$. Also give the direction cosines of ON.

10. A straight line is drawn through the origin meeting perpendicularly the given straight line $(x - a)/l = (y - b)/m = (z - c)/n$. Prove that its direction cosines are proportional to $a - lk$, $b - mk$, $c - nk$, where $k = al + bm + cn$, and find the length of the perpendicular from the origin upon the given straight line. (L.U., Sc.)

11. Find the equation of the line of greatest slope through the point $(1, 1, 1)$ on the plane $2x + 3y - 4z = 1$. (L.U.)

12. Shew that the equations of the planes bisecting the angles between $a_1x + b_1y + c_1z + d_1 = a_2x + b_2y + c_2z + d_2 = 0$ are

$$(a_1x + b_1y + c_1z + d_1)/k_1 = \pm(a_2x + b_2y + c_2z + d_2)/k_2,$$

where $$k_s = \sqrt{a_s^2 + b_s^2 + c_s^2} \quad (s = 1, 2).$$

13. Shew that the sum of the projections of the sides of any closed polygon upon a given straight line is zero. Find the equation of a plane in terms of its perpendicular distance from the origin and the angles this perpendicular makes with the axes. Calculate the distance of the point $(2, 3, 1)$ from the plane $7x + 24y + 60z = 16$.

14. Given two points $P(3, 5, 1)$, $Q(5, 2, 4)$; shew that the sum of the squares of the projections of PQ on the coordinate axes is equal to the square on PQ. A plane contains the line PQ and passes through the point $(7, 6, 2)$; find (1) the equation of the plane, (2) its distance from the origin, (3) the direction cosines of a normal to the plane, and (4) the intercepts made by the plane on the axes.

15. Define the direction cosines of a straight line and prove the formulae for the cosine and the sine of the angle between the straight lines whose direction cosines are given.

Find the direction cosines of the line of intersection of the planes

$$3x - 2y - 4z = 0, \quad 4x + y + z - 1 = 0.$$

Find the angles made with these planes by the line

$$(x-1)/3 = y + 3 = 1 - z. \qquad \text{(L.U.)}$$

16. Shew that the three planes $x - 2y + z - 7 = 0$, $3x + 4y - 2z + 5 = 0$, $8x - 6y + 3z - 30 = 0$, have a common line of intersection.

A plane is drawn through $P(2, -3, 0{\cdot}2)$ to contain this common line; shew that this plane is perpendicular to the plane $x - 2y + z = 7$.

17. Obtain the equation of a plane in the form $x/a + y/b + z/c = 1$. What form does the equation take when $c = 0$ and when $c = \infty$?

The planes $3x - y + z + 1 = 0$, $5x + y + 3z = 0$ intersect in the line PQ, and a plane is drawn through the point $(2, 1, 4)$ perpendicular to PQ. Shew that this plane contains the point $(2, 3, 5)$.

18. Shew that the straight lines $(x+1)/(-3) = (y-3)/2 = z + 2$, and $x = (y-7)/(-3) = (z+7)/2$, intersect. Find the coordinates of their point of intersection and the equation of the plane containing them.

(L.U., Sc.)

19. Find the direction cosines of the line of intersection of the planes $6x - 4y - z + 12 = 0$, $3x - 4y + 19 = 0$, and shew that the point $(3, 7, 2)$ lies on this line. Find also the coordinates of the two points P, Q on this line which are at a distance 13 from the point $(3, 7, 2)$, and shew that PQ subtends a right angle at the point $(54/5, -17/5, 2)$.

20. Prove that if $\theta + \phi + \psi = \dfrac{\pi}{2}$, the planes

$$x = y \sin \psi + z \sin \phi,$$
$$y = z \sin \theta + x \sin \psi,$$
$$z = x \sin \phi + y \sin \theta,$$

intersect in a line. (L.U., Sc.)

21. Shew that an equation of the first degree represents a plane and find the perpendicular distance from the origin to the plane

$$5x - 4y + 2z - 6 = 0,$$

and also the angle of inclination of the axis of y to the plane. (L.U., Sc.)

22. Find the condition that two straight lines in space shall intersect. Find the equation of the surface generated by a straight line which intersects the three straight lines

$$y - 2z = 0,\ x = a; \quad z - 2x = 0,\ y = a; \quad x - 2y = 0,\ z = a.$$

(L.U., Sc.)

23. Find the length of the shortest distance between the lines

$$(x-1)/4=(y+2)/3=z \quad \text{and} \quad -(x-4)/2=(y+5)/6=(z-5)/7.$$

(L.U., Sc.)

24. Find the length of the shortest distance between the lines

$$(x+14)/-2=y/0=z+23 \quad \text{and} \quad (x+2)/4=(y-5)/3=(2-z)/3.$$

25. Find the equations of the line along which lies the shortest distance between the straight lines $(x+1)/4=(y-1)/2=-(z-9)/6$ and $(x-3)/2=-(y+15)/7=(z-9)/5$.

26. Find the shortest distance of the line $(x-7)/2=(y+4)/3=z-2$ from the intersection of the planes

$$2x+5y-8z-52=0, \quad 3x-3y+2z+27=0.$$

27. Find the equation to the circle in the plane $z=0$ which has the points $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ as extremities of a diameter. Find the condition that the straight line $(x-f)/l=(y-g)/m=(z-h)/n$ should intersect the circumference of this circle. (L.U., Sc.)

28. With given rectangular axes, the line $x/2=-y/3=z$ is vertical. Find the direction cosines of the line of greatest slope in the plane $3x-2y+z=5$ and the angle this line makes with the horizontal plane. (L.U.)

29. A tripod has its feet A, B, C on three walls, C being 2 ft. higher than B, and B 3 ft. higher than A. In plan $ab=18$ ft., $bc=20$ ft., $ca=21$ ft., and d, the plan of the apex of the tripod, is equidistant from a, b, and c. If D is 25 ft. higher than C, find the lengths of the legs and the true values of the three plane angles at D. (L.U.)

30. The vertical angle of each of the isosceles triangular faces of a right pyramid on a square base is 28°. A plane cuts the pyramid, the section being a quadrilateral $ABCD$ such that the distances of A, B and C from the vertex are 4, 8 and 6 inches respectively. Find the distance of D from the vertex. (L.U.)

31. A tetrahedron has its vertices at the points $(1, 0, 0)$, $(0, 0, 1)$, $(0, 0, 2)$, $(1, 2, 3)$ respectively. Find the lengths and the direction cosines of the six edges, the equations of the four faces and the volume of the tetrahedron. (L.U.)

32. The coordinates of the angular points A, B, C, D of a tetrahedron are $(-2, 1, 3)$, $(3, -1, 2)$, $(2, 4, -1)$ and $(1, 2, 3)$ respectively. Calculate to the nearest minute the angle between the edges AC and BD. (L.U.)

33. Three of the vertices of a tetrahedron are at the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Prove that if the perpendiculars from the vertices on the opposite faces are concurrent, the fourth vertex lies upon the line $ax=by=cz$. (L.U., Sc.)

34. The vertices of a tetrahedron are (0, 1, 2), (3, 0, 1), (2, 3, 2) and $(\lambda+1, \lambda, 2\lambda)$; its volume is 6 ; find the numerical value of λ.

35. A sphere of radius a rests in the positive octant in contact with the three planes of rectangular coordinates. Write down the equation of its surface.

Shew that there are two such spheres passing through the point (2, 1, 3), one of radius 4·41... and the other of radius 1·59... . Obtain the equation of the tangent plane to each sphere at the given point. (L.U.)

36. Prove that the equation of the sphere described on the line joining the points (2, −1, 4) and (−2, 2, −2) as diameter is

$$(x-2)(x+2)+(y+1)(y-2)+(z-4)(z+2)=0.$$

Find the area of the circle in which this sphere is intersected by the plane $2x+y-z=3$. (L.U.)

37. The rectangular coordinates of a point are (1, 3, 2) ; what are its polar and cylindrical coordinates ?

A sphere is of 3 in. radius, its centre is (0, 2, 1) in rectangular coordinates ; find the equation to the surface in rectangular, polar and cylindrical coordinates.

Determine in rectangular coordinates the equation to the sphere which passes through the origin and the points (0, 0, 1), (1, 2, −1) and (1, 0, 3). (L.U.)

38. Describe with sketches the surfaces represented by the following equations in three dimensions :

(1) $x^2+2y^2+3z^2=12$, (2) $x^2+4y^2=1$,

(3) $x^2+3y^2-4z^2=0$, (4) $xyz=c^3$. (L.U.)

39. Express in a symmetrical form the equations of the straight line given by $8x-3y-z-16=0$, $3x-y-z+1=0$.

Give the distance of the point (x, y, z) from the straight line

$$(x-a)/l=(y-b)/m=(z-c)/n,$$

and deduce the equation of a cone of semi-vertical angle 60°, having its vertex at (1, 3, 2) and its axis parallel to the line $x/3=y/4=z/2$.

40. What surface is represented by the equation $3x^2+4y^2+z^2=20$? Find the value of p such that the plane $lx+my+nz-p=0$ may touch this surface.

Shew that there are two tangent planes parallel to the plane

$$3x+2y+z=0,$$

and give the coordinates of their points of contact with the surface.

41. Shew that $Ax^2+By^2+Cz^2=0$ is the equation of a cone having its vertex at the origin, and that $lx+my+nz=0$ is a tangent plane to this cone if $l^2/A+m^2/B+n^2/C=0$.

42. Find the condition that the line $(x-f)/l=(y-g)/m=(z-h)/n$ should touch the quadric $ax^2+by^2+cz^2=1$, and deduce the equation of the tangent cylinder with its axis in the direction l, m, n. (L.U., Sc.)

***43.** A circular cylinder of radius 2 has for its axis the line

$$(x-1)/2=y=3-z.$$

Find (1) its equation, (2) the lengths of the axes of the ellipse in which it is cut by the plane $x=y$. (L.U.)

***44.** Find the equation of a right circular cylinder of radius 2, whose axis is the line $(x-3)/5=(y-1)/4=(z+2)/2$. Find also the equations of the tangent planes to the cylinder from the origin. (L.U.)

***45.** A right circular cylinder is cut by a sphere whose centre is on one of the generators of the cylinder. Shew that the projection of the curve of intersection on the plane containing the axis of the cylinder and the centre of the sphere is a parabola whose latus rectum is twice the radius of the cylinder. (L.U.)

***46.** Shew that any plane section of an ellipsoid is in general an ellipse. The ellipsoid $x^2+5y^2+2z^2=12$ is cut by the plane

$$2x+y\sqrt{6}+z=0.$$

Find the area of the elliptic section thus made, and prove that its eccentricity is $\sqrt{14/33}$. (See pp. 413-416.)

47. Shew that the plane $lx+my+nz=p$ will touch the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, if $p^2=l^2a^2+m^2b^2+n^2c^2$.

The ellipsoid $2x^2+3y^2+z^2=1$ is cut by parallel planes $3x+2y+4z=1$ and $3x+2y+4z=2$. Find the ratio of the areas of the sections made on the planes.

48. Find the eccentricity of the section of the ellipsoid $x^2+4y^2+7z^2=1$ made by the plane $x+y+z=0$. (L.U., Sc.)

49. Find the equation of the tangent plane at any point of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Find the equation of the plane which intersects the ellipsoid

$$x^2/4+y^2/9=z^2/16=1$$

in an ellipse whose centre is the point $(1, 1, 1)$. (L.U.)

50. Find the equations of the tangent planes to the ellipsoid

$$x^2/4+y^2/16+z^2/9=1$$

at the points in which it is met by the line $x/2=y/2=z/3$. Find the equation of the projection on the xy-plane of the section of the ellipsoid made by the plane through the centre parallel to the above tangent planes. (L.U.)

51. Shew that there are on the ellipsoid $x^2/25+y^2+z^2/9=1$ two sets of circular sections. Find the coordinates of the points of contact of the tangent planes parallel to these sections, and find the diameter of the circular section made by a plane whose perpendicular distance from the centre of the ellipsoid is 1.

52. Prove that an ellipsoid has two real circular sections passing through the ends of the mean axis.

A sphere is drawn to meet the ellipsoid in plane sections; prove that, if the sections are real, the radius of the sphere must lie between ab/c and bc/a inclusive. (L.U., Sc.)

53. Shew that the spheres $x^2+y^2+z^2+6y+2z+8=0$ and

$$x^2+y^2+z^2+6x+8y+4z+20=0$$

intersect at right angles.

Find the equation of the tangent plane to each sphere furthest from and parallel to the plane of intersection of the spheres. (L.U.)

***54.** Shew that the middle points of parallel chords of an ellipsoid lie on a plane.

Obtain the equation of the diametral plane of chords having direction cosines l, m, n in the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

In the ellipsoid $x^2+3y^2+2z^2=1$, find the equations of the diameter in the plane $z=0$ conjugate to the line $x/2=2y=z$, and also the equations of the third conjugate diameter. (L.U.)

***55.** If the tangent plane to the surface $ax^2+by^2+cz^2=1$ makes equal angles with the three coordinate axes, shew that it forms with the coordinate planes a tetrahedron of volume $(1/a+1/b+1/c)^{3/2}/6$. (Li.U.)

***56.** Shew that the straight line $x=a^2\xi/(a^2+\mu)$, $y=b^2\eta/(b^2+\mu)$, $z=\zeta+\mu$ is a normal to the elliptic paraboloid $x^2/a^2+y^2/b^2=2z$, drawn from the point (ξ, η, ζ); hence prove that five normals can be drawn from this point to the surface, and that they lie on the cone

$$\xi/(x-\xi)-\eta/(y-\eta)+(a^2-b^2)/(z-\zeta)=0.$$

***57.** Shew that through every point on the surface $3x^2-4y^2+z^2=1$, two straight lines can be drawn lying wholly on the surface.

A straight line is at a distance 2 from the axis of z and is inclined at 60° to that axis; shew that the surface generated by the revolution of this line about the axis of z is a hyperboloid of revolution.

***58.** The normal at a point P on the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=1$ meets the coordinate planes in G_1, G_2, G_3, and

$$PG_1^{\,2}+PG_2^{\,2}+PG_3^{\,2}=\text{constant}=k^2;$$

shew that the point P lies also on the ellipsoid

$$x^2/a^4+y^2/b^4+z^2/c^4=k^2/(a^4+b^4+c^4). \qquad \text{(Li.U.)}$$

CHAPTER XIV

PERIMETERS, AREAS AND VOLUMES

106. Length of Arc in Cartesian Coordinates. Let ds be the length of an infinitesimal element of a plane curve whose Cartesian equation is given, then the integral $\int ds$, taken between two points on the curve between which it is continuous, gives the length of the arc between those points. This process is sometimes called **Rectification.**

Since

$$ds^2 = dx^2 + dy^2 = \{1 + (dy/dx)^2\} \cdot dx^2 = \{(dx/dy)^2 + 1\} \, dy^2 ;$$

$$\therefore \; ds = \sqrt{1 + (dy/dx)^2} \cdot dx \quad \text{or} \quad \sqrt{1 + (dx/dy)^2} \cdot dy,$$

so that the length of a plane curve, which is continuous, between the points (x_1, y_1) and (x_2, y_2) is given by the formulae

$$\mathbf{s} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx \quad \text{or} \quad \mathbf{s} = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy, \; \ldots\ldots\ldots\textbf{(116)}$$

the integral chosen depending upon whether the equation of the curve be given in the form $y = f(x)$ or $x = \phi(y)$.

107. Area of a Curve. Let da_1 be an element of area between a plane curve PQ (Fig. 41) and the x-axis, and da_2 an element of area between the curve and the y-axis, then by considering narrow strips parallel to each of the axes and of width dx, dy respectively,

$$da_1 = y \cdot dx = f(x) \cdot dx \quad \text{and} \quad da_2 = x \cdot dy = \phi(y) \cdot dy,$$

so that the areas between the arc, the axes and the ordinates or abscissae at $P(x_1, y_1)$, $Q(x_2, y_2)$ are given by

$$a_1 = \int_{x_1}^{x_2} f(x) \cdot dx \quad \text{or} \quad a_2 = \int_{y_1}^{y_2} \phi(y) \cdot dy. \;\ldots\ldots\ldots\ldots\ldots\ldots\textbf{(117)}$$

If the curve cuts both axes by a continuous arc, like an ellipse, then, in general, $a_1 = a_2$. It often happens, however, that the areas a_1, a_2 lie on opposite sides of the curve, as in Fig. 41, in

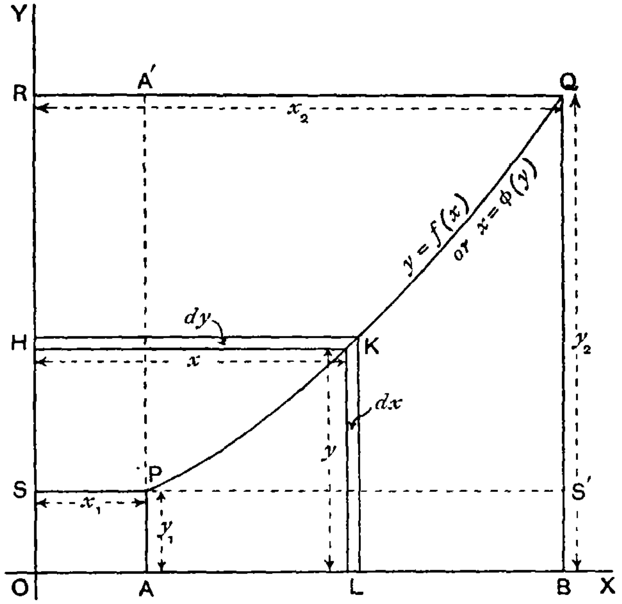

FIG. 41 Area of a curve.

which case they are said to be complementary, because their sum makes up the rectangle $PA'QS'$, formed by joining the points

$$P(x_1, y_1), \quad A'(x_1, y_2), \quad Q(x_2, y_2), \quad S'(x_2, y_1),$$

$$i.e. \quad a_1 + a_2 = (x_2 - x_1)(y_2 - y_1).$$

In every case a rough graph should be drawn to ensure the correct area being calculated.

It should also be observed that if the curve cuts itself, thus forming a loop or loops, the area of each loop must be found separately if such falls within the given limits.

When the curve is defined by parametric equations of the form $x = F(t)$, $y = f(t)$, then

$$da_1 = y \, . \, dx = f(t) \, . \, (dx/dt) \, . \, dt, \text{ and } da_2 = x \, dy = F(t) \, . \, (dy/dt) \, . \, dt.$$

Ex. 1. *Find the length of the curve*

$$20y = 3(4x^2 - 20x + 9)$$

between the ordinates where $x = 0{\cdot}5$ *and* $x = 4{\cdot}5$.

Determine also the area of the curve bounded by the arc, the x-axis and the given ordinates.

By plotting a rough sketch of the curve, it will at once be seen that it is a parabola and the given points are those where it crosses the x-axis.

From the equation,

$$dy/dx = 3(8x - 20)/20 = 3(2x - 5)/5\,;$$

$\therefore$ from (116), $\quad ds = \sqrt{1 + 9(2x-5)^2/25}.$

Let $z = 3(2x - 5)/5$, then $dx = 5\,.\,dz/6$,

and $$ds = (5\sqrt{1 + z^2}/6)\,.\,dx.$$

Now when $x = 0{\cdot}5$, $z = -2{\cdot}4$, and when $x = 4{\cdot}5$, $z = 2{\cdot}4$;

$$\therefore\ s = \frac{5}{6}\int_{-2\cdot4}^{2\cdot4}\sqrt{1+z^2}\,.\,dx = \frac{5}{6}\left[\tfrac{1}{2}z\sqrt{1+z^2} + \tfrac{1}{2}\log\left(z + \sqrt{1+z^2}\right)\right]_{-2\cdot4}^{2\cdot4},$$

by (48),

$$= \tfrac{5}{6}\{3{\cdot}12 + \tfrac{1}{2}\log 5 + 3{\cdot}12 + \tfrac{1}{2}\log 5\}$$

$$= \tfrac{5}{6}(6{\cdot}24 + \log 5) = \tfrac{5}{6}(6{\cdot}24 + 1{\cdot}6094) = \mathbf{6{\cdot}541}.$$

To find the area,

$$A = \frac{3}{20}\int_{0\cdot5}^{4\cdot5}(4x^2 - 20x + 9)\,dx$$

$$= \frac{3}{20}\left[\tfrac{4}{3}x^3 - 10x^2 + 9x\right]_{0\cdot5}^{4\cdot5}$$

$$= \frac{3}{20}\left(-40{\cdot}5 - \frac{13}{6}\right) = -6{\cdot}4.$$

The minus sign indicates, as is obvious from the graph, that the area lies in the fourth quadrant, *i.e.* on the negative side of the y-axis; hence, the magnitude of the area is **6·4 square units.**

Ex. 2. *Find the length and the area of the curve*

$$y = \cosh^{-1} x$$

between the ordinates where $x = 1$ *and* $x = 2{\cdot}6$.

The curve is shown in Fig. 42, where it will be observed that it is symmetrical about the x-axis.

Since $$y = \cosh^{-1} x,$$

$$\therefore\ y = \log(x + \sqrt{x^2 - 1}),\ \text{from (26)},$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

and $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{x^2 - 1};$$

$$\therefore\ ds = \frac{x}{\sqrt{x^2 - 1}} \,.\, dx,$$

since x and s increase together.

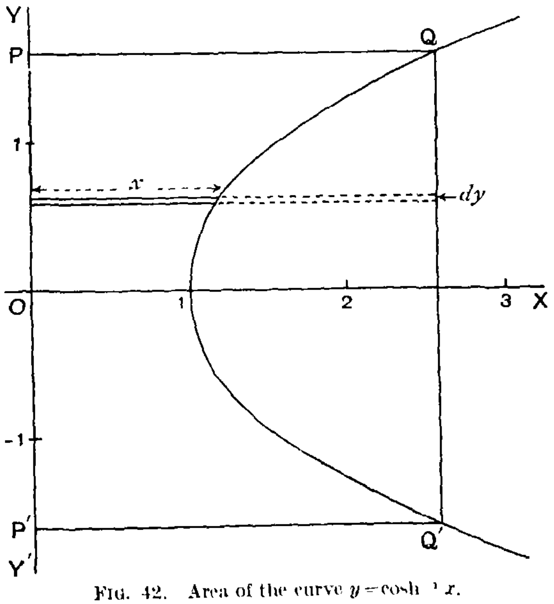

FIG. 42. Area of the curve $y = \cosh^{-1} x$.

Put $x^2 - 1 = z^2$, then $x \,.\, dx = z \,.\, dz$;

$$\therefore\ ds = dz.$$

For the given limits, when $x = 1$, $z = 0$, and when $x = 2{\cdot}6$, $z = 2{\cdot}4$; hence, if s be the semi-length between these limits,

$$\therefore\ s = 2\int_0^{2\cdot4} dz = 2\Big[z\Big]_0^{2\cdot4} = \mathbf{4{\cdot}8}.$$

This result may be verified without using the logarithmic form for the inverse function; thus, from the equation of the curve,

$$x = \cosh y, \quad \text{so that } \frac{dx}{dy} = \sinh y,$$

$$ds^2 = (1 + \sinh^2 y)dy^2 = \cosh^2 y \cdot dy^2;$$

$$\therefore\ ds = \cosh y \cdot dy.$$

Now when $x = 1$, $y = 0$, and when $x = 2{\cdot}6$, $y = \log 5$;

$$\therefore\ s = 2\int_0^{\log 5} \cosh y \cdot dy = 2\Big[\sinh y\Big]_0^{\log 5} = 2 \sinh (\log 5)$$

$$= (5 - \tfrac{1}{5}) = 4{\cdot}8 \text{ as before.}$$

To find the area,

$$A' = 2\int_0^{\log 5} x \cdot dy = 2\int_0^{\log 5} \cosh y \cdot dy = 2\Big[\sinh y\Big]_0^{\log 5} = 4{\cdot}8.$$

From Fig. 42, it will readily be seen that this is the area between the curve, the axis of y and the abscissae at $y = +\log 5$.

If A be the area between the axis of x and the double ordinate at $x = 2{\cdot}6$, then

$$A + A' = \text{area of rectangle } PQQ'P'$$
$$= 5{\cdot}2 \log 5;$$
$$\therefore\ A = 5{\cdot}2 \log 5 - A' = 5{\cdot}2 \log 5 - 4{\cdot}8.$$

Taking the area as

$$A = 2\int_1^{2{\cdot}6} y\, dx,$$

the strip $y \cdot dx$ is now between the curve, the x-axis and the given ordinates, so that the value of the integral will give the required area.

Hence,

$$A = 2\int_1^{2{\cdot}6} y \cdot dx = 2\int_1^{2{\cdot}6} \cosh^{-1} x \cdot dx = 2\int_1^{2{\cdot}6} \log (x + \sqrt{x^2 - 1}) \cdot dx$$

$$= 2\Big[x \cdot \log (x + \sqrt{x^2 - 1}) - \int \frac{x}{\sqrt{x^2 - 1}} \cdot dx\Big]_1^{2{\cdot}6}, \text{ integrating by parts,}$$

$$= 5{\cdot}2 \log 5 - 2\int_0^{2{\cdot}4} dz, \text{ where } z^2 = x^2 - 1,$$

$$= 5{\cdot}2 \log 5 - 4{\cdot}8, \text{ as before.}$$

108. Length of Arc in Polar Coordinates. When the equation of the curve is given in polar coordinates, and is therefore of the form $r=f(\theta)$, then, from (90*a*),

$$ds=\sqrt{r^2+(dr/d\theta)^2}\,.\,d\theta=\sqrt{1+r^2(d\theta/dr)^2}\,.\,dr,$$

so that the length of arc between the points (r_1, θ_1) and (r_2, θ_2) is

$$\mathbf{s}=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,.\,d\theta \quad \text{or} \quad \mathbf{s}=\int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,.\,dr, \quad \text{......(118)}$$

the former being generally much more convenient.

109. Area in Polars. Let dA be the area of a small sector POQ (Fig. 43), bounded by two radii vectores inclined at an angle $d\theta$ to each other at the pole, and an infinitesimal arc ds, then

$$dA=\tfrac{1}{2}r^2\,.\,d\theta.$$

Hence, for a continuous curve, the **area** between two points (r_1, θ_1) and (r_2, θ_2) is

$$\mathbf{A}=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}[f(\theta)]^2\,.\,d\theta. \quad \text{..............................(119)}$$

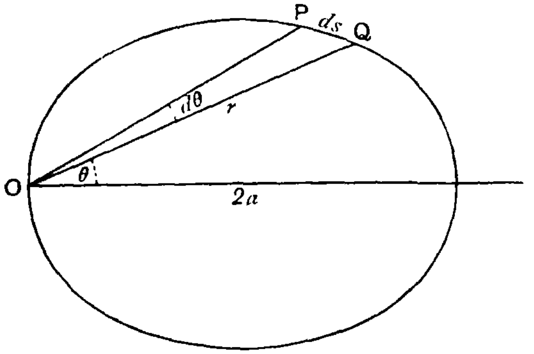

FIG. 43. Area in polars.

Ex. 3. *Shew that the length of one loop of the curve*

$$r=2a\cos^2\theta$$

is $\frac{2}{3}a\,.\,\sqrt{3}\{2\sqrt{3}+\log(2+\sqrt{3})\}$, *and find its area.*

One loop of the curve will be traced as θ varies from $\theta=-\dfrac{\pi}{2}$ to $\theta=\dfrac{\pi}{2}$. This is shewn in Fig. 43; since $\cos\theta=\cos(-\theta)$, the

axis divides the loop in half. Hence the integration may range from 0 to $\frac{\pi}{2}$, and the result doubled.

Now $$\frac{dr}{d\theta} = -4a\cos\theta\,.\,\sin\theta;$$

$$\therefore\ \left(\frac{ds}{d\theta}\right)^2 = r^2 + 16a^2\cos^2\theta\,.\,\sin^2\theta = 4a^2\cos^4\theta + 16a^2\cos^2\theta\,.\,\sin^2\theta$$

$$= 4a^2\cos^2\theta\,(\cos^2\theta + 4\sin^2\theta)$$

$$= 4a^2\cos^2\theta(1 + 3\sin^2\theta);$$

$$\therefore\ ds = 2a\cos\theta\sqrt{1 + 3\sin^2\theta}\,.\,d\theta$$

$$= 2a\sqrt{1 + 3u^2}\,.\,du, \text{ on putting } u = \sin\theta;$$

the limits $0, \frac{\pi}{2}$ for θ, then become 0, 1 for u; hence, the whole length of the loop becomes

$$s = 4a\int_0^1\sqrt{1 + 3u^2}\,.\,du$$

$$= \frac{4}{\sqrt{3}}a\,.\left[\frac{\sqrt{3}}{2}u\,.\,\sqrt{1 + 3u^2} + \tfrac{1}{2}\log(\sqrt{1 + 3u^2} + u\sqrt{3})\right]_0^1$$

$$= \frac{2}{3}a\sqrt{3}\{2\sqrt{3} + \log(2 + \sqrt{3})\}$$

as given.

To find the area, A,

$$\tfrac{1}{2}A = \tfrac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,.\,d\theta = 2a^2\int_0^{\frac{\pi}{2}}\cos^4\theta\,.\,d\theta.$$

Since the limits are $0, \frac{\pi}{2}$, Gamma functions may conveniently be used; thus, by (51),

$$A = 4a^2\int_0^{\frac{\pi}{2}}\cos^4\theta\,.\,d\theta = \frac{4a^2\,.\,\Gamma(\frac{1}{2})\,.\,\Gamma(\frac{5}{2})}{2\Gamma(3)} = \frac{4a^2\,.\,\sqrt{\pi}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,.\,\sqrt{\pi}}{2\,.\,2}$$

$$= \frac{3}{4}\pi a^2.$$

Without Gamma functions, the integration may be effected as follows :

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)\,;$$

$$\therefore\ \cos^4\theta = \tfrac{1}{4} + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{4}\cos^2 2\theta = \tfrac{1}{4} + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{8}(1+\cos 4\theta)$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{8}\cos 4\theta.$$

$$\therefore\ A = a^2\int_0^{\frac{\pi}{2}}(\tfrac{3}{2} + 2\cos 2\theta + \tfrac{1}{2}\cos 4\theta)\,d\theta = a^2\Big[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\Big]_0^{\frac{\pi}{2}}$$
$$= \frac{3}{4}\pi a^2$$

as before.

Ex. 4. *Transform the equation*

$$(x^2+y^2)(3ay - x^2 - y^2) = 4ay^3$$

into polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$; *hence, shew that the area of one loop is one-third that of a circle whose diameter is a.*

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

$\therefore$ The given equation becomes

$$r^2(3ar\sin\theta - r^2) = 4ar^3\sin^3\theta.$$

Divide out by r^3, assuming it is not zero,

$$3a\sin\theta - r = 4a\sin^3\theta$$
$$= 3a\sin\theta - a\sin 3\theta\,;$$
$$\therefore\ r = a\sin 3\theta.$$

Now r will be zero for $\theta = 0$, and for $\theta = \frac{\pi}{3}$; these are therefore the limits of integration for one loop.

$$\therefore\ A = \tfrac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,.\,d\theta = \tfrac{1}{2}a^2\int_0^{\frac{\pi}{3}}\sin^2 3\theta\,.\,d\theta = \tfrac{1}{4}a^2\int_0^{\frac{\pi}{3}}(1-\cos 6\theta)\,d\theta$$
$$= \tfrac{1}{4}a^2\Big[\theta - \tfrac{1}{6}\sin 6\theta\Big]_0^{\frac{\pi}{3}} = \frac{1}{12}\pi a^2,$$

which is $\frac{1}{3}$ the area of a circle whose radius is $\frac{1}{2}a$.

EXERCISES 14A.

Wherever possible, a rough graph of the curve should be sketched on squared paper.

1. Find the length and the area of the parabola $y^2=25x$ between the points where $x=0$ and $x=36$.

2. Find the length of the parabola $20y=3(2x^2-3x-5)$ between the points where it intersects the axis of x.
Find also the area of the curve between these points and the x-axis.

3. Find the length of the curve $y^2=x^3$ between the ordinates where $3x=4$ and $x=5$.

4. Shew that the length of the curve $y=a\log(x^2-a^2)$ between the points where $x=1{\cdot}5a$ and $x=11{\cdot}5a$ is $(10+\log 4{\cdot}2)a$.

5. Find the area of the curve $y^2=(13-x)(3+x)$ between the x-axis and the ordinates where $x=-3$ and $x=5$.

6. Calculate the area of the curve $y^2=(7-x)(5+x)$ between the x-axis and the ordinates where $x=-5$ and $x=1$.

7. Shew that the area of the curve $y^2(x^2+6x-55)=1$ between the x-axis and the ordinates where $x=7$ and $x=14$ is $\log 2$.

8. Plot the curve $y=2x^3-15x^2+24x+25$ between $x=0$ and $x=4$; then calculate the area enclosed by the ordinates at these points, the axis of x, and the arc of the curve.

9. Plot the curve $9y^2=x^2(9-x^2)$, and calculate the area of one of its loops.

10. Trace the curve $y=11x-x^2-18$ from $x=2$ to $x=9$, and calculate the area enclosed by it and the x axis between those points.

11. Find the length and the area of the cycloid

$$x=r(\theta-\sin\theta),\quad y=r(1-\cos\theta)$$

from cusp to cusp.

12. Calculate the length of the epicycloid

$$x=(a+b)\sin\theta-b\sin\left(\frac{a+b}{b}\,.\,\theta\right),$$

$$y=(a+b)\cos\theta-b\cos\left(\frac{a+b}{b}\,.\,\theta\right),$$

for one revolution of the rolling circle whose radius is b, whilst a is the radius of the fixed circle.

13. Find the perimeter of the cardioide

$$x=2a\sin\theta(1-\cos\theta),\quad y=2a\cos\theta-a\cos 2\theta.$$

***14.** By putting $x = a\sin^3\theta$, $y = a\cos^3\theta$, find the length of the four-cusped hypocycloid between the limits where $\theta = 0$ and $\theta = \frac{\pi}{2}$.

15. Find the length of the catenary $y = 5\cosh\frac{x}{5}$, the values of x at the points of suspension being $\pm 8{\cdot}047$.

***16.** The evolute of the parabola, $y^2 = 4ax$, is given by the equation,

$$4(x - 2a)^3 = 27ay^2.$$

Shew that this curve meets the parabola at $x = 8a$, and find the length of the evolute between this point and the point where it crosses the x-axis.

***17.** The evolute of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is given by the equation

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

Find the length of one arc of this evolute, the limits being $x = 0$ and the ordinate where $y = 0$.

Shew that the length of this arc is equal to the difference between the radii of curvature of the ellipse at the extremities of the minor and major axes respectively. See p. 273.

***18.** Shew that the length of the curve $y = \log\sec x$ between the points where $x = 0$ and $x = \frac{\pi}{3}$ is $\log(2 + \sqrt{3})$.

***19.** Find the length of the curve

$$x = \log(\coth y - \operatorname{cosech} y)$$

between the points where $y = \log 2$ and $y = \log 5$.

***20.** Shew that the length s, of the arc of the spiral of Archimedes

$$r = 2\theta,$$

between $\theta = 0$ and $\theta = \alpha$, where $\alpha < 2\pi$, is given by the relation

$$\alpha = \sinh(s - \alpha\sqrt{1 + \alpha^2}).$$

Calculate s when $\alpha = \pi$.

***21.** In the equiangular spiral, $r = ae^{\theta\cot\alpha}$, prove that

$$\phi\cot\alpha = \log\left(\frac{s}{a}\cos\alpha + 1\right),$$

where s is the length of arc from $\theta = 0$ to $\theta = \phi$.

Hence, shew that if b is the increase in the length of r, for one revolution, then $s = ab\sec\alpha$.

22. Find the length of the arc of the parabola $a = r\cos^2\frac{\theta}{2}$, cut off by the latus rectum, *i.e.* the perpendicular to the axis at the origin. Work this out for $a = 5$.

***23.** Shew that the length s of the arc of the hyperbolic spiral,

$$r\theta = a,$$

is given by
$$s = a\int_{\theta_1}^{\theta_2} \frac{\sqrt{1+\theta^2}}{\theta^2} \, . \, d\theta.$$

Transform this integral into the form

$$s = a\int_{u_1}^{u_2} \frac{du}{u^2(1-u^2)},$$

by means of the substitution $\theta^2 = \frac{u^2}{1-u^2}$; hence, by resolving into partial fractions, evaluate s, between $\theta = \frac{5}{12}$ and $\theta = \frac{21}{20}$, taking $a = 2{\cdot}1$.

***24.** BOA is a quadrant of a circle whose centre is O; M is a point in OA such that $OM = 8$ inches, and MP is drawn parallel to OB. If the radius $OA = 17$ inches. find the area of the figure bounded by OB, OM, MP and the arc BP.

25. The work done in expanding a gas from a volume v to a volume V is measured by the area of the curve whose general equation is

$$x^n y = c$$

between the x-axis and the ordinates at $x = v$ and $x = V$, n and c being constants and y the pressure of the gas.

When the expansion is isothermal, $n = 1$ and $c = a$; when it is adiabatic, $n = 1 + \gamma$, where γ is a constant depending upon the gas used, and $c = b$.

Shew that the difference between the work done in expanding a gas isothermally and adiabatically is

$$\frac{1}{\gamma}(pv - PV) - pv \log \frac{V}{v},$$

where p, P are the values of y corresponding to $x = v$ and $x = V$ respectively.

Calculate the work done in each case when $V = 276$, $P = 75$, $V = 2v$ and $\gamma = 0{\cdot}4$.

***26.** The area of the curve $y = ax^2 + 2bx - 15$ between the x-axis and the ordinates at $x = 1$, $x = 5$, is 32 square units, and between the x-axis and the ordinates at $x = 2$, $x = 4$, it is 22 square units. Find the numerical values of a and b.

***27.** Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

The area of the parabola $16y^2 = 121x$, between the curve and the double ordinate at $x = 9$, is equal to half the area of the ellipse

$$\frac{x^2}{81} + \frac{y^2}{b^2} = 1;$$

find the semi-minor axis of the ellipse, taking $\pi = 22/7$.

***28.** Sketch roughly, and find the area of each of the following curves,

$$\text{(i) } y = e^{ax}\sin bx, \qquad \text{(ii) } y = e^{ax}\cos bx,$$

between the ordinates where $x=0$ and $x=\omega$.

If the curves intersect for the first time on the positive sides of the axes at $x=\omega$, find the value of ω, and shew that the area enclosed between the two curves and the y-axis is

$$\frac{b(e^z\sqrt{2}-1)-a}{a^2+b^2},$$

where $z=\dfrac{a\pi}{4b}$. Calculate this area when $a=7$, $b=5{\cdot}5$ and $7\pi=22$.

***29.** Prove that the perimeter s, of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

is given by the equation

$$s=4a\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{\frac{1}{2}}\,d\theta,$$

where $a^2e^2=a^2-b^2$ and $x=a\sin\theta$.

Hence, by applying the binomial expansion to the integrand, shew that s is given approximately by the formula

$$32s=\pi a(64-16e^2-3e^4).$$

Calculate s to the nearest tenth of an inch, when $a=25$ in. and $b=24$ in., and shew that this perimeter is less than that of the circle whose radius is a by π very nearly.

***30.** Shew that the length of the curve $y=a\sin x$, from $x=0$ to $x=\pi$, is given by the integral

$$s=2a\int_0^{\frac{\pi}{2}}(1+a^2\cos^2 x)^{\frac{1}{2}}\,.\,dx.$$

Assuming that $a<1$, prove, by expansion, that

$$64s=(64+16a^2-3a^4)\pi$$

approximately.

Calculate s when $a=0{\cdot}25$.

***31.** Shew, by eliminating t, that the lemniscate, defined by the equations

$$x=\frac{a^2t(t^2+a^2)}{a^4+t^4}, \quad y=\frac{a^2t(t^2-a^2)}{a^4+t^4},$$

may also be defined by the equation

$$(x^2+y^2)^2=a^2(x^2-y^2);$$

and from this shew, by putting $x=r\cos\theta$, $y=r\sin\theta$, that its polar equation is $r^2=a^2\cos 2\theta$.

Hence, sketch the curve roughly and find the area of one loop.

***32.** Trace the curve $(x^2+y^2)(x^2+y^2-3ay)+4ay^3=0$, and find the area enclosed by one of its loops. (L.U., Sc.)

***33.** Shew that the common chord joining the points where the parabola $y^2=4ax$

intersects the parabola $\qquad y^2=4b(h-x)$,

is the double ordinate at $\qquad x=\dfrac{hb}{a+b}$.

Hence prove that the area bounded by the arcs of these parabolas is

$$\frac{8}{3}\left(\frac{h^3ab}{a+b}\right)^{\frac{1}{2}}.$$

Calculate this area when $a=4$, $b=9$ and $h=13$.

***34.** If s be the length of arc, measured from the origin to the ordinate where $x=a$ of the curve $y^2=x(1-\frac{1}{3}x)^2$,

shew that $\qquad 9s^2=a(a+3)^2$.

Find also the area of the loop of this curve.

***35.** If s, A denote the perimeter and area respectively of the curve

$$r^3=a^3\cos 3\theta,$$

prove that $\qquad 2As=3\pi a^3\sqrt{3}$.

***36.** Shew that the whole area of the curve

$$a^6y^2=x^6(a^2-x^2)$$

is $8a^2/15$.

37. Obtain the area in the first quadrant bounded by the curve whose equation is $b^4y^2=(a^2-x^2)^3$ and the line $x=0$. (L.U.)

38. The chain of a suspension bridge has the form of the curve $x^2=b^2y/h$, where the origin of coordinates is taken as the lowest point; the axis of y is vertical, b is half the span, and h the dip of the chain. Write down an expression for the length of the chain in the form of an integral.

Shew that when h is much smaller than b, the radical under the integral sign may be expanded by the binomial theorem and that the length of the chain is approximately $2b+4h^2/3b$. (L.U.)

***39.** If a rod move with its ends on a closed curve of area A and make a complete revolution, prove that a point P on the rod at distances c and c' from the ends will trace out an area of $A-\pi cc'$. (L.U.)

40. Find the area of the loop of the curve whose equation is

$$ay^2=(x-a)(x-5a)^2. \qquad \text{(L.U.)}$$

41. At each point of a curve the gradient varies inversely as the cube of the abscissa of the point, and the curve passes through the points (2, 0) and (4, 3). Prove that the equation of the curve is $x^2y=4(x^2-4)$, and find the area bounded by the curve, the axis of x and the line $x=4$. (D.U.)

***42.** Shew that the area included by the parabola $r=a\sec^2\dfrac{\theta}{2}$ and the focal vectors of lengths a and r is $\frac{1}{3}\sqrt{a(r-a)}\,.\,(2a+r)$. (Br.U.)

110. Surface Areas and Volumes of Solids of Revolution. Let S be the area of a surface generated by the revolution of the arc PQ (Fig. 44), of a plane curve $y=f(x)$, about the x-axis,

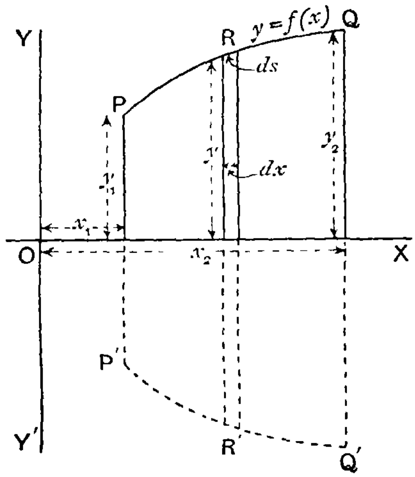

FIG. 44. Surface and volume of a solid of revolution.

between the points $P(x_1, y_1)$, $Q(x_2, y_2)$; then an element of arc ds, whose ordinate is y, traces out an area, $2\pi y\,.\,ds$ in one revolution; hence the **whole area** generated is

$$\mathbf{S}=2\pi\int_{x=x_1}^{x=x_2} y\,.\,ds=2\pi\int_{x_1}^{x_2} f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx, \quad\text{............}\mathbf{(120a)}$$

or, in some cases, it may be easier to evaluate the integral,

$$\mathbf{S}=2\pi\int_{y_1}^{y_2} y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,.\,dy. \quad\text{........................}\mathbf{(120b)}$$

Let V be the **volume** generated by the revolution about the x-axis of the area bounded by the arc PQ, the axis of x, and the ordinates at x_1, x_2, then the volume of a thin circular slice RR' perpendicular to the axis of revolution, of radius y and thickness dx, is $\pi y^2 \,.\, dx$; hence

$$\mathbf{V} = \pi\int_{x_1}^{x_2} y^2 \,.\, dx = \pi\int_{x_1}^{x_2}\{f(x)\}^2 dx. \qquad \text{(121a)}$$

For the **volume of a regular solid** which is not generated by the revolution of a plane curve, the area of a slice of thickness dx, perpendicular to the axis of symmetry, must be found by the methods of the previous article. Denoting this area by A,

$$\mathbf{V} = \int_{y_1}^{y_2} A \,.\, dx. \qquad \text{(121b)}$$

Ex. 5. *The curve $y = x(6-x) - 7{\cdot}56$ revolves about the axis of x between the points where it crosses the axis; find* (i) *the surface area, and* (ii) *the volume of the solid thus generated.*

The curve crosses the x-axis when $y = 0$, *i.e.* when

$$x(6-x) - 7{\cdot}56 = 0, \quad \text{or} \quad 25x^2 - 150x + 189 = 0,$$

i.e.

$$(5x-21)(5x-9) = 0,$$

so that

$$\mathbf{x = 4{\cdot}2 \text{ or } 1{\cdot}8.}$$

By plotting the curve between these values, it will be seen that the graph is the arc of a parabola, and the ordinate

$$x = \tfrac{1}{2}(4{\cdot}2 + 1{\cdot}8) = 3$$

cuts the arc in half; hence the integration may be taken from $x = 1{\cdot}8$ to $x = 3$, and the result doubled.

Since

$$y = 6x - x^2 - 7{\cdot}56,$$

$$\therefore \frac{dy}{dx} = 6 - 2x = 2(3-x);$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + 4(3-x)^2};$$

$$\therefore S = 2\pi\int_{x=1{\cdot}8}^{x=4{\cdot}2} y \,.\, ds = 4\pi\int_{1{\cdot}8}^{3}(6x - x^2 - 7{\cdot}56)\sqrt{1+4(3-x)^2} \,.\, dx.$$

Put $2(3-x) = \tan z$, then $-2 \,.\, dx = \sec^2 z \,.\, dz$, and when $x = 1{\cdot}8$. $z = \tan^{-1} 2{\cdot}4$; denote the principal value of this angle by z_1, when $x = 3$, $z = \tan^{-1} 0 = 0$.

Finally, $$6x - x^2 - 7{\cdot}56 = -7{\cdot}56 + 9 - (3 - z)^2$$
$$= 1{\cdot}44 - 0{\cdot}25 \tan^2 z$$
$$= 1{\cdot}69 - 0{\cdot}25 \sec^2 z\,;$$

$$\therefore\ S = -2\pi \int_{z_1}^{0} (1{\cdot}69 - 0{\cdot}25 \sec^2 z) \,.\, \sec^3 z \,.\, dz$$
$$= \left\{ 3{\cdot}38 \int_0^{z_1} \sec^3 z \,.\, dz - 0{\cdot}5 \int_0^{z_1} \sec^5 z \,.\, dz \right\} \pi\,:$$

$$\therefore\ 400S = \Big[601 \{\sec z \,.\, \tan z + \log(\tan z + \sec z)\} - 50 \sec^3 z \,.\, \tan z) \Big]_0^{z_1} \,.\, \pi,$$

on applying the results of Ex. 25, p. 133.

Since $\tan z_1 = 2{\cdot}4$, $\sec z_1 = 2{\cdot}6$, hence

$$400S = (1611{\cdot}12 + 601 \log 5)\pi,$$

giving $$S = 6{\cdot}52\pi = 20{\cdot}47.$$

For the volume V,

$$V = \pi \int_{1\cdot 8}^{4\cdot 2} y^2 \,.\, dx = 2\pi \int_{1\cdot 8}^{3} (6x - x^2 - 7{\cdot}56)^2 \,.\, dx$$
$$= 2\pi \int_0^{1\cdot 2} (1{\cdot}2^4 - 2 \,.\, 1{\cdot}2^2 v^2 + v^4) \,.\, dv, \text{ on putting } v = 3 - x,$$
$$= 2\pi \Big[1{\cdot}2^4 v - \tfrac{2}{3} \,.\, 1{\cdot}2^2 \,.\, v^3 + \tfrac{1}{5} v^5 \Big]_0^{1\cdot 2}$$
$$= (16 \times 1{\cdot}2^5/15)\pi = 8{\cdot}339.$$

Ex. 6. *Shew that the volume of a right pyramid is equal to one-third of the volume of a regular prism of the same height standing on the same base.*

Prove also that the volume V of a frustum whose parallel areas are A and A/n^2, and whose distance apart is h, is given by the formula

$$3n^2 V = A(n^2 + n + 1)h.$$

Hence find n when $V = 65$, $A = 15$ and $h = 9$.

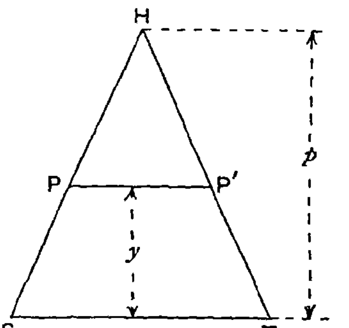

FIG 45. Volume of a pyramid.

A right pyramid is one whose centroidal axis is perpendicular to its base.

Let PP' (Fig. 45) be a thin slice, of area a and thickness dy,

parallel to, and distant y from the base ST, whose area is A. Then if p be the height of the pyramid,

$$a/A = (p-y)^2/p^2,$$

or

$$a = (p-y)^2A/p^2.$$

Hence the volume $V = \int_0^p a \, . \, dy$

$$= (A/p^2)\int_0^p (p^2 - 2py + y^2)\,dy$$

$$= A/p^2\Big[p^2y - py^2 + y^3/3\Big]_0^p$$

$$= Ap/3.$$

But Ap = volume of a regular prism of height p standing on the base ST; hence, **the volume of a right pyramid = one-third of that of a regular prism of the same height standing on the same base.**

For the frustum, the limits of integration are now h to 0, so that the volume $V = Ah(p^2 - ph + h^2/3)$.

But since $a = (p-y)^2A/p^2$, and the area of the upper face is A/n^2,

$$\therefore \; A/n^2 = (p-h)^2A/p^2,$$

from which

$$p = nh/(n-1).$$

Substituting this value of p in the expression for the volume,

$$V = \frac{(n-1)h}{n^2}\left\{\frac{n^2}{(n-1)^2} - \frac{n}{n-1} + \frac{1}{3}\right\} . A = Ah(n^2+n+1)/(3n^2)\,;$$

$$\therefore \; 3n^2V = Ah(n^2+n+1).$$

When $V = 65$, $A = 15$, $h = 9$,

$$195n^2 = 135(n^2+n+1),$$

or

$$4n^2 - 9n - 9 = 0\,;$$

i.e.

$$(n-3)(4n+3) = 0,$$

giving

$$n = 3 \quad \text{or} \quad -0{\cdot}75.$$

The negative value is inadmissible in this case;

$$\therefore \; \mathbf{n = 3.}$$

Ex. 7. *Prove that the area of surface of a prolate spheroid of eccentricity e and major axis* $2a$ *is*

$$4\pi a^2(1-e^2)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}}(1 - e^2\cos^2\phi)^{\frac{1}{2}}\sin\phi \, . \, d\phi\,;$$

hence shew that if powers of e above the second may be neglected this area is that of a sphere which has the same volume as the spheroid.
(L.U.)

When an ellipse revolves about its major axis, the solid generated is called a **prolate spheroid**; when it revolves about its minor axis, the solid generated is called an **oblate spheroid**.

Let S be the surface of the prolate spheroid, then $S = 2\pi\int y \, . \, ds$, and if ϕ be the eccentric angle of any point (x, y) on the generating ellipse, $x = a \cos \phi$, $y = b \sin \phi$, from (73c), b being the semi-minor axis;

$$\therefore \ (ds/d\phi)^2 = a^2 \sin^2 \phi + b^2 \cos^2 \phi = a^2(1 - e^2 \cos^2 \phi),$$

since $a^2e^2 = a^2 - b^2$.

The limits over half the spheroid are $\phi = 0$, and $\phi = \pi/2$; hence for the whole surface,

$$S = 4\pi ab\int_0^{\frac{\pi}{2}} (1 - e^2 \cos^2 \phi)^{\frac{1}{2}} \, . \, \sin \phi \, . \, d\phi,$$

which, on replacing b by $a(1 - e^2)^{\frac{1}{2}}$, gives the required expression.

Since powers of e above the second may be neglected, and $e^2 \cos^2 \phi < 1$, the root may be expanded by the binomial theorem, giving

$$S = 4\pi ab\int_0^{\frac{\pi}{2}} (\sin \phi - \tfrac{1}{2}e^2 \sin \phi \, . \, \cos^2 \phi + \dots)\, d\phi$$

$$= 4\pi a^2(1 - \tfrac{1}{2}e^2)(1 - \tfrac{1}{6}e^2) = 4\pi a^2(1 - \tfrac{2}{3}e^2).$$

Now if r be the radius of a sphere of the same volume as the spheroid,

$$\tfrac{4}{3}\pi r^3 = 2\pi\int_0^{\frac{\pi}{2}} y^2 \, . \, dx = 2ab^2\pi\int_0^{\frac{\pi}{2}} \sin^3 \phi \, . \, d\phi = \tfrac{4}{3}\pi ab^2;$$

$$\therefore \ r = (ab^2)^{\frac{1}{3}},$$

and the surface of the sphere $S' = 4\pi r^2 = 4\pi a^{\frac{2}{3}} b^{\frac{4}{3}}$

$$= 4\pi a^2(1 - e^2)^{\frac{2}{3}}$$

$= 4\pi a^2(1 - \frac{2}{3}e^2)$, neglecting powers of e above the second,

$= S$, from above;

so that the surface of the spheroid = that of a sphere equal in volume when powers of e above the second can be neglected.

EXERCISES 14B.

1. The curve $y=ae^{bx}$ passes through the points $x=1$, $y=3{\cdot}5$; $x=10$, $y=12{\cdot}6$; determine the values of a and b.

This curve rotates about the axis of x, thus generating a solid of revolution. Find the volume of this solid between the ordinates at $x=1$ and $x=10$.

2. The curve $x^2y=(3-x)(3+x)$ revolves about the axis of x; find the volume of the solid generated between the ordinates at $x=1$ and $x=3$.

3. Find the surface generated by the revolution of the curve

$$y=x^3$$

about the x-axis between the ordinates at $x=0{\cdot}5$ and $x=0$.

4. In the curve
$$y=a+bx^{\frac{1}{2}},$$
if $y=1{\cdot}82$ when $x=1$, and $y=5{\cdot}32$ when $x=4$, find a and b.

Let this curve rotate about the axis of x.

Find the volume enclosed by the surface of revolution between the two sections at $x=1$ and $x=4$.

5. Find the volume of the paraboloid generated by the revolution of the parabola

$$y^2=4a(h-x)$$

about the x-axis.

On the flat circular end of this paraboloid a hemisphere of the same diameter is fastened, and the total volume of the solid thus formed is equal to that of a cylinder whose length and radius are each equal to the radius of the common section. Prove that

$$9h=16a.$$

***6.** Shew that the surface area S, and volume V, of the solid generated by the revolution of the parabola

$$2(a-x)=\tan\theta, \quad 4y+1=\sec^2\theta,$$

about the x-axis between the ordinates at $x=0$ and $x=2a$, are given by

$$(a)\ \ 16S=\pi\{2a(1+8a^2)\sqrt{1+4a^2}-\sinh^{-1}2a\},$$

$$(b)\ \ 5V=2\pi a^5.$$

7. The curve $y^2(x-8)=x(x-6)$ revolves about the x-axis; find the volume generated between the ordinates at $x=0$ and $x=6$.

8. A tank has plane uniformly sloping sides, the top and bottom being horizontal rectangles of sides a, b, and α, β respectively, whilst the vertical depth is h. Find the capacity of the tank, and shew that if the tank is the frustum of a pyramid, then

$$a:b=\alpha:\beta.$$

Calculate the capacity in gallons when $a=12$ ft., $b=9$ ft., $\alpha=8$ ft., $\beta=6$ ft. and $h=10$ ft., it being given that 25 gallons occupy four cubic feet.

9. Find the whole surface of the paraboloid generated by the revolution of the parabola

$$y^2=25x$$

about the x-axis between the ordinates at $x=0$ and $x=36$.

10. Find the surface area and the volume of the solid generated by the revolution of the cycloid

$$x=r(\theta-\sin\theta),\quad y=r(1-\cos\theta),$$

about the x-axis.

***11.** Find the surface generated by the revolution of an arc of a circle of radius r, subtending an angle 2α at the centre, about a line in its plane parallel to its chord and at a distance d from it.

Deduce from the result the surface generated

(a) when the axis coincides with the chord,

(b) when the axis passes through the centre,

(c) when the arc is a semicircle revolving about its diameter.

***12.** Find the surface generated by the revolution of a semicircular arc of radius r about the tangent at its middle point.

13. $ABCD$ is a four-sided figure having AB, CD parallel and each perpendicular to BC. The figure revolves about BC; find the volume of the solid thus generated when $AB=10$ ft., $CD=6$ ft. and $BC=8$ ft.

14. Find the volume of a frustum of a sphere of radius 5 ft. lying between two parallel planes on opposite sides of the centre at distances 3 ft. and 2 ft. respectively from the centre.

15. A bowl, in the form of a spherical segment, is two feet in diameter at the top, and is 9 inches deep. Find the radius of the sphere of which it forms part, and, neglecting the thickness of the material, find how much water the bowl will hold. Give the answer in gallons, having given that one gallon of water occupies 0·16 cubic feet.

16. Find the volume cut off from a hemisphere of radius r, by a plane parallel to the flat surface and at a distance d from it.

Deduce, from the result, the volume

(a) when $d=\frac{1}{2}r$.

(b) when the segment cut off has a depth h and radius of flat surface a.

Calculate how many gallons of water a bowl in the form of a hemispherical segment would hold when its upper diameter is 2·5 ft. and its depth 10 inches, taking 6·25 gallons of water to one cubic foot.

***17.** A sphere is cut by a plane intersecting the diameter perpendicular to it at P. If $PB = n \cdot AP$, where $n < 1$, V_1, S_1, are the volume and surface area of the larger segment, and V, S, are the volume and surface area of the whole sphere, prove that

$$(a)\ V_1 : V = 1 + 3n : (1+n)^3,$$
$$(b)\ S_1 : S = 1 : 1 + n,$$
$$(c)\ 36\pi V_1^2 : S_1^3 = (1+3n)^2 : (1+n)^3.$$

If $S_1 = 990$ square feet, and $n = 0{\cdot}4$, find V_1 and the radius of the sphere. Take $\pi = 22/7$.

Prove the following formulae where S = surface area, and e = eccentricity of the generating ellipse, and is given by $a^2e^2 = a^2 - b^2$.

***18.** For a prolate spheroid :

$$S = 2\pi a^2 \left\{ 1 - e^2 + \frac{\sqrt{1-e^2}}{e} \cdot \sin^{-1} e \right\}.$$

***19.** For an oblate spheroid,

$$S = 2\pi a^2 \left\{ 1 + \frac{1-e^2}{2e} \log \frac{1+e}{1-e} \right\}.$$

***20.** Calculate the surface areas in Exs. 18 and 19 when $a = 5$ ft., and $b = 4$ ft.

If S_1, S_2 denote the respective areas of the prolate and oblate spheroid, shew that $S_2 = 1{\cdot}16 S_1$ approximately.

21. Calculate the volume of the oblate spheroid generated by the ellipse $4x^2 + 81y^2 = 324$.

22. The ratio of the volumes of an oblate and a prolate spheroid is $1{\cdot}6$. Find the area of the generating ellipse if the sum of its semi-axes is $22{\cdot}75$ ft.

***23.** An anchor ring is a solid formed by the revolution of a circle of radius r about an axis distant R from its centre and in its plane, R being greater than r; if A = area of the circle, p = its perimeter, P = circumference of the circle described by the centre of the revolving circle, prove that

$$(a)\ S = AP, \qquad (b)\ V = Ap,$$

where S, V are the surface area and volume of the ring respectively.

***24.** $ABCD$ is a square whose side is of length $2a$; on AB a semicircle is described outside the square. The whole figure then revolves about the side CD; find the volume of the solid thus generated.

If V denote this volume, and $4V = 9\pi(14 + 3\pi)$, find the side of the square.

***25.** A uniform sphere, whose radius is a, has a cylindrical hole of radius b bored through it so that the axis of the hole is a diameter of the sphere. Shew that the ratio of the volume of the solid to that of the whole sphere is $(a^2 - b^2)^{\frac{3}{2}} : a^3$.

If $a = p^2 + q^2$, and $b = 2pq$, shew that the volume removed is

$$\tfrac{8}{3}\pi q^2(3p^4 + q^4).$$

Calculate the volume of the pierced sphere when $a = 13$ ft. and $b = 5$ ft.

***26.** ABC is a triangle having a right angle at C; on CB a quadrant of a circle is drawn outside the triangle meeting AC produced in D. The whole figure revolves about AD, thus generating a solid whose volume is $1296\frac{1}{6}$ cubic feet. If $AC : CD = 45 : 28$, find (a) the lengths of AC, CD, and AB, (b) the surface area of the solid, taking $7\pi = 22$ and without using tables.

A right circular cylinder whose axis is vertical is represented in elevation by a rectangle $ANSB$, SB being the base. It is cut by a plane perpendicular to the paper through N, which intersects AB at C. The height of the cylinder, $SN = h$, the radius of its base $= r$, and $CB = c$.

***27.** Find the volume of the solid $CNSB$.

***28.** If the solid $CNSB$ be hollow and made of thin sheet iron, whose thickness is negligible, find the area of iron plate needed.

Calculate this area in square feet when $r = 7{\cdot}5$ in., $h = 1$ ft. 8 in., and $c = 1$ ft.

***29.** The solid $CNSB$ is cut by another plane MR perpendicular to the base BS and the plane of the paper, which intersects CN at M. Find the volume of the solid $MNSR$, if $RS = r - b$, and $MR = a$.

Calculate this volume in cubic feet when $r = 1$ ft. 1 in., $a = 11$ in., $h = 21$ in., and $b = 5$ in.

***30.** Shew from the result of Ex. 29, that if the plane MR passes through the geometrical axis of the cylinder, the volume of the solid is

$$\tfrac{1}{6}r^2(4h + 3\pi a - 4a).$$

Hence find r, when $a = 7$ in., $h = 14{\cdot}5$ in., and this volume is 484 cubic inches.

31. Find the surface generated by the revolution of the catenary

$$y = c \cosh \frac{x}{c},$$

about the y-axis, from $x = 0$ to $x = c$.

Calculate also the volume generated.

***32.** Two right circular cylinders, each of base radius r, intersect with their axes cutting each other at right angles. Find the surface area and the volume of the solid common to both.

***33.** Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and find the radius of the sphere of equal volume when $a = 9$, $b = 6$, and $c = 4$.

***34.** Find the volume generated between the point where the curve crosses the axis of x on the positive side and the ordinate where $x=12$, by the hyperbola $4x^2-9y^2=36$ revolving about the x-axis.

35. The curve $ay^2=x^3$ revolves about the axis of y; find the surface area and the volume generated between the planes perpendicular to the axis of revolution at the origin and through the point where $27y=8a$.

36. Find the volume generated by the revolution of an equilateral triangle of side a, about one of its sides.

37. A segment of a circle whose chord is 6 inches and height 1 inch is revolved round its chord; find the number of cubic inches in the solid spindle so formed. (L.U.)

***38.** Prove that the volume of a paraboloid of revolution cut off by a plane perpendicular to its axis is half the volume of the surrounding cylinder.

A closed cylindrical vessel containing water is rotating with uniform angular velocity ω about its axis, which is vertical. The free surface of the water is known to be a paraboloid formed by the revolution of the parabola $y=\omega^2x^2/(2g)$ about its axis. Shew that the difference in the heights of the lowest and highest points of the paraboloid varies as ω or ω^2 according as the water is or is not in contact with the top of the cylinder, provided that the cylinder contains sufficient water for the base to remain covered. (L.U.)

***39.** Sketch the curve $y(1+x^2)=(3-x)(x-2)$, and shew that the volume generated by revolving it about the axis of x is

$$\tfrac{1}{2}\pi(5\log 2-2-10\tan^{-1}\tfrac{1}{7}). \qquad \text{(Br.U.)}$$

***40.** Trace the curve $8a^2y^2=x^2(a^2-x^2)$, and prove that the length of the arc from the origin to the point (x, y) is $y+\frac{1}{2}a\sqrt{2}\,.\sin^{-1}(x/a)$. If the curve revolve about the axis of x, prove the total area of the surface generated is $\frac{1}{2}\pi a^2$. (D.U., Sc.)

111. Approximate Methods. In many practical problems the precise relationship between the variables concerned is unknown, and it is, therefore, not possible to evaluate an integral of the form $\int y\,.\,dx$ by any of the preceding methods. In such cases, however, approximate methods of calculation have been devised which will yield results to a degree of accuracy quite sufficient for the purpose. These methods depend upon plotting an approximate curve through a series of discrete points whose coordinates are observed values of the variables under consideration; hence the process is often called **graphical integration.**

112. The Trapezoidal Rule. Suppose a series of corresponding pairs of observed values of two variables x, y, are plotted and an approximate continuous curve drawn through the points, assuming there are no discontinuities in the series, then to find the area bounded by the curve, the axis of x and two given ordinates, the simplest method of approximation is as follows. Divide the area into n strips of equal width h by drawing $n+1$ ordinates, n being so large that each strip may be regarded as a trapezium. The required area is therefore the sum of the areas of the n trapezia. Denote the ordinates by y_s ($s=0, 1, 2, \ldots n$), then the area becomes

$$\tfrac{1}{2}h(y_0+y_1)+\tfrac{1}{2}h(y_1+y_2)+\ldots+\tfrac{1}{2}h(y_{n-1}+y_n)$$

$$= h\{\tfrac{1}{2}(y_0+y_n)+y_1+y_2+\ldots+y_{n-1}\}. \qquad \ldots\ldots\ldots\ldots\ldots(122)$$

This rule is known as the **Trapezoidal Rule.**

Ex. 8. *The following table gives corresponding measurements of two quantities, x and y. Plot these on squared paper and calculate the area of the curve between the axis of x and the extreme ordinates.*

x	0	2	3	4	5	7	8
y	10	28	33	35	32	24	16

The approximate curve is shown in Fig. 46.

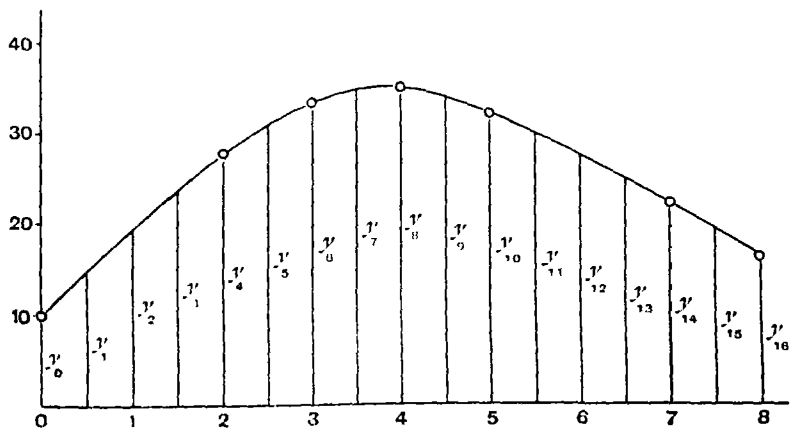

FIG. 46. Area of a curve by the Trapezoidal Rule.

Divide the area into 16 strips by drawing ordinates at intervals of 0·5 unit; denote the lengths of the ordinates by

$$y_s \ (s=0, 1, 2, \ldots 16),$$

then from the table

$$y_0=10, \quad y_4=28, \quad y_6=33, \quad y_8=35, \quad y_{14}=24, \quad y_{16}=16.$$

Reading from the graph, the lengths of the remaining ordinates are approximately,

$$y_1=15, \quad y_2=19, \quad y_3=23{\cdot}5, \quad y_5=31, \quad y_7=34{\cdot}5, \quad y_9=31,$$
$$y_{10}=32, \quad y_{11}=29{\cdot}5, \quad y_{12}=27{\cdot}5, \quad y_{13}=24{\cdot}5, \quad y_{15}=19.$$

$\therefore$ Sum of ordinates y_1 to $y_{15}=409{\cdot}5$, and $\frac{1}{2}(y_0+y_{16})=13$.

$\therefore$ By (122), the approximate area is

$$0{\cdot}5(13+409{\cdot}5)=\mathbf{211{\cdot}25 \text{ sq. units.}}$$

113. Simpson's Rule. In 1750 Thomas Simpson devised a rule with a view to securing greater accuracy than is obtainable by the Trapezoidal Rule. It depends upon finding a rational integral function which will express an approximate relationship between the variables It is therefore founded upon the method of § 63.

Let $y=f(x)$ be a function defined by a series of discrete points through which a smooth, continuous curve is drawn. Consider the area bounded by this curve, the axis of x, and the ordinates y_0, y_2, assuming that the whole area is divided into $2n$ strips of equal width h. For convenience let the ordinate y_0 coincide with the y-axis. As in § 63, let $f(x)$ take the form $y=a+bx+cx^2$, then to determine a, b, c,

$$\begin{aligned} y_0&=a, && \text{since } x_0=0. \\ y_1&=a+bh+ch^2, && \text{since } x_1=h. \\ y_2&=a+2bh+4ch^2, && \text{since } x_2=2h. \end{aligned}$$

Solving the last two equations for b and c,

$$b=(4y_1-3y_0-y_2)/(2h), \quad c=(y_2-2y_1+y_0)/(2h^2).$$

Now the area between y_0 and y_2

$$=\int_0^{2h} y\,.\,dx=\int_0^{2h}(a+bx+cx^2)\,dx=2h\,(a+bh+4ch^2/3)$$
$$=\tfrac{1}{3}h\,(y_0+4y_1+y_2)$$

on inserting the values of a, b, c.

In the same way, the area between y_2 and $y_4 = \frac{1}{3}h(y_2+4y_3+y_4)$, and so on, until the area of the last two strips

$$=\tfrac{1}{3}h(y_{2n-2}+4y_{2n-1}+y_{2n}).$$

Hence the total area between y_0 and y_{2n}

$$=\tfrac{1}{3}h\{y_0+y_{2n}+4(y_1+y_3+\dots+y_{2n-1})+2(y_2+y_4+\dots+y_{2n-2})\}.$$

This is **Simpson's Rule,** which may be stated as follows.

Divide the area to be calculated into 2n strips of equal width h by drawing 2n + 1 ordinates, then if

A = sum of end ordinates,
B = „ „ even ordinates,
C = „ „ odd ordinates,

the required area $=\frac{1}{3}$h(A + 4B + 2C). ....(123)

Ex. 9. *Find the area of the curve $xy=1$, between the ordinates at $x=1$ and $x=3$, (a) by Simpson's Rule, and (b) by the Trapezoidal Rule, and verify the results by integration.*

(*a*) In numerical calculations it is convenient, when employing Simpson's Rule, to use the following tabular method, which is largely adopted in practice.

Take ordinates at intervals of 0·2.

No. of ordinate.	Length of ordinate.	Simpson's multiplier.	Products.
1	1	1	1·000
2	0·833	4	3·332
3	0·714	2	1·428
4	0·625	4	2·500
5	0·556	2	1·112
6	0·500	4	2·000
7	0·455	2	0·910
8	0·417	4	1·668
9	0·385	2	0·770
10	0·357	4	1·428
11	0·333	1	0·333
		Total	16·481

$$\therefore\ \text{Area}=\tfrac{1}{3}\,.\,0{\cdot}2\,.\,16{\cdot}481=1{\cdot}0987.$$

Hence (*b*) By the Trapezoidal Rule,

$$\text{Area}=0{\cdot}2(0{\cdot}667+4{\cdot}842)=1{\cdot}1018.$$

By integration, area $=\displaystyle\int_1^3\frac{dx}{x}=\Big[\log x\Big]_1^3=\log 3=1{\cdot}0986.$

Thus the Trapezoidal Rule gives a result 0·0032 too high, whilst Simpson's Rule gives a result only 0·0001 too high. These results could be even nearer the true value if the number of ordinates were increased.

114. Application to the Determination of Volume. The above rules may equally well be applied to determine the approximate volume of an irregular body, such as a tree trunk, whose areas of cross-section perpendicular to its axis are known at regular intervals along that axis. For let the ordinates of a curve denote areas, then the volume V between an area of section a_0 and an area a_{2n}, is $\int a\,.\,dx$ taken between the limits $x_{2n}=x_0+2nh$ and x_0, which is, by Simpson's Rule, $\frac{h}{3}(A+4B+2C)$, where $A=a_0+a_{2n}$, $B=\sum_{s=1}^{n}a_{2s-1}$, and $C=\sum_{s=1}^{n-1}a_{2s}$. This is sometimes called the **Prismoidal Formula.**

Ex. 10. *The following are cross-sectional areas, A square feet, of a body 12 feet long at distances x feet from one end :—*

x	0	1·3	3·2	4·6	6	7·3	10	12
A	2·9	3·1	3·6	4·2	4·8	5·5	6·1	6

Draw the graph of A and x; give the probable cross-section at $x=8·5$, and find the approximate volume of the body.

The graph is shewn in Fig. 47. Draw ordinates at intervals of 6 inches, then denoting their lengths by A_0, A_1, A_2, ... A_{24}, and reading from the graph, the following gives the approximate values :

$A_0=2·9$, $A_5=3·35$, $A_{10}=4·35$, $A_{15}=5·6$, $A_{20}=6·1$,

$A_1=2·95$, $A_6=3·5$, $A_{11}=4·55$, $A_{16}=5·75$, $A_{21}=6·07$,

$A_2=3·0$, $A_7=3·7$, $A_{12}=4·8$, $A_{17}=5·9$, $A_{22}=6·06$,

$A_3=3·15$, $A_8=3·9$, $A_{13}=5·05$, $A_{18}=6·0$, $A_{23}=6·05$,

$A_4=3·25$, $A_9=4·25$, $A_{14}=5·3$, $A_{19}=6·05$, $A_{24}=6·0$.

Hence, at $x = 8{\cdot}5$, the probable area of cross-section

$$= A_{17} = 5{\cdot}9 \text{ sq. ft.}$$

By the Trapezoidal Rule,

$$\text{Volume of body} = 0{\cdot}5\{(\tfrac{1}{2}A_0 + A_{24}) + A_1 + A_2 + \dots + A_{23}\}$$
$$= 0{\cdot}5(4{\cdot}45 + 108{\cdot}68)$$
$$= 56{\cdot}57 \text{ cubic feet.}$$

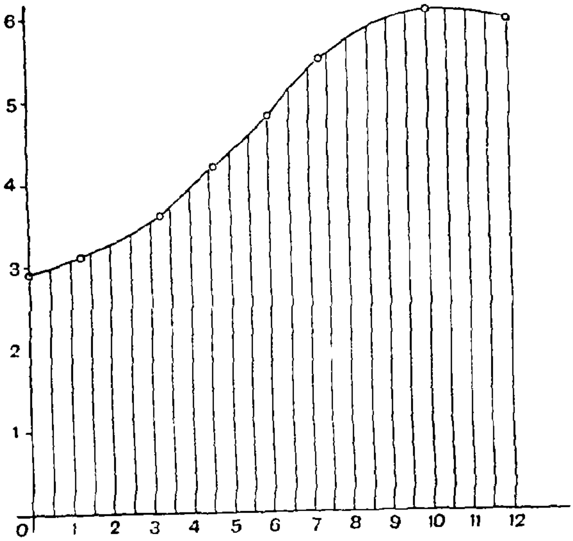

FIG. 47. Area by Simpson's Rule.

By Simpson's Rule,

$$\text{Volume} = \tfrac{1}{6}\{A_0 + A_{24} + 4(A_1 + A_3 + \dots + A_{23}) + 2(A_2 + A_4 + \dots + A_{22})\}$$
$$= \tfrac{1}{6}(8{\cdot}9 + 4 \times 56{\cdot}67 + 2 \times 52{\cdot}01)$$
$$= \tfrac{1}{6}(8{\cdot}9 + 226{\cdot}68 + 104{\cdot}02)$$
$$= 56{\cdot}60 \text{ cubic feet.}$$

This result should be checked by using the tabular method of Ex. 9.

Exercises 14c.

1. The following table gives corresponding values of x and y:

x	5	16	26	35	50
y	13	22	24	27	31

Plot x and y on squared paper, and find the area of the curve between the x-axis and the ordinates at $x=5$ and $x=50$,

2. A series of soundings taken across a river from shore to shore is given by the following table, x ft. being the distance from one shore and y ft. the corresponding depth.

x	0	1	2	3	4	5	6	7	8	9	10
y	0	5·4	10·6	12·6	14·4	15	13·4	11·6	8·3	4·7	0

Draw the section, and calculate its area by Simpson's Rule.

3. The following table gives corresponding values of x and y:

x	0	2	3	5	6	8	10	11	13	15	16
y	10	10·9	11·2	11·7	12·6	11·5	10·9	10·5	11·2	11·7	12·6

Find the approximate value of the integral $\int_0^{16} y \, . \, dx$.

4. x is the distance in chains measured along a straight line AB from the point A, the values of y are offsets or distances in chains measured at right angles to AB to the border of a field. Draw the shape of this border and find the area in square chains between the first and last offset, the straight line AB and the border.

y	0	1·5	3	5	7·5	9
y	0·53	0·27	0·46	0·42	0·35	0·52

5. A series of soundings taken across a river channel is given by the following table, x ft. being the distance from one shore and y ft. the corresponding depth. Draw the section and find its area.

x	0	10	16	23	30	38	43	50	55	60	70	75	80
y	5	10	13	14	15	16	14	12	8	6	4	3	0

6. The following corresponding values of x and y are given; find the approximate value of $\int_0^1 y \,.\, dx$ by Simpson's Rule.

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
y	1·27	1·36	1·47	1·58	1·70	1·84	2·00	2·19	2·43	2·73	2·93

7. x is the distance in feet across a river from one shore, and y is the corresponding depth in feet. Draw the section from the following measurements, and calculate its area.

x	0	4	9	10	13	16	18
y	0	56	81	80	65	32	0

8. The following are the areas of cross-section of a body at right angles to its straight axis.

A in square inches	250	292	310	273	215	180	135	120
x in. from one end	0	22	41	70	84	102	130	145

Find the whole volume from $x=0$ to $x=145$.

9. The area of a horizontal section of a pond is A sq. ft. at the height h ft. from the lowest point. When h is 30 what is the total volume of the water?

h	0	2·5	5	7·5	10	12·5	15	17·5	20	22·5	25	27·5	30
A	0	2510	3860	4670	5160	5490	5810	6210	6890	7680	8270	8620	8780

10. The cross-section of a tree is A sq. in. at a distance x in. from one end. Corresponding values of A and x are:

x	10	30	50	70	90	110	130	150
A	120	123	129	129	131	135	142	156

What is the volume of the tree in cubic inches, its total length being 160 inches?

11. The following values represent the areas of cross-section of a body perpendicular to its axis:

Area in sq. in.	250	292	310	273	215	180	135	120
x in. from end	0	22	41	70	84	102	130	145

Plot A and x on squared paper, and find the whole volume.

12. The area of a horizontal section of a tank is A sq. ft. at a height h feet from the lowest point. The following table gives a series of values of A and h:

h	0	1	2	3	4	5	6	7
A	76·8	77·24	77·66	78·04	78·38	78·7	78·96	79·20
h	8	9	10	11	12	13	14	
A	79·42	79·6	79·74	79·86	79·94	79·98	80	

Use the Trapezoidal Rule to calculate the total volume of the tank.

13. Shew that the area of the curve $x^2 = -4(y - 9)$, above the x-axis, is given exactly by Simpson's Rule.

14. Find the area of the curve $y = \cos x$, between the axis of x and the ordinates at $x = \frac{\pi}{2}$ and $x = 0$, by Simpson's Rule, taking 5° as the common interval between consecutive ordinates, and shew that the result is within 0·00005 of the exact value.

15. An odd number of parallel ordinates is drawn to a curve at equal intervals h. Prove that the area of the curve between the extreme ordinates is approximately

$$\tfrac{1}{3}h(A + 4B + 2C),$$

where A is the sum of the extreme ordinates, B the sum of the even ordinates, and C the sum of the remaining ordinates.

The under-water portion of a vessel is divided by horizontal planes, one foot apart, of the following areas: 472, 398, 302, 198, 116, 60, 34, 12, 4 sq. ft. Find the volume in cubic feet between the extreme areas. (L.U.)

16. Plot the function $4y - x^2 = 64$ for values of x between 8 and -8, then determine the area bounded by the curve and the x-axis,

(*a*) by either the Trapezoidal or Simpson's Rule,

(*b*) by direct integration.

***17.** If $f(x) = a + bx + cx^2$, shew that

$$\int_0^{2h} f(x)\,.\,dx = \tfrac{1}{3}h\{f(0) + 4f(h) + f(2h)\}.$$

Deduce Simpson's Rule, and use it to shew that 1·62 is an approximate value of $\int_1^5 \frac{dx}{x}$, taking ordinates at integral values of x. (Br.U.)

CHAPTER XV

CENTROIDS AND MOMENTS OF INERTIA

115. Determination of Centroids. The **centroid** or **centre of gravity** of a system of particles is the point where the whole mass of the system may be considered to be concentrated. Suppose a system consists of n particles whose masses are m_s $(s=1, 2, \ldots n)$. Let the coordinates of these particles with reference to three mutually perpendicular planes be (x_s, y_s, z_s) $(s=1, 2, \ldots n)$; then if M be the total mass of the system, on taking moments about each of planes of reference, the position of the centroid (x, y, z) is given by the equations,

$$Mx=\sum_{s=1}^{n} m_s x_s, \quad My=\sum_{s=1}^{n} m_s y_s, \quad Mz=\sum_{s=1}^{n} m_s z_s, \quad M=\sum_{s=1}^{n} m_s;$$

from which

$$x=\left(\sum_{s=1}^{n} m_s x_s\right)/M, \quad y=\left(\sum_{s=1}^{n} m_s y_s\right)/M, \quad z=\left(\sum_{s=1}^{n} m_s z_s\right)/M. \quad \ldots\ldots\ldots(124a)$$

When the particles form a solid continuous body of uniform density, then the above summations become integrations, and the formulae assume the forms,

$$\bar{x}=\left(\int_{x_1}^{x_2} x \,.\, dm\right)\Big/\left(\int_{x_1}^{x_2} dm\right), \quad y=\left(\int_{y_1}^{y_2} y \,.\, dm\right)\Big/\left(\int_{y_1}^{y_2} dm\right),$$

$$z=\left(\int_{z_1}^{z_2} z \,.\, dm\right)\Big/\left(\int_{z_1}^{z_2} dm\right). \quad \ldots(124b)$$

In solving problems, however, it is better in general to work from first principles rather than to apply the above formulae. The method is illustrated in the following examples.

Ex. 1. *Bodies of weights* 18, 11, 12, 26 *and* 33 *lb. have their centroids at points whose coordinates are* (3, 2), (5, −4), (−2, 4), (−1, 3) *and* (7, 4) *respectively with reference to rectangular axes. Find the coordinates of the centroid of the system.*

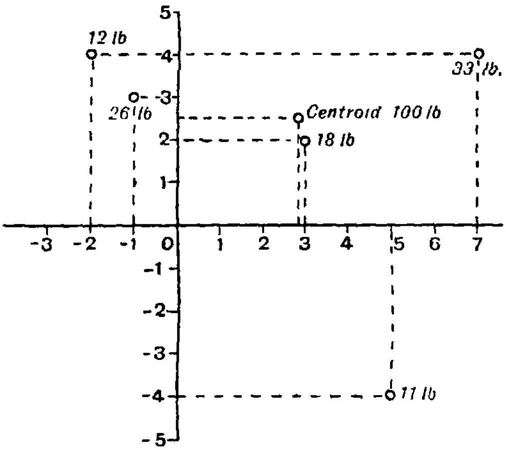

FIG. 48. Centroid of a system of particles

The points should first be plotted on squared paper, as shewn in Fig. 48. Then if (x, y) are the coordinates of the centroid, taking moments about the y-axis

$$(18+11+12+26+33)x$$
$$=(18\times 3)+(11\times 5)-(12\times 2)-(26\times 1)+(33\times 7),$$

or $$100\bar{x}=54+55-24-26+231=290;$$

$$\therefore\ x=2.9.$$

Similarly, taking moments about the x-axis,

$$100y=(18\times 2)-(11\times 4)+(12\times 4)+(26\times 3)+(33\times 4)=250;$$
$$\therefore\ y=2.5.$$

Ex. 2. *A uniform triangular lamina ABC has a circular hole of radius* 3·5 *in. punched out of it. If $AB=17$ in., $BC=28$ in., $CA=25$ in., and the centre of the hole is ten inches from A measured along the perpendicular from A to BC, find the distance of the centroid of the plate from BC.*

The figure should be drawn to scale, marking O as the centre of the circle, and D as the foot of the perpendicular on BC from A, as shewn in Fig. 49.

The area of triangle $ABC = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{35 \,.\, 7 \,.\, 10 \,.\, 18}$$

$$= 210 \text{ sq. in.}$$

But area $= \frac{1}{2}BC \,.\, AD = 14 \,.\, AD = 210$;

$$\therefore\ AD = 15 \text{ in.},$$

and area of circle $= \pi \times 3{\cdot}5^2 = 38{\cdot}5$ sq. in.

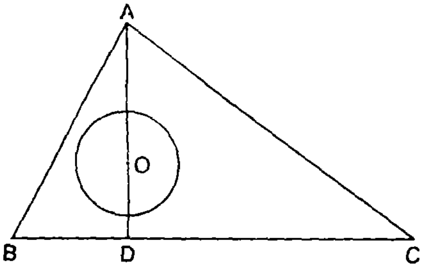

FIG. 49. Centroid of a pierced triangular lamina.

Now the centroid of the whole triangle is distant $AD/3 = 5$ in. from BC, and the centroid of the circle is at O, so that

$$OD = AD - 10 = 5 \text{ in.}$$

Hence if y be the distance of the centroid of the plate from BC, moments about BC give

$$210 \times 5 = (210 - 38{\cdot}5)\,y + 38{\cdot}5 \times 5\,;$$

$$\therefore\ 171{\cdot}5y = 857{\cdot}5,$$

from which $y = 5$ in.

Ex. 3. *Find, by integration, the centroids of* (*a*) *a quadrant of a circular area of radius a*, (*b*) *the portion of a parabola of latus rectum* 4*a*, *bounded by the arc, the axis and an ordinate distant b from the vertex.* (L.U.)

(*a*) Let y be the length of a strip of width dx, parallel to one of the bounding radii and distant x from it, then if $\bar{x}$ be distance of the centroid from this radius, moments about it give

$$\pi a^2 \bar{x}/4 = \int_0^a xy\,dx.$$

Putting $x = a\cos\theta$, $y = a\sin\theta$, this becomes

$$\pi \bar{x} = 4a\int_0^{\frac{\pi}{2}} \sin^2\theta \,.\, \cos\theta \,.\, d\theta$$

$$= 4a\Gamma(\tfrac{3}{2}) \,.\, \Gamma(1)/\{2\Gamma(\tfrac{5}{2})\}, \quad \text{by (51)},$$

$$= 4a/3\,;$$

$$\therefore\ \bar{x} = 4a/(3\pi).$$

Similarly, if $\bar{y}$ be the distance of the centroid from the other radius, $\bar{y} = 4a/(3\pi)$.

Hence taking the bounding radii as axes, the centroid of the quadrant is the point $4a/(3\pi)$, $4a/(3\pi)$.

(*b*) With the vertex as origin and the axis of the parabola as the x-axis, the equation of the curve is $y^2 = 4ax$.

Let y be the length of a strip of width dx, parallel to the y-axis and distant x from it. Taking moments about both axes, and $(\bar{x}, \bar{y})$ as the centroid,

$$\bar{x}\int_0^b y \,.\, dx = \int_0^b xy \,.\, dx \quad \text{and} \quad \bar{y}\int_0^b y \,.\, dx = \tfrac{1}{2}\int_0^b y^2 \,.\, dx.$$

$\therefore$ Putting $y^2 = 4ax$ or $y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$,

$$\bar{x}\int_0^b x^{\frac{1}{2}} \,.\, dx = \int_0^b x^{\frac{3}{2}}\, dx \quad \text{and} \quad \bar{y}\int_0^b x^{\frac{1}{2}} dx = a^{\frac{1}{2}}\int_0^b x \,.\, dx,$$

i.e. $$\tfrac{2}{3}\bar{x} \,.\, b^{\frac{3}{2}} = \tfrac{2}{5}b^{\frac{5}{2}} \quad \text{and} \quad \tfrac{2}{3}\bar{y}b^{\frac{3}{2}} = \tfrac{1}{2}a^{\frac{1}{2}}b^2.$$

$\therefore$ Centroid lies at the point $(3b/5,\ 3a^{\frac{1}{2}}b^{\frac{1}{2}}/4)$.

Ex. 4. *A bowl in the shape of a hemispherical segment is* 18 *in. deep, and* 4 *ft. in diameter at the top; find the position of the centroid, assuming it to be made of uniform thin material.*

If the bowl is filled with water to a depth of 15 *in., find how much the centroid is raised, assuming that the density of the material of the bowl is* 8·75 *times that of the water.*

Let ABY (Fig. 50) be the bowl, O the centre of the sphere, then $AC = 24$ in., $CY = 18$ in., so that $OC = OY - 18 = r - 18$, where $r =$ radius of sphere.

Now $OA^2 = AC^2 + OC^2$ or $r^2 = 24^2 + (r - 18)^2 = 900 - 36r + r^2$;

$$\therefore\ 36r = 900, \text{ or } r = 25 \text{ in.}$$

Consider a thin band PQ round the bowl parallel to AB, of width ds, radius x, and distant y from O, then

$$\text{Area of band} = 2\pi x\, ds.$$

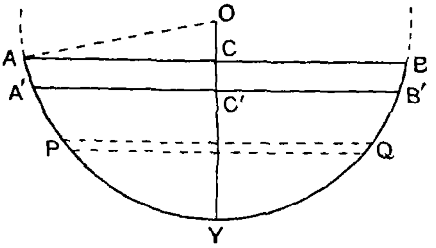

FIG. 50. Centroid of a hemispherical segment.

But $\quad x^2 + y^2 = OP^2 = r^2 = 625$, so that $x \cdot dx + y \cdot dy = 0$,

and $\quad ds^2 = dx^2 + dy^2 = (y^2/x^2 + 1)dy^2 = 625 \cdot dy^2/x^2$;

$$\therefore \quad x \cdot ds = 25dy.$$

$$\therefore \quad \text{Area of band} = 50\pi\, dy,$$

$$\text{and area of bowl} = 50\pi \int_7^{25} dy = 900\pi.$$

Let y_1 = distance of centroid of bowl from O, then taking moments about O,

$$900\pi y_1 = 50\pi \int_7^{25} y \cdot dy = 50\pi \left[\tfrac{1}{2} y^2 \right]_7^{25} = 25 \times 576\pi\,;$$

$$\therefore \quad y_1 = 16 \text{ in.}$$

Now consider the water only, and suppose $A'B'$ is its level, then $C'Y = 15$ in. and $OC' = 25 - 15 = 10$ in.

Let PQ represent a disc of water, of width dy, radius x, and distant y from O; then mass of water $= \pi x^2 \cdot dy = \pi(625 - y^2)dy$, taking density as 1.

$\therefore$ Total mass of water

$$= \pi \int_{10}^{25} (625 - y^2)\, dy = \pi \left[625y - \tfrac{1}{3} y^3 \right]_{10}^{25} = 4500\pi.$$

If y_2 = distance of centroid of water from O, then moments about O give

$$4500\pi \bar{y}_2 = \pi \int_{10}^{25} (625 - y^2)\, y \cdot dy = \pi \left[625 \cdot \tfrac{1}{2} y^2 - \tfrac{1}{4} y^4 \right]_{10}^{25} = 68906\tfrac{1}{4} \cdot \pi\,;$$

$$\therefore \quad y_2 = 15\tfrac{5}{16} \text{ in.}$$

For the bowl and water, let $\bar{y}$ = distance of centroid from O, then taking moments about O,

$$(900 \times \tfrac{35}{4} \,.\, \pi + 4500\pi)\bar{y} = 900 \times \tfrac{35}{4} \times 16\pi + 4500\pi \times \tfrac{245}{16}.$$

Dividing out by 900π,

$$(\tfrac{35}{4} + 5)\bar{y} = 140 + \tfrac{1225}{16} = \tfrac{3465}{16};$$

$$\therefore \; \bar{y} = \tfrac{3465}{16} \times \tfrac{4}{55} = \tfrac{63}{4} = 15\tfrac{3}{4} \text{ in.};$$

$\therefore$ **the water raises the centroid by a quarter of an inch.**

116. The Theorems of Pappus. When an arc of length s, measured between two given ordinates at x_1, x_2, revolves about the x-axis, it generates a surface of revolution whose area $S = 2\pi\int_{x_1}^{x_2} y \,.\, ds$, taking the axis of revolution as the x-axis. If, however, $\bar{y}$ be the ordinate of the centroid of the arc, then, by (124*b*), $s\bar{y} = \int_{x_1}^{x_2} y \,.\, ds$.

Eliminating the definite integral between this and the expression for S,

$$\mathbf{S = 2\pi \bar{y} \,.\, s}, \qquad \text{(125a)}$$

i.e. **the area of a surface of revolution is equal to the product of the length of the generating arc and the path described by its centroid.**

Again, the volume of the solid generated by the revolution of the plane area bounded by the arc, the x-axis and the given ordinates is $V = \pi\int_{x_1}^{x_2} y^2 \,.\, dx$, but if A be the plane area and $\bar{y}$ the ordinate of its centroid, then

$$A\bar{y} = \tfrac{1}{2}\int_{x_1}^{x_2} y^2 \,.\, dx.$$

Eliminating the integral as before,

$$\mathbf{V = 2\pi \bar{y} \,.\, A}, \qquad \text{(125b)}$$

i.e. **the volume of a solid of revolution is equal to the product of the generating area and the path described by its centroid.**

The two theorems were first given by Pappus, a mathematician of Alexandria, in the latter half of the fourth century.

Ex. 5. *Prove that if a plane area revolve about an axis in its plane, not intersecting it, the volume generated is equal to the area multiplied by the length of the path of its mean centre.*

Find the surface and volume of the annular solid generated by the revolution of a quadrant of a circle whose bounding radii are AB, AC, about an axis in the plane of the quadrant parallel to the chord BC. The distance of the axis from A is twice the radius r of the circle and the convexity of the quadrant is towards the axis.

(L.U.)

The position of the centroid is often called the **mean centre,** so that this is the second theorem of Pappus (125*b*), already proved above.

For the second part, since the convexity is towards the axis of revolution, the figure will be as shewn in Fig. 55 (p. 402), where AO is perpendicular to the axis OX; if AO meets the arc BC in D, then $OD=r$, and clearly D is the mid-point of the arc.

To find the centroid of the perimeter, it is evident that the centroid of the two radii lies in AO at a distance $r\sqrt{2}/4$ from A, or $(8-\sqrt{2})r/4$ from O. The centroid of the arc is at D, so that if $\bar{y}$ be the distance from O of the centroid of the perimeter, moments about O give

$$(\pi r/2+2r)\,\bar{y}=\pi r^2/2+(8-\sqrt{2})\,r^2/2,$$

from which $$\bar{y}=(\pi+8-\sqrt{2})\,.\,r/(\pi+4)\,;$$

hence, from (125*a*), the surface

$$S=2\pi\bar{y}(\pi+4)\,r/2=\pi(\pi+8-\sqrt{2})r^2.$$

From Ex. 3, the centroid of the area of the quadrant lies in OA distant $4r\sqrt{2}/(3\pi)$ from A, *i.e.* $2(3\pi-2\sqrt{2})\,r/(3\pi)$ from O; hence from (125*b*), the volume

$$\begin{aligned} V&=(\pi r^2/4)\,.\,2\pi\,.\,2(3\pi-2\sqrt{2})r/(3\pi)\\ &=\pi(3\pi-2\sqrt{2})r^3/3. \end{aligned}$$

Exercises 15a.

In each of the following systems of particles, the weight and position referred to rectangular axes of every particle is given. Plot these positions on squared paper, and calculate the position of the centroid for each system.

1. 8·5, (2, 0); 10·5, (4, 2); 15, (1, 3); 16, (3, −1).

2. 21, (6, 8); 13, (5, −4); 56, (−1, −1); 10, (−3·5, 4).

3. 12, (1, 2); 14, (2, 1); 8, (3, −1); 16, (4, 5).

4. 23, (1, 1); 15, (1, 3); 162, (2, 7); 87, (−4, 6); 135, (0, −8); 78, (7, 2).

5. 15, (−1, −2); 21, (3, −3); 8, (2, 2); 16, (5, 5). Where must an additional weight of 3 lb. be placed so that the centroid of the whole system may be at the origin?

6. Weights of 11 lb., 13 lb., 17 lb., and 7 lb. are placed at the corners A, X, O, Y respectively of a square $AXOY$, whose side is 2·5 feet long. Find the position of the centroid of the system with reference to the sides OX, OY.

7. Weights of 6 lb., 8 lb., 12 lb., 9 lb., 5 lb., and 15 lb. are placed at the angular points A, B, C, D, E, F, of a regular hexagon $ABCDEF$, whose side is two feet long. Find the position of the centroid.

8. Weights of 1, 4, 2, and 3 lb. are placed at the corners Y, A, X, O of a rectangle $YAXO$, having OX 15 inches long and OY 12 inches long. Find the position of the centroid with reference to the sides OX, OY.

9. Seven equal weights are placed at the angular points B, C, D, E, F, G, H, respectively of a regular octagon $ABCDEFGH$. Find the position of the centroid with reference to the axes OGH, OFE, where O is the point of intersection of HG and EF produced.

10. Weights of 56 lb., 13 lb., 16 lb., are suspended from a rigid rod OX whose weight may be neglected. The 56-lb. weight is hung from O, the 13-lb. weight at a distance 3 feet from O, and the 16-lb. weight 6 feet from O. The rod is 12 feet long, and a fourth weight hung from X causes the rod to balance horizontally when supported at a point distant two feet from O. Find the magnitude of this weight.

11. ABC is a triangle having AB, 10 inches long, BC, 21 inches long, and CA, 17 inches long. 7 lb. is placed at A, 5 lb. at B, and 2 lb. at C. Find the position of the centroid with reference to AD, DC as axes, where D is the foot of the perpendicular on BC from A.

12. Find the position of the centroid of a uniform triangular lamina ABC, whose base BC is a inches long and height h inches.

Shew that it is the same as that of three equal particles, each one-third of the mass of the plate, placed at the angular points.

If D is the mid-point of BC, calculate the distance of the centroid from D along DA, when AB is 8 in., BC is 18 in., and CA is 14 in.

13. $PQRS$ is a uniform plate in the shape of a trapezium, having PQ, SR as its parallel sides. If D, E, are the mid-points of PQ, RS, respectively, find the distance of the centroid of the plate along ED measured from E, having given that SR is 40·5 in. long, PS is 9 in., QR is 15·75 in., and PQ is half the length of SR.

14. $PQYOX$ is a uniform plate in which each of the angles at Q, O, X, are right angles, and OY, PX are equal. PQ is 15 inches long, YQ is 20 inches long, and the centroid of the whole plate lies on a line parallel to OY and distant 12·64 inches from it. Find the length of the side PX and the distance of the centroid from OX.

15. ABC is a triangular wedge whose thickness, measured perpendicular to the plane ABC, varies uniformly from zero at A to t along BC. Prove that the centroid lies three-quarters of the way along the line from A to the centre of the rectangular base of which BC is the edge.

16. Find the value of y from the equation

$$y \,.\int_0^\pi y \,.\, dx = \int_0^\pi y^2 \,.\, dx,$$

when $y = a \,.\, \sin^2 \dfrac{x}{2} \cos \dfrac{x}{2}$. What is the meaning of this value ?

Find the position of the centroid in each of the following cases :

17. A uniform circular arc of radius r, subtending an angle θ at the centre.

18. A uniform semi-circular area of radius r.

19. A uniform hemispherical shell of radius r.

20. A uniform solid hemisphere of radius r.

21. A uniform lamina in the shape of a quadrant of a circle of radius r.

22. A uniform semi-elliptical area whose semi-axes are a and b.

23. The area formed by the curve of the parabola, $y^2 = 4ax$, and the double ordinate at $x = h$.

24. A uniform right circular cone of height h and base radius r.

25. The surface of the frustum of a right circular cone, whose flat circular surfaces are distant h from each other and whose radii are a and b respectively, a being greater than b.

26. A circular disc of one foot radius has a circular hole of radius three inches cut out of it, the centre of the hole being at a distance of one and a half inches from the centre of the disc. Find the position of the centroid of the disc.

27. A solid is formed by joining the flat surface of a hemisphere to the base of a cylinder of the same radius. Find the centroid of the solid, the height of the cylinder being equal to the diameter of its base, which is 9·6 inches.

28. ABC is a uniform lamina in the shape of a sector of a circle whose centre is O: if r is its radius and 2ϕ the angle AOB, find the distance of the centroid from O along the bisecting radius of

(1) the whole sector,

(2) the segment bounded by the chord AB.

Deduce the position of the centroid of a semicircle.

29. Find the position of the centroid of the area enclosed by the cycloid,

$$x=r(\theta-\sin\theta), \quad y=r(1-\cos\theta),$$

and the x-axis from cusp to cusp.

30. Find the position of the centroid of a bowl in the form of a hemispherical segment whose top diameter is $2a$ and depth h.

Deduce the position of the centroid of a hemispherical shell whose radius is r.

***31.** $OPQR$ is a uniform square lamina of side a; a triangular portion DHP is cut off such that D is the mid-point of PQ and H is the point of trisection of the side OP, HP being the smaller segment. Find the position of the centroid of the figure $RQDHO$ with reference to OP, OR as axes.

***32.** $ABCD$ is a uniform rectangular plate whose sides AB, BC are 28 and 21 inches long respectively. At a distance 14·4 inches from A measured along AC, a circular hole of radius r inches is cut out, and the centroid of the plate is thereby shifted to a point on AC, distant 18·6 inches from A. Find the value of r, taking $7\pi=22$, and without using tables.

***33.** A uniform triangular lamina ABC whose base BC is two feet and height three feet has its vertex A removed by cutting through a line PQ parallel to BC; shew that if y is the distance of the centroid of $BPQC$ from BC, then

$$y=\frac{2+p-p^2}{2+p},$$

where p is the length of PQ.

***34.** Find the position of the centroid of an octant of a uniform ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

***35.** Find the centroid of the arc of the catenary

$$y=c\,.\cosh(x/c),$$

between the points where $x=c$ and $x=-c$.

***36.** Find the position of the centroid of a hollow conical vessel of height h and base radius r, made of uniform thin metal plate,

(1) when there is no base,

(2) when there is a base of the same material.

***37.** In a disc of brass 40 cm. in diameter, a circular hole of 8 cm. diameter is cut out and replaced by lead of the same thickness, the distance between the two centres being 8·73 cm. Find the distance of the centroid of the whole plate from its centre, having given that the densities of brass and lead are 8·62 and 11·37 grams per cubic centimetre respectively.

***38.** A uniform plate $ABCDEFG$ is shaped like the letter L, being formed by two rectangular strips $ABFG$, $CDEF$, the point C lying on FB, and GF, FE being in the same straight line. If the lengths of AB, AG, DE, FE are x, y, a, b inches respectively, shew that, if the centroid of the whole plate lies on a straight line passing through C, then

$$ax - by + 2ab = 0.$$

Find y when a is 18, b is 24 and x is 12.

Hence, with these values, find the position of the centroid of the plate with reference to GE, GA as axes.

On the strip $ABFG$ another strip of the same material and width is fixed symmetrically, so that the centroid of the whole plate is situated at C; find the length of this strip.

***39.** Regarding a walking stick as made up of a long right cylinder of length na and radius b, together with a spherical knob of the same material whose radius is a, shew that the distance of the centroid from the bottom of the stick is

$$\frac{8a^3(n+1) + 3n^2ab^2}{2(3nb^2 + 4a^2)}.$$

Work out this distance when a is $1\frac{1}{4}$ in., b is $\frac{5}{8}$ in., and n is 28.

***40.** Find the position of the centroid of:

(1) A solid regular tetrahedron, *i.e.* a pyramid on a triangular base, all of whose faces are equilateral triangles.

(2) The surface, including the base, of a regular tetrahedron, the length of an edge in each case being a.

***41.** $YZXO$ is a uniform rectangular sheet of metal, having YZ and OY, x and a inches respectively. The corner Z is turned over so that it lies on the side OX, and the centroid of the plate thus folded lies on a line parallel to OY and distant b inches from it. Prove that x is the positive solution of the equation

$$3x^2 - 6bx - a^2 = 0.$$

Hence, find x and the distance of the centroid from OX, when a is 12 in. and b is 8·8 in.

42. Find the coordinates of the mass centre of the area in the first quadrant bounded by the curve whose equation is $b^4y^2=(a^2-x^2)^3$ and the line $x=0$. (L.U.)

***43.** A long vertical tapering rod of circular section has to bear a load W at its end; the rod weighs w lb. per unit volume, and the tensile stress f over every section has to be constant. Determine the law giving the radius of the section at any distance y from the smaller end, and find the position of the centre of gravity of the rod when of length h. (L.U.)

***44.** Find the centre of gravity of the solid cut off from the cylinder $x^2+y^2=a^2$ by the planes $z=0$, $x\sin\theta-(z-h)\cos\theta=0$, where

$$h\cot\theta>a.$$

A cylindrical shaft, 1 foot diameter, bevelled at one end, is to be mounted on a horizontal axis forming a diameter of a circular section so that it will hang with the bevelled face horizontal. If the greatest length of the shaft is 4 ft. and the shortest length 3 ft., determine the distance of the axis from the circular end. (L.U.)

***45.** Find the area and centroid of the portion of a plane bounded by a parabola $y^2=ax$, the line $x=b$ and the axis $y=0$. The area is revolved about the axis of y so as to form a solid ring. Find the volume of the ring. (L.U.)

***46.** A plate in the form of a quadrant of the ellipse $x^2/a^2+y^2/b^2=1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes; shew that the coordinates of the centroid are $8a/15$, $8b/15$. (L.U.)

***47.** The coordinates of the vertices A, B, C, D of a quadrilateral with reference to the diagonals as axes are $(a, 0)$, $(0, b)$, $(-c, 0)$ and $(0, -d)$. Prove that the centroid of the quadrilateral is at the point $(a-c)/3$, $(b-d)/3$. Hence shew that if O is the point of intersection of the diagonals, and E and F are points on OA and OB such that $AE=OC$ and $BF=OD$, the centroid of OEF coincides with that of the quadrilateral. (L.U.)

48. Find the area and the abscissa of the centroid of the plane surface bounded by the curve

$$y=\frac{5}{(4+x^2)(1-x)},$$

the axes of x and y, and the line $x=\frac{1}{2}$. (L.U.)

49. A semi-circular bend of lead pipe has a mean radius of one foot; the internal diameter of the pipe is 4 in. and the thickness of the lead $\frac{1}{2}$ in. Find its weight, given that a cubic inch of lead weighs 0·41 lb. (L.U.)

50. AEB is the diameter of a semicircle of radius a, and CD is a line parallel to AB and lying on the same side of AB as the semicircle.

Shew that the volume of the solid formed by revolving the semicircle round CD is $2\pi^2a^3 - 4\pi a^3/3$, the perpendicular distance between the lines AB and CD being $2a$. (S.U.)

***51.** Find the coordinates of the centroid of the smaller segment of the ellipse $(x/a)^2+(y/b)^2=1$ cut off by the line $bx+ay=ab$. If the segment revolves about its chord, prove by the theorem of Pappus that the volume of the solid generated is

$$\frac{\pi(10-3\pi)}{6} \cdot \frac{a^2b^2}{\sqrt{a^2+b^2}}. \qquad \text{(D.U., Sc.)}$$

***52.** A groove of semicircular section, of radius b, is cut round a cylinder of radius a. Prove that the volume removed is $\pi^2ab^2 - 4\pi b^3/3$, and that the surface of the groove has an area of $2\pi^2ab - 4\pi b^2$. (Br.U.)

117. Moments of Inertia. Suppose a particle be describing a circle of radius r with uniform angular velocity ω about the centre, then the angle swept out in any time t is θ or ωt. If, however, v be the corresponding linear velocity along the circumference, the arc described in time t is vt; but vt/r is the circular measure of the angle described at the centre, so that $vt/r = \omega t$, or

$$\mathbf{v} = \boldsymbol{\omega}\mathbf{r}. \qquad \textbf{(126)}$$

Suppose, further, that a system of particles of mass m_s $(s=1, 2, \ldots n)$, moving with uniform angular velocity ω about a fixed axis, their distances from this axis being r_s, and their corresponding linear velocities being v_s, then the kinetic energy of the system

$$=\frac{1}{2}\sum_{s=1}^{n} m_s v_s^2, \quad = \tfrac{1}{2}\omega^2 \sum_{s=1}^{n} m_s r_s^2, \text{ by (126).}$$

The expression $\sum_{s=1}^{n} m_s r_s^2$ is called the **Second Moment of Mass or** the **Moment of Inertia** of the system about the axis of revolution, and is usually denoted by I. When the particles form a uniform, continuous body, the summation then becomes an integration, so that

$$\left.\begin{aligned} &\mathbf{I} = \sum_{s=1}^{n} \mathbf{m_s r_s^2} \quad \text{or} \quad \int_{m_1}^{m_2} \mathbf{r}^2 \,.\, \mathbf{dm}, \\ &\text{and } \textbf{the Kinetic Energy} = \tfrac{1}{2}\mathbf{I}\boldsymbol{\omega}^2 \textbf{ or } \tfrac{1}{2}\mathbf{I}\left(\frac{\mathbf{d\theta}}{\mathbf{dt}}\right)^2, \\ &\textbf{when the angular velocity is not uniform.} \end{aligned}\right\} \qquad \textbf{(127a)}$$

Again, if $M=\sum_{s=1}^{n} m_s$, *i.e.* the total mass of the system, and k be such a distance that $Mk^2=I$, k is called the **radius of inertia** or, more often, the **radius of gyration** of the body about the axis of revolution ; hence

$$\left.\begin{aligned} \mathbf{k}^2&=\Big(\sum_{s=1}^{n} \mathbf{m}_s\mathbf{r}_s^2\Big)\Big/\mathbf{M} \quad \text{or} \quad \Big(\int_{\mathbf{m}_1}^{\mathbf{m}_2}\mathbf{r}^2 \,.\, \mathbf{dm}\Big)\Big/\mathbf{M}, \\ \text{where} \qquad \mathbf{M}&=\sum_{s=1}^{n}\mathbf{m}_s \quad \text{or} \quad \int_{\mathbf{m}_1}^{\mathbf{m}_2}\mathbf{dm}. \end{aligned}\right\} \qquad \text{(127b)}$$

The momentum of the system is $\sum_{s=1}^{n} m_s v_s$, and therefore the moment of the momentum of the system about the axis of revolution is $\sum_{s=1}^{n} m_s v_s r_s=\omega\sum_{s=1}^{n} m_s r_s^2$.

∴ **Moment of momentum about the axis of revolution**

$$=\mathbf{I}\omega \quad \text{or} \quad \mathbf{I}\frac{\mathbf{d}\theta}{\mathbf{dt}}, \qquad \text{(127c)}$$

when ω is not uniform.

Ex. 6. *Find the moment of inertia of a thin uniform circular plate of mass m and radius r about* (1) *any diameter, and* (2) *the axis perpendicular to the plate passing through its centre.*

Hence shew that when a disc of radius r rolls down an inclined plane along the line of greatest slope without sliding, its acceleration is $\frac{2}{3}g \sin a$, *where* a *is the inclination of the plane to the horizontal.*

Let the centre O be the origin, and the diameters XOX', YOY' (Fig. 51) the axes.

Taking a strip PQ of length $2x$, width dy, parallel to and distant y from XX', it is clear that its moment of inertia about XX' is $2xy^2\rho \,.\, dy$, where ρ is the surface density.

Let $\angle XOQ=\theta$, then $x=r\cos\theta$, $y=r\sin\theta$, and $dy=r\cos\theta \,.\, d\theta$.

$$\therefore\ I \text{ for plate about } X'X=4\rho\int_0^r xy^2 \,.\, dy$$

$$=4\rho r^4\int_0^{\frac{\pi}{2}}\cos^2\theta \,.\, \sin^2\theta \,.\, d\theta$$

$$=\rho r^4\pi/4, \text{ by (50) and (51).}$$

But $$m = \pi r^2 \rho\,;$$

$$\therefore\ I = \tfrac{1}{4}\mathrm{m}r^2.$$

It is obvious that this result will be true for any diameter, since both m and r are constant.

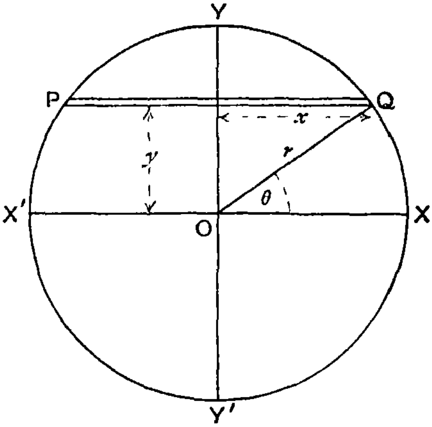

FIG. 51. Moment of Inertia of a circular plate.

For an axis through O perpendicular to the plate, usually called the z-axis, take a circular strip bounded by two concentric circles of radii x and $x+dx$ respectively, then the area of this strip $= 2\pi x\,.\,dx$;

$$\therefore\ I_z = 2\pi\rho\int_0^r x^3\,.\,dx = \tfrac{1}{2}\pi\rho r^4 = \tfrac{1}{2}\mathrm{m}r^2.$$

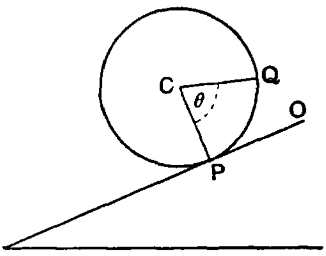

FIG. 52. Disc rolling down an inclined plane.

Suppose the disc starts from O (Fig. 52), the point Q being then coincident with O; let it traverse the distance $OP = x$ in time t, the angle PCQ being θ, then the kinetic energy of the disc due to its translational motion from O to $P = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$; and the kinetic energy due to its rotation about $C = \frac{1}{2}I_z\left(\frac{d\theta}{dt}\right)^2$, by (127$a$).

But $$\frac{dx}{dt} = r\,\frac{d\theta}{dt},\ \text{by (126)};$$

$$\therefore\ \text{Total kinetic energy} = \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}I_z\left(\frac{dx}{dt}\right)^2\Big/ r^2,$$

and by the previous result, $I_z = \frac{1}{2}mr^2$;

$$\therefore \text{ Total kinetic energy} = \tfrac{3}{4} \cdot m \cdot \left(\frac{dx}{dt}\right)^2,$$

and this is equal to the work done by gravity

$$= mg \times \text{vertical distance fallen through}$$
$$= mgx \sin \alpha;$$

$$\therefore\ 3 \cdot \left(\frac{dx}{dt}\right)^2 = 4gx \sin \alpha.$$

Hence, by differentiation,

$$3 \cdot \frac{d^2x}{dt^2} = 2g \sin \alpha;$$

$$\therefore \text{ Acceleration down plane} = \frac{d^2x}{dt^2} = \tfrac{2}{3}g \sin \alpha.$$

118. Theorems of Perpendicular and Parallel Axes. Consider a *system of n particles of mass m_s ($s = 1, 2, \ldots n$), and let $P(x_P, y_P, 0)$* be the position of one of them referred to three mutually perpendicular axes, then the distance of P from the z-axis is

$$\sqrt{x_P^2 + y_P^2},$$

and
$$\sum_{P=1}^{n} m_P x_P^2 + \sum_{P=1}^{n} m_P y_P^2 = \sum_{P=1}^{n} m_P (x_P^2 + y_P^2),$$

i.e. if I_x, I_y, I_z be the moments of inertia and k_x, k_y, k_z, the corresponding radii of gyration about the x, y, z-axes respectively,

$$\mathbf{I_x + I_y = I_z}, \quad \text{or} \quad \mathbf{k_x^2 + k_y^2 = k_z^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(128a)$$

Also, if P be in the position (x, y, z), then since

$$OP^2 = x^2 + y^2 + z^2,$$

it follows that the

$$\textbf{Moment of Inertia about the Origin} = \mathbf{I_x + I_y + I_z}. \quad \ldots\ldots\ldots(128b)$$

These important results are known as the **Theorem of Perpendicular Axes.**

Again, let XX' (Fig. 53) be any axis through the centroid of a body, and SS' a parallel axis distant λ from XX'.

Suppose P be any particle of the body whose mass is m and whose distance from XX' is $r = PR$.

Then $PQ = r - \lambda$; $\therefore$ $PQ^2 = r^2 - 2\lambda r + \lambda^2$;

$$\therefore\ m \, . \, PQ^2 = m(r^2 - 2r\lambda + \lambda^2).$$

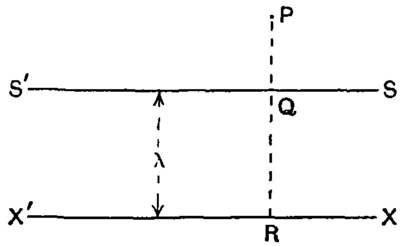

FIG. 53. Theorem of parallel axes.

Hence, summing over the whole body, and denoting the moments of inertia about XX', SS' by I_0, I_λ respectively,

$$I_\lambda = I_0 - 2\lambda\Sigma mr + \lambda^2 \, . \, \Sigma m.$$

Now since XX' passes through the centroid, $\Sigma mr = 0$, by (124a), and $\Sigma m =$ total mass of body; denote it by M, then

$$\left.\begin{aligned} \mathbf{I}_\lambda &= \mathbf{I}_0 + \mathbf{M}\lambda^2, \\ \mathbf{k}_\lambda{}^2 &= \mathbf{k}_0{}^2 + \lambda^2. \end{aligned}\right\} \qquad \textbf{(129)}$$

This is the theorem of **parallel axes.**

Ex. 7. *Determine the radius of gyration of a heavy uniform cylinder of radius r and length l about an axis through its centroid perpendicular to its geometrical axis, and calculate this radius when* $r = 4$ *in.* and $l = 4$ *ft.*

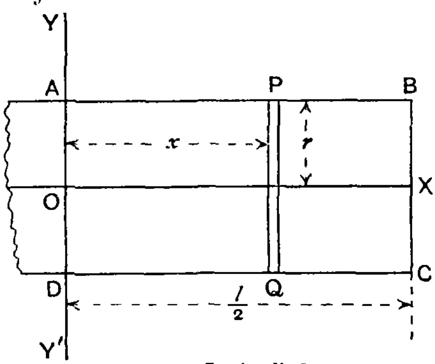

FIG. 54. I. of cylinder.

Let $ABCD$ be half the cylinder (Fig. 54), OX, the geometrical axis, and O, the centroid.

Consider a disc PQ perpendicular to OX, of thickness dx, and distant x from O.

Then I. of PQ about its own diameter is $\frac{1}{4}\pi\rho r^4 dx$, by Ex. 1, ρ being the density of the material.

Hence, by (44*a*),

$$\text{I. of } PQ \text{ about } Y'Y = \tfrac{1}{4}\pi\rho r^4 dx + \pi\rho r^2 x^2 dx$$
$$= \pi\rho r^2(\tfrac{1}{4}r^2 + x^2) \,.\, dx.$$

$\therefore$ If k = radius of gyration about $Y'Y$,

$$Mk^2 = \pi\rho r^2 \int_{-\frac{1}{2}l}^{\frac{1}{2}l} (\tfrac{1}{4}r^2 + x^2)\, dx.$$

But $M = \pi r^2 l\rho$, and taking the integral over half the cylinder and doubling the result, since it is uniform,

$$lk^2 = 2\int_0^{\frac{1}{2}l} (\tfrac{1}{4}r^2 + x^2)\, dx = 2\left[\tfrac{1}{4}r^2 x + \tfrac{1}{3}x^3\right]_0^{\frac{1}{2}l} = \tfrac{1}{4}r^2 l + \tfrac{1}{12}l^3\,;$$

$$\therefore\ k^2 = \tfrac{1}{4}(r^2 + \tfrac{1}{3}l^2).$$

When $r = 4$ in., and $l = 4$ ft. $= 48$ in.,

$$k^2 = \tfrac{1}{4}(16 + \tfrac{1}{3}\ 48^2) = 4(1 + 48) = 196\,;$$
$$\therefore\ k = 14 \text{ in.}$$

119. Ellipsoids of Inertia. Let $P(x, y, z)$ be the position of any particle of a body whose total mass is M, referred to three mutually perpendicular axes, OX, OY, OZ. Denote the moments of inertia of the body about OX, OY, OZ respectively, by

$$A = \Sigma\rho(y^2 + z^2), \quad B = \Sigma\rho(z^2 + x^2), \quad C = \Sigma\rho(x^2 + y^2),$$

where $M = \Sigma\rho$, the summations including integrations also.

Let $D = \Sigma\rho yz$, $E = \Sigma\rho zx$, $F = \Sigma\rho xy$. These expressions are called **products of inertia.**

Now take any other axes OX', OY', OZ', mutually perpendicular to each other, through the origin, and let the direction cosines of OZ' be l, m, n. If (x', y', z') be the coordinates of P with respect to these axes, then from Ex 8 of § 101, or by solving the equations of (111) for x' and y',

$$kx' = mx - ly \quad \text{and} \quad ky' = -lnx - mny + k^2 z,$$

where $k^2 = l^2 + m^2 = 1 - n^2$.

Hence $k^2(x'^2+y'^2)=(-lnx-mny+k^2z)^2+(mx-ly)^2$

$$=(l^2n^2+m^2)x^2+(m^2n^2+l^2)y^2+k^4z^2$$
$$-2k^2lnzx-2k^2mnyz-2lm(1-n^2)xy.$$

Now $l^2n^2+m^2=l^2(1-k^2)+m^2=k^2(1-l^2)=(m^2+n^2)k^2.$

Similarly, $m^2n^2+l^2=(l^2+n^2)k^2.$

Hence, after dividing out by k^2,

$$x'^2+y'^2=(m^2+n^2)x^2+(l^2+n^2)y^2+(l^2+m^2)z^2-2mnyz-2lnzx$$
$$-2lmxy$$
$$=l^2(y^2+z^2)+m^2(z^2+x^2)+n^2(x^2+y^2)-2mnyz-2lnzx$$
$$-2lmxy.$$

Multiplying out by p, and summing over the whole of the body, the moment of inertia I about $OZ'=\Sigma p(x'^2+y'^2)$; hence:

The moment of inertia of a body about any line through the origin whose direction cosines are l, m, n, is given by

$$\mathbf{I=l^2A+m^2B+n^2C-2mnD-2lnE-2lmF,} \quad \text{.... ....(130)}$$

where A, B, C, are the moments of inertia about the axes of x, y, z respectively, and D, E, F the products of inertia for the coordinate planes.

Let a point Q on OZ' be chosen so that the moment of inertia about OQ is inversely proportional to the square on OQ; if $OQ=r$, then (130) gives

$$l^2A+m^2B+n^2C-2mnD-2nlE-2lmF=\mu^2/r^2,$$

where μ is a constant.

If (x, y, z) be the coordinates of Q, then $x=lr$, $y=mr$, $z=nr$; $\therefore$ the locus of Q is the surface

$$\mathbf{Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=\mu^2.} \quad \text{.... ....(131)}$$

Since A, B, C are essentially positive, this equation represents an ellipsoid, and is called the **ellipsoid of inertia** or the **momental ellipsoid**, and its axes are called the **principal axes** of the body at the point O.

120. Principal Moments at O. The moments of inertia about the principal axes are known as the **principal moments of inertia**, and their determination is an important practical problem. It

is obvious that the principal axes are normal to the surface, and, by (114), the direction cosines of any normal to a quadric $Q(x, y, z)=0$ are proportional to $\partial Q/\partial x$, $\partial Q/\partial y$, $\partial Q/\partial z$ respectively. Applying this to the equation of the ellipsoid of inertia (131),

$$(Ax - Fy - Ez)/l = (-Fx + By - Dz)/m = (-Ex - Dy + Cz)/n,$$

or replacing x, y, z by lr, mr, nr,

$$\begin{aligned}(Al - Fm - En)/l &= (-Fl + Bm - Dn)/m = (-El - Dm + Cn)/n \\ &= \{l(Al - Fm - En) - m(Fl - Bm + Dn) \\ &\qquad - n(El + Dm - Cn)\}/(l^2 + m^2 + n^2) \\ &= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm \\ &= I, \text{ by (130).}\end{aligned}$$

Hence

$$\begin{aligned}(I - A)l + Fm + En &= 0, \\ Fl + (I - B)m + Dn &= 0, \\ El + Dm + (I - C)n &= 0,\end{aligned}$$

which, on eliminating l, m, n, gives

$$\begin{vmatrix} \mathbf{I-A} & \mathbf{F} & \mathbf{E} \\ \mathbf{F} & \mathbf{I-B} & \mathbf{D} \\ \mathbf{E} & \mathbf{D} & \mathbf{I-B} \end{vmatrix} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (132)$$

a cubic in I whose roots give the three principal moments of inertia of the body for the point O. The direction cosines l, m, n may then readily be derived from the above equations on inserting the values of I and solving, thus the positions of the principal axes become known.

Ex. 8. *Find the equation of the ellipsoid of inertia and the principal radii of gyration at one of the vertices of a uniform cube whose edge is of length a.*

Take the selected vertex as origin and the three edges concurrent at that vertex as axes. Let ρ be the density of the cube, then considering a rectangular slice, of width dy, parallel to and distant y from the zx-plane, its moment of inertia about an axis parallel to OX through its centroid is $a^4\rho \cdot dy/12$.

Hence, by (129), the moment of inertia about OX is

$$\begin{aligned}a^4\rho \cdot dy/12 + (a^4/4 + y^2)a^2\rho \cdot dy \\ = a^2\rho(a^2/3 + y^2)dy.\end{aligned}$$

∴ In the notation of § 119,

$$A = d^2\rho\int_0^a (a^2/3 + y^2)\,dy = 2a^5\rho/3 = 2Ma^2/3,$$

where M = mass of cube $= a^3\rho$.

Similar considerations, as well as those of symmetry, shew that

$$A = B = C = 2Ma^2/3.$$

Referring again to the rectangular slice, its product of inertia for the yz and zx-planes is clearly

$$a^2\rho \,.\, dy \,.\, (ay/2)\,;$$

$$\therefore\quad F = \tfrac{1}{2}a^3\rho\int_0^a y\,.\,dy = a^5\rho/4 = Ma^2/4.$$

Similarly, $\qquad D = E = F = Ma^2/4.$

Hence, by (131), putting $\mu^2 = M\lambda^4$, the equation of the ellipsoid of inertia becomes, on multiplying out by $6/M$,

$$4(x^2 + y^2 + z^2) - 3(yz + zx + xy) = 6\lambda^4/a^2.$$

If $I = Mk^2$ be the moment of inertia about any line through O, then, by (132), after division throughout by M, the principal radii of gyration are given by the cubic,

$$\begin{vmatrix} k^2 - 2a^2/3 & a^2/4 & a^2/4 \\ a^2/4 & k^2 - 2a^2/3 & a^2/4 \\ a^2/4 & a^2/4 & k^2 - 2a^2/3 \end{vmatrix} = 0,$$

or $\qquad (k^2 - 2a^2/3)^3 - 3\,.\,(a^2/4)^2(k^2 - 2a^2/3) + 2(a^2/4)^3 = 0,$

which factorises into

$$(k^2 - 2a^2/3 - a^2/4)^2(k^2 - 2a^2/3 + a^2/2) = 0\,;$$

$$\therefore\quad {k_1}^2 = {k_2}^2 = 11a^2/12, \quad {k_3}^2 = a^2/6.$$

Exercises 15b.

1. A thin uniform rod has a length l and mass m : find I about an axis perpendicular to its length, (1) through its centre, (2) through one end.

If the rod is two feet long and it rotates about a point in its length such that its radius of gyration is eight inches, find the distance of this point from the nearer end.

2. A uniform wire of length l and mass m is bent into a plane rectangle. Find I about an axis through the centroid perpendicular to its plane. If the radius of gyration about this axis is $2\sqrt{3}$, find the length of the wire.

3. A uniform wire is bent into a circle of radius r and mass m; find the moment of inertia about (1) any diameter, and (2) the axis through the centroid perpendicular to the plane of the wire.

The circle has two diameters of similar wire fixed to it, and these are perpendicular; find the radius of gyration about the axis through the centre, perpendicular to the plane. If this radius of gyration is ten inches, find the radius of the circle, taking $7\pi = 22$.

4. A uniform rectangular plate has the length l and breadth b; find I about:

(1) an axis through the centroid parallel to the breadth;

(2) one of the shorter sides;

(3) an axis through the centroid perpendicular to the area;

(4) an axis through one of the angular points perpendicular to the area.

Calculate the radius of gyration in each of the above cases when l is 31·5 inches and b is 30 inches.

5. Given a uniform triangular plate whose altitude is h and base a; find I about:

(1) the base;

(2) an axis through the centroid parallel to the base;

(3) an axis parallel to the base passing through the vertex opposite the base.

Calculate each of these moments when the base is 48 inches long and the other two sides are 29 and 35 inches long respectively.

6. An annular area is enclosed by two concentric circles having radii a and b respectively, a being greater than b; find I about:

(1) any diameter;

(2) an axis through the common centre, perpendicular to the plane of the area.

Find the value of b such that when a is 15, the radius of gyration about a diameter is 8·5.

7. Find I for a thin uniform circular disc of radius r, about:

(1) any tangent;

(2) an axis through any point on its circumference perpendicular to its plane.

If a circular hole of radius x be drilled at the centre of the disc so that the radius of gyration about any tangent is 3·5 in. when r is 3 in., find x.

8. A uniform elliptical plate of mass m has semi-major and minor axes a and b respectively; find I about (1) the minor axis, (2) an axis through the centroid perpendicular to its plane.

Find the minor axis so that the radius of gyration about (2) shall be 13·75 when the major axis is 44 inches long.

9. A thin uniform parabolic plate of mass m is bounded by the y-axis and the parabola $y^2=4a(h-x)$; find I about the y-axis.

Calculate the radius of gyration when a is 35, and the straight edge of the plate is 14 feet long.

10. A uniform thin plate is made up of a square $ABCD$ and a semicircle CED of radius a; find I about the side AB, the total mass of the plate being m.

Calculate the radius of gyration when a is 13 in.

11. A uniform hollow cylinder of mass m has radii a, b, a being greater than b; find I about its geometrical axis.

Deduce from the result the moment of inertia of a solid cylinder of radius r about its geometrical axis.

If the radius of gyration of the hollow cylinder be $53\sqrt{2}$ in., and a is 7·5 ft., find the radius b.

12. Find I for a uniform sphere of radius r and mass M, about any diameter.

A uniform sphere of radius 8 in., with centre A, is attached to a thin rod OA of length 65 in., and the system revolves about O. Find the radius of gyration of the sphere, neglecting that of the rod, when the vertical distance between O and A is 33 in. If the vertical distance between O and A increases to 63 in., shew that the radius of gyration is decreased in the ratio of 10 to 3 very approximately, and find the change in the angle at which OA is inclined to the vertical.

13. A uniform right circular cone has height h, base radius r, and mass M; find I about:

(1) its geometrical axis;

(2) a base diameter.

If the radii of gyration about the given axes respectively are in the ratio of $2:\sqrt{3}$, prove that $r:h=2:\sqrt{3}$.

14. Find I for the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

about each of the principal axes.

Calculate the radii of gyration for each axis when a is 24 in., b is 18 in., and c is 7 in.

15. Find the radius of gyration of a uniform triangular plate ABC about any line through C distant p from A and q from B.

Shew that this is the same as the radius of gyration about the same axis of three particles, each one-third the mass of the plate, placed at the mid-points of the sides.

16. Find the moment of inertia of a uniform triangular lamina ABC whose sides are a, b, c and mass M, about an axis through the centroid perpendicular to the area.

Calculate the radius of gyration when a is 24 in., b is 27 in., and c is 36 in.

17. Find the radius of gyration about an axis through the centre parallel to a side of a uniform square plate of side a, having a circular hole of radius r cut out at the centre.

Calculate this radius when a is 22 in. and r is 7 in.

18. A uniform rectangular plate 16 in. by 25 in. has another small uniform plate, which is square, fastened to it so that their centres are coincident. The radius of gyration k, about an axis through their common centre, parallel to the shorter side is given by

$$12k^2 = 481\ ;$$

find the side of the square plate.

19. *Two thin uniform discs of radii a, b respectively, a being greater* than b, are fastened together so that the distance between their centres is c. Find the radius of gyration of the whole about an axis through the centre of the larger disc and perpendicular to its plane.

If a is 3 feet, and c is 1 foot, find the values of b which will give $\sqrt{4 \cdot 2}$ feet as the radius of gyration about this axis.

20. A uniform solid sphere of radius a has a cylindrical hole of radius b drilled centrally through it. Prove that the square of the radius of gyration about the axis of the hole is $(2a^2 + 3b^2)/5$. Shew also that for a diameter perpendicular to the above axis the result would be $(4a^2 + b^2)/10$. (S.U.)

21. A uniform heavy cylinder having hemispherical ends, of mass M and radius r, rotates about an axis through the centroid perpendicular to the geometrical axis. Find the radius of gyration. The length of the cylinder exclusive of the hemispherical ends is l.

22. $ABCD$ is a square whose side is of length $2a$; on AB a semicircle is described outside the square. The whole figure then revolves about the side CD ; find the radius of gyration of the solid thus generated about the axis of revolution.

***23.** Find the radius of gyration of a uniform plate shaped as a trapezium whose parallel lengths are h feet apart and of lengths a, b feet respectively about the longer side a.

If k be the radius of gyration about an axis through the centroid of the plate, parallel to the side a, prove that

$$18 \, . \, k^2 \, . \, (a+b)^2 = (a^2 + 4ab + b^2)h^2,$$

and calculate both radii when a is $3 \cdot 6$, b is $2 \cdot 4$, and h is $1 \cdot 8$.

***24.** Prove that if k_x, k_y, k_z, are the radii of gyration of a particle about the x, y and z-axes respectively, and k is the radius of gyration of the particle about the origin, then

$$2k^2 = k_x^2 + k_y^2 + k_z^2.$$

Apply this theorem to find the moment of inertia of a uniform hollow sphere of radius a, containing a concentric cavity of radius b, and shew that if p, q be the radii of gyration about a diameter and the centre respectively, *when b is zero, then p, q, a are the lengths of the sides of a* right-angled triangle. Hence find a when p is 9·5 in. and q is 16·8 in.

***25.** Find the moment of inertia of a rectangular thin lamina about a diagonal, its sides being a and b.

Shew that the radii of gyration about the diagonal and an axis through the centroid perpendicular to the lamina are the square roots of the values of x which satisfy the equation:

$$72(a^2+b^2)x^2 - 6(a^4+4a^2b^2+b^4)x + a^2b^2(a^2+b^2) = 0.$$

***26.** The radius of gyration about the x-axis of a solid paraboloid generated by the revolution of the parabola

$$y^2 = 12x,$$

about the axis of x, is five feet. Find the radius of the flat circular end of the solid.

27. Find the moment of inertia about its centre of a hollow sphere containing a concentric cavity of radius b, the radius of the sphere being a.

***28.** A T-section consists of two rectangles $ABCD$, $EFGH$, E, H lying in CD such that DE is equal to HC. When AB is 2 ft., BC is 9 in., and EF is 1 ft. 6 in., the radius of gyration about the neutral axis, *i.e.* the line parallel to AB through the centroid of the area, is $3\sqrt{6}$ in. Find the width of the rectangle $EFGH$.

***29.** Find the moment of inertia about the axis of revolution of a uniform anchor ring of mass M, formed by the revolution of a circle of radius r about an axis distant R from its centre, R being greater than r.

Calculate the radius of gyration when r is $8\sqrt{3}$ in. and R is 35 in.

***30.** Find the radius of gyration of the area of the curve

$$(x^2+y^2)^2 = 144(x^2-y^2)$$

about a line in its plane through the origin perpendicular to its axis.

31. A cylindrical shell whose external diameter is 3 ft. and internal diameter 2 ft. rolls down a plane inclined at 30° to the horizon. If it starts at rest, determine its speed when it has described 20 ft. of the plane. (L.U.)

***32.** Shew that the principal radii of gyration at the centroid of a triangle are given by the equation

$$108k^4 - 3(a^2+b^2+c^2)k^2 + \Delta^2 = 0,$$

where a, b, c are the sides and Δ the area of the triangle. (L.U., Sc.)

***33.** Prove that if a plane closed curve, which is symmetrical about any line in its plane, revolves about a parallel axis not intersecting the curve, then the moment of inertia of the solid generated about the axis of revolution is given by

$$I = M(h^2 + 3k^2),$$

where k is the radius of gyration of the generating area about its axis of symmetry and h is the distance between this axis and the axis of revolution.

34. Prove that the moment of inertia about its axis of symmetry of a solid frustum of a cone of mass M is $\frac{3}{10}M(b^5 - a^5)/(b^3 - a^3)$, where a and b are the radii of its circular ends. (S.U.)

***35.** Prove that the radius of gyration of a uniform lamina in the form of a parallelogram about a diagonal is $S/c\sqrt{6}$, where S is the area, and c the length of the diagonal in question. (M.U., Sc.)

***36.** A mass of 10 lb. hangs from a string wrapped round the horizontal axle of a flywheel of mass 200 lb.; the radius of the axle is 2 in. and the mass falls 15 ft. from rest in 16 secs. Find the radius of gyration of the flywheel. (Li.U.)

***37** A fly-wheel in the form of a uniform circular disc of radius 8 in. and weight 54 lb. is free to rotate about a horizontal axle. A weight of 20 lb. is suspended by a rope coiled round the circumference of the wheel. If the system starts from rest, find the velocity of the weight when it has descended 14 ft. (Le.U.)

38. Verify the following rule for finding the moments of inertia of symmetrical bodies, first given by Dr. Routh. The moment of inertia about an axis of symmetry is

$$\text{Mass} \times (\text{sum of squares of perpendicular semi-axes})/d,$$

where d is 3, 4 or 5 according as the body is rectangular, elliptical or ellipsoidal.

***39.** Find the equation of the momental ellipsoid at the vertex of a right circular cone of height h and base radius r. (Li.U., Sc.)

***40.** Prove that the equation of the momental ellipsoid at a point on the edge of the circular base of a right solid cone of height 1 ft. 6 in. and base diameter 1 ft. is $21x^2 + 41y^2 + 26z^2 - 30zx = \mu^2$, where μ is a constant. Hence calculate the radius of gyration of the cone about a generator.

CHAPTER XVI

DOUBLE AND TRIPLE INTEGRATION

121. Volume by Double Integration. In determining the volumes of solids, the general method employed is to find the **area** a of a convenient slice and then integrate over the whole solid. In general, however, the area a also requires an integration, and if the slice lies in a plane parallel to that of yz, then

$$a = \int z \,.\, dy,$$

taken between the proper limits, and the volume V becomes

$$V = \int a \,.\, dx = \iint z \,.\, dx \,.\, dy,$$

the second integration being taken over the whole solid.

Such a symbol is known as a **double integral.**

If $z = \phi(x, y)$, then in evaluating

$$\int z \,.\, dy \quad \text{or} \quad \int \phi(x, y) \,.\, dy,$$

x is regarded as a constant, as it certainly will be for an area parallel to the yz-plane. The resulting integral taken over the region considered will then become a function of x, and the second integration performed with respect to x.

In the notation of double integrals, the right-hand differential always applies to the first integration, and the other differential to the second integration. The following examples will illustrate the method of dealing with double integrals.

Ex. 1. *Find the volume of the annular solid generated by the revolution of a quadrant of a circle whose bounding radii are AB, AC about an axis in the plane of the quadrant parallel to the chord BC. The distance of the axis from A is twice the radius r of the circle, and the convexity of the quadrant is towards the axis.* (L.U.)

The solution to this question was intended to be based on the second proposition of Pappus, and such a solution is given on p. 381, but an independent method, though longer, will exemplify the use of double integrals.

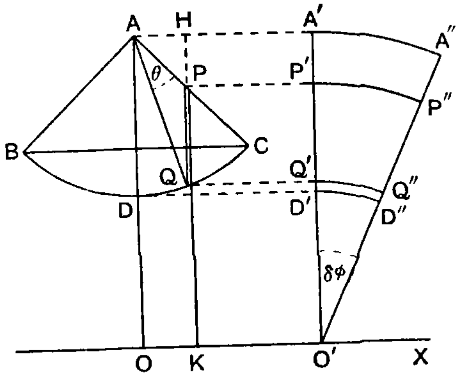

FIG. 55. Volume of an annular solid.

The quadrant is shewn in Fig. 55, O being the origin on the axis of revolution. Suppose the quadrant revolve through a small angle $d\phi$ about OX, so that $O'A''$ be its new position in the plane perpendicular to OX, containing OA.

Consider a slice PQ of thickness dx parallel to OA, then the volume of this slice is

$$\begin{aligned}&\text{Area } P'P''Q''Q' \,.\, dx\\&=\tfrac{1}{2}(O'P'^2 - O'Q'^2)\,.\,d\phi\,.\,dx\\&=\tfrac{1}{2}\,.\,PQ\,.\,(KP+KQ)\,.\,d\phi\,.\,dx.\end{aligned}$$

Let $\angle QAP = \theta$, then since $\angle PAH = \pi/4$,

$$\begin{aligned}PQ = QH - PH = QH - AH &= r\{\sin(\theta+\pi/4) - \cos(\theta+\pi/4)\}\\&=2r\sin\theta/\sqrt{2}.\end{aligned}$$

$$KP = KH - PH = 2r - r\cos(\theta+\pi/4) = 2r - r(\cos - \sin\theta)/\sqrt{2}.$$

$$\begin{aligned}KQ = KP - PQ &= 2r - r(\cos\theta - \sin\theta)/\sqrt{2} - 2r\sin\theta/\sqrt{2}\\&=2r - r(\cos\theta + \sin\theta)/\sqrt{2}\,;\end{aligned}$$

and $$x = r\sin\left(\frac{\pi}{4} - \theta\right) = r(\cos\theta - \sin\theta)/\sqrt{2},$$

so that $$dx = -r(\sin\theta + \cos\theta)/\sqrt{2}.$$

∴ Volume generated by the quadrant in revolving through the angle $d\phi$

$$= r^3\sqrt{2}\int_0^{\frac{\pi}{4}}\{2\sqrt{2}\,(\sin^2\theta + \cos\theta\sin\theta)$$
$$-\cos\theta\,.\,\sin^2\theta - \cos^2\theta\sin\theta\}d\theta\,.\,d\phi,$$

$d\phi$ being constant.

∴ Volume generated by the whole revolution

$$= r^3\sqrt{2}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\{2\sqrt{2}\,(\sin^2\theta + \cos\theta\,.\,\sin\theta)$$
$$-\cos\theta\sin^2\theta - \cos^2\theta\sin\theta\}d\phi\,.\,d\theta$$
$$= r^3\int_0^{2\pi}(3\pi - 2\sqrt{2})\,.\,d\phi/3 = \pi(3\pi - 2\sqrt{2})r^3/3,$$

which agrees with the result previously obtained from the second proposition of Pappus.

Ex. 2. $x^2/a^2 + y^2/b^2 = 1$ *gives the contour of the base of a right cylinder, and the height is* c. *The cylinder is bevelled down so that the height* z *at any point* (x, y) *of the base, in the first quadrant, is given by* $z = c(1 - x/a)(1 - y/b)$. *Express the volume left in this quadrant as a double integral and shew that its value is*

$$abc(\pi - 13/6)/4. \qquad \text{(L.U.)}$$

Consider a rectangular prism of height z, whose axis is parallel to the z-axis, and the coordinates of whose base on the xy-plane are (x, y), $(x + dx, y)$, $(x + dx, y + dy)$, $(x, y + dy)$; then the volume of the prism is $z\,.\,dx\,.\,dy$.

Hence the volume of a slice of the solid, parallel to the yz-plane, is

$$\int_0^y z\,.\,dx\,.\,dy = c\int_0^y(1 - x/a)(1 - y/b)\,.\,dx\,.\,dy, \text{ where } y = b\sqrt{a^2 - x^2/a}.$$

∴ The volume of the whole bevelled solid in the first quadrant is

$$c\int_0^a\int_0^y(1 - x/a)(1 - y/b)\,.\,dx\,.\,dy.$$

It is obvious, in evaluating the first integral, that x is constant, since the slice is parallel to the yz-plane; hence, if V denote the required volume, the first integration gives

$$V = -\frac{bc}{2}\int_0^a\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)^2\right]_0^{b\sqrt{a^2-x^2}/a} . dx$$

$$= -\frac{bc}{2}\int_0^a (1-x/a)\{2\sqrt{a^2-x^2}/a - (a^2-x^2)/a^2\} . dx.$$

Putting $x = a \sin\theta$,

$$V = \tfrac{1}{2}abc\int_0^{\frac{\pi}{2}}(2\cos^2\theta - \cos^3\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^3\theta) . d\theta$$

$$= \tfrac{1}{2}abc(\pi/2 - 2/3 - 2/3 + 1/4) = \mathbf{abc(\pi - 13/6)/4.}$$

122. Area of a Surface. Let $z = f(x, y)$ be a quadric surface, and suppose it to be intersected by a rectangular prism whose axis is parallel to OZ and whose base on the xy-plane is a rectangle having the points (x, y), $(x+dx, y)$, $(x+dx, y+dy)$, $(x, y+dy)$ as its vertices. If (x, y, z) be one of the points of intersection of the quadric and the prism, then the tangent plane at this point will cut a section of the prism whose area will approximate to that of the section cut off on the quadric by the prism, these two areas being equal in the limit.

Let dS be the area of the section of the prism cut off by the tangent plane, then since this section is a rectangle, by (108)

$$dx . dy = n . dS,$$

l, m, n being the direction cosines of the tangent plane.

Now, by (114), these are proportional to $\partial z/\partial x$, $\partial z/\partial y$, and 1.

Let $p = \partial z/\partial x$, $q = \partial z/\partial y$, and $l = kp$, $m = kq$, $n = k$, then since $l^2 + m^2 + n^2 = 1$, $k^2(p^2 + q^2 + 1) = 1$ or $1/n = 1/\sqrt{1+p^2+q^2}$.

Hence **the surface S of the quadric $\mathbf{z = f(x, y)}$, is given by**

$$\mathbf{S} = \iint\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} . dx . dy. \quad \ldots\ldots\ldots\ldots\ldots(\mathbf{133})$$

It is clear from this that if y, z, or z, x be taken as the independent variables, corresponding formulae involving the partial derivatives $\partial x/\partial y$, $\partial x/\partial z$, and $\partial y/\partial z$, $\partial y/\partial x$ respectively may be used.

Ex. 3. *Shew that the area cut out of the cylinder $z^2=4ax$ by the cylinder $x^2+y^2=a^2$ is $5\pi a^2/2$.* (L.U., Sc.)

The vertex of the parabolic cylinder $z^2=4ax$ is the axis of y, and the axis of the cylinder $x^2+y^2=a^2$ is the axis of z.

Now, since $z^2=4ax$,

$$\therefore \ \frac{\partial z}{\partial x}=\frac{2a}{z}=\sqrt{\frac{a}{x}} \quad \text{and} \quad \frac{\partial z}{\partial y}=0;$$

$$\therefore \ S=4\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{1+a/x}\,.\,dx\,.\,dy=4\int_0^a\sqrt{(a^2-x^2)(a+x)/x}\,.\,dx$$

$$=8a^2\int_0^{\frac{\pi}{2}}(\cos^2\theta+\sin^2\theta\,.\,\cos^2\theta)\,d\theta, \text{ on putting } x=a\sin^2\theta,$$

$$=8a^2(\pi/4+\pi/16)=5\pi a^2/2.$$

123. Centroids and Moments of Inertia. There are many cases where double integration is required in the determination of centroids and moments of inertia. The following example will illustrate the method of dealing with such cases.

Ex. 4. (*a*) *A right prism standing on a rectangular base is cut by a plane so that the lengths of parallel edges are z_1, z_2, z_3, z_4. Shew that the centroid of the solid thus cut off between the xy-plane and the intersecting plane is at a height above the base equal to*

$$\{2z_4+z_2+(z_2^2-z_1z_3)/(z_1+z_3)\}/6.$$

Calculate this height when $z_1=5$ ft., $z_2=3$ ft. and $z_3=4$ ft.

(*b*) *The thickness z of a plate in the form of the parabola $y^2=4ax$ between the vertex and the ordinate at $x=h$, is given by the relation $z=x^2+y^2$. Find its radius of gyration about the axis of the parabola, assuming it to be made of material of uniform density, and with edges perpendicular to the xy-plane.*

(*a*) Let $lx+my+nz=p$ be the plane of intersection, and suppose the base of the prism to be on the xy-plane, having $(a, 0)$, $(0, 0)$, $(0, b)$, (a, b) as its vertices; then if z_1, z_2, z_3, z_4 be the corresponding heights to the plane, the equation of the plane through $(a, 0, z)$, $(0, 0, z_2)$ and $(0, b, z_3)$ may, by (104*e*), be written

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & z_1 & 1 \\ 0 & 0 & z_2 & 1 \\ 0 & b & z_3 & 1 \end{vmatrix}=0,$$

from which it is evident that

$$l/p=(z_2-z_1)/az_2, \quad m/p=(z_2-z_3)/bz_2, \quad n/p=1/z_2.$$

Hence, taking a prism whose base is a rectangle of sides dx, dy and height z, between the xy-plane and the intersecting plane, its volume $=z \,.\, dx \,.\, dy$, and its moment about the xy-plane is

$$\tfrac{1}{2}z^2 \,.\, dx \,.\, dy,$$

so that, if $\bar{z}$ be the height of the centroid,

$$\bar{z}\int_0^a\int_0^b z \,.\, dx \,.\, dy = \tfrac{1}{2}\int_0^a\int_0^b z^2 \,.\, dx \,.\, dy.$$

Putting $\quad z=(p-lx-my)/n$,

$$2n\bar{z}\int_0^a\int_0^b (p-lx-my)\,dx \,.\, dy = \int_0^a\int_0^b (p-lx-my)^2 \,.\, dx \,.\, dy.$$

$$\therefore\ 2n\bar{z}\int_0^a\{(p-lx)-\tfrac{1}{2}mb\}\,dx = \int_0^a\{(p-lx)^2-m(p-lx)b+\tfrac{1}{3}m^2b^2\}\,dx;$$

$$\therefore\ 2n\bar{z}(p-\tfrac{1}{2}mb-\tfrac{1}{2}la)=p^2-mbp+\tfrac{1}{3}m^2b^2+\tfrac{1}{2}(mb-2p)la+\tfrac{1}{3}l^2a^2.$$

Dividing throughout by p^2, and inserting the values of l/p, m/p, n/p already found, a simple reduction gives

$$6(z_1+z_3)\bar{z}=2(z_1+z_3)^2-z_2(z_1+z_3)+z_2{}^2-z_1z_3;$$

$$\therefore\ 6z=2z_1+2z_3-z_2+(z_2{}^2-z_1z_3)/(z_1+z_3).$$

Now since (a, b, z_4) also lies on the plane $lx+my+nz=p$,

$$la/p+mb/p+nz_4/p=1,$$

so that $$z_4=z_1-z_2+z_3.$$

Hence $$z=\{2z_4+z_2+(z_2{}^2-z_1z_3)/(z_1+z_3)\}/6,$$

as given.

It is easy to transform this expression into the form

$$\bar{z}=(z_1{}^2+z_2{}^2+z_3{}^2+z_4{}^2)/\{6(z_1+z_2+z_3+z_4)\}+(z_1+z_2+z_3+z_4)/12.$$

With the given values, $z_4=6$, and

$$\bar{z}=62/27=\mathbf{2{\cdot}296\ ft.}$$

(*b*) Let p be the density of the plate, then the mass of an elementary prism of length z is $pz \,.\, dx \,.\, dy$, and its moment of

inertia about the x-axis is $py^2z \,.\, dx \,.\, dy$; hence, if k be the radius of gyration about this axis,

$$k^2p\int_0^h\int_0^y z \,.\, dx \,.\, dy = 2p\int_0^h\int_0^y y^2z \,.\, dx \,.\, dy,$$

i.e.
$$k^2\int_0^h\int_0^y (x^2+y^2)\,dx \,.\, dy = 2\int_0^h\int_0^y y^2(x^2+y^2)\,dx\,dy;$$

$$\therefore\ k^2\int_0^h (x^2y+y^3/3)\,dx = 2\int_0^h (x^2y^3/3+y^5/5)\,dx.$$

Since y is now an ordinate of the parabola $y^2=4ax$,

$$k^2\int_0^h (x^{\frac{5}{2}}+\tfrac{4}{3}ax^{\frac{3}{2}})\,dx = 8a\int_0^h (\tfrac{1}{3}x^{\frac{7}{2}}+\tfrac{4}{5}ax^{\frac{5}{2}})\,dx;$$

$$\therefore\ k^2(h/7+4a/15) = 8ah\,(h/27+4a/35),$$

giving
$$k = \frac{2}{3}\sqrt{\frac{2(35h+108a)ah}{15h+28a}}.$$

124. Centres of Pressure. The force exerted on each unit of area of a surface by a fluid in contact with it is called the pressure of that fluid. Let δa be a small element of area, and let δp be the force exerted by the fluid upon it, then $\delta p/\delta a$ is the pressure at this point on the surface, and the sum of the pressures over the whole area is called the resultant pressure. The point on the surface where a single force acts, equivalent to the resultant pressure, is called the **centre of pressure** of that area. The determination of resultant pressures and centres of pressure requires, in general, the use of double integrals.

Ex. 5. (*a*) *Explain the use of double integrals in finding centres of pressure of areas under fluid thrust.* (L.U.)

(*b*) *Determine the coordinates of the centre of pressure of a uniform thin lamina in the shape of a quadrant of a circle of radius r, when immersed vertically in a homogeneous liquid with one bounding radius in the surface. Compare the position of this point with that of the centroid.*

(*a*) Let a plane surface be completely immersed in a homogeneous liquid whose pressure at any point is p, then the pressure on an infinitesimal rectangular area whose vertices are the points (x, y), $(x+dx, y)$, $(x+dx, y+dy)$, $(x, y+dy)$, is $p \,.\, dx \,.\, dy$.

Hence the resultant pressure over the area $=\iint p\,.\,dx\,.\,dy$.

Let $(\bar{x}, \bar{y})$ be the centre of pressure, then taking moments about each axis in turn,

$$\bar{x}\iint p\,.\,dx\,.\,dy=\iint px\,.\,dx\,.\,dy \quad \text{and} \quad \bar{y}\iint p\,.\,dx\,.\,dy=\iint py\,.\,dx\,.\,dy,$$

or

$$\bar{\mathbf{x}}=\frac{\iint \mathbf{px}\,.\,\mathbf{dx}\,.\,\mathbf{dy}}{\iint \mathbf{p}\,.\,\mathbf{dx}\,.\,\mathbf{dy}}, \quad \bar{\mathbf{y}}=\frac{\iint \mathbf{py}\,.\,\mathbf{dx}\,.\,\mathbf{dy}}{\iint \mathbf{p}\,.\,\mathbf{dx}\,.\,\mathbf{dy}}, \qquad \text{(134)}$$

where the integrals are taken so as to embrace the whole area of the surface immersed.

If the weight of the fluid alone be considered, and taking ρ = density of the liquid, and h the depth of the point (x, y) below the surface, $p=\rho g h$.

This analysis shews clearly the use of double integrals in determining centres of pressure.

(*b*) Since the lamina is vertical with one edge in the surface, taking this edge as the axis of x, $h=y$, and $p=\rho g y$.

Hence (134) becomes

$$\bar{x}\int_0^r\int_0^y y\,.\,dy\,.\,dx=\int_0^r\int_0^y xy\,.\,dy\,.\,dx$$

and

$$\bar{y}\int_0^r\int_0^y y\,.\,dy\,.\,dx=\int_0^r\int_0^y y^2\,.\,dy\,.\,dx,$$

i.e.

$$\bar{x}\int_0^r y^2\,.\,dx=\int_0^r xy^2\,.\,dx, \quad \tfrac{1}{2}\bar{y}\int_0^r y^2\,.\,dx=\frac{1}{3}\int_0^r y^3\,.\,dx.$$

But y is now an ordinate of the quadrant, and is therefore equal to $\sqrt{r^2-x^2}$;

$$\therefore\ \bar{x}\int_0^r (r^2-x^2)dx=\int_0^r x(r^2-x^2)\,.\,dx,$$

$$\tfrac{1}{2}\bar{y}\int_0^r (r^2-x^2)dx=\frac{1}{3}\int_0^r (r^2-x^2)^{\frac{3}{2}}\,.\,dx;$$

i.e.

$$\tfrac{2}{3}\bar{x}\,.\,r^3=\tfrac{1}{4}r^4, \quad \bar{y}r^3=r^4\int_0^{\frac{\pi}{2}}\cos^4\theta\,.\,dx,$$

where

$$x=r\sin\theta=\pi/16;$$

$$\therefore\ \bar{x}=\mathbf{3r/8}, \quad \bar{y}=\mathbf{3\pi r/16}.$$

If (x', y') be the centroid, then from Ex. 3, p. 377,

$$x' = 4r/(3\pi) = y' \quad \text{and} \quad y - y' = r(9\pi^2 - 64)/(48\pi),$$

which is obviously positive, so that $\bar{y} > y'$, and therefore the centre of pressure is below the centroid. Since the pressure increases with the depth, it is clear that this will always be the case for plane areas which are immersed in any position not horizontal. For a horizontal position the pressure is uniform over each element, and therefore the centre of pressure will coincide with the centroid.

125. Triple Integrals. In some practical problems it is necessary to use triple integration. If, for instance, the density at any point (x, y, z) of a solid be a given function $\phi(x, y, z)$ of the coordinates, then the mass of an infinitesimal prism of volume dv whose edges are dx, dy, dz, is $\phi(x, y, z) \,.\, dv$ or $\phi(x, y, z) \,.\, dx\,dy\,dz$.

Hence the mass of the whole solid is

$$\int \phi(x, y, z) \,.\, dv, \quad \text{or} \quad \iiint \phi(x, y, x)\,dx\,dy\,dz,$$

the integrations extending over the whole solid. Triple integrals may be evaluated in precisely the same way as double integrals.

Ex. 6. *The solid ellipsoid,* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, *is such that the density at any point* (x, y, z) *is* $\mu(x^{2n} + 1)$. *Calculate its mean density.*

Let $dx\,dy\,dz$ be the volume of an infinitesimal prism situated in the limit at the point (r', y', z'), then the mass M of the ellipsoid is given by

$$M = \mu \int_{-a}^{a} \int_{-y'}^{y'} \int_{-z}^{z} (x^{2n} + 1)\,dx\,dy\,dz$$

$$= 2\mu \int_{-a}^{a} \int_{-y'}^{y'} (x^{2n} + 1)\,z\,dx\,dy.$$

z is now the ordinate of the ellipse $y^2/y'^2 + z^2/z'^2 = 1$, on the plane $x = x'$; so that $z = z'\sqrt{y'^2 - y^2}/y'$;

$$\therefore\ M = 2\mu \int_{-a}^{a} \int_{-y'}^{y'} (x^{2n} + 1) z' \sqrt{y'^2 - y^2}/y' \,.\, dx\,dy$$

$$= 2\mu \int_{-a}^{a} (x^{2n} + 1) z' \left[\frac{y'^2}{2} \sin^{-1}\frac{y}{y'} + \frac{y}{2}\sqrt{y'^2 - y^2}\right]_{-y'}^{y'} \,.\, dx$$

$$= \pi\mu \int_{-a}^{a} (x^{2n} + 1)\,y'z'\,dx.$$

But y' is the ordinate of the ellipse $x^2/a^2+y^2/b^2=1$ on the xy-plane, and z' is the ordinate of the ellipse $x^2/a^2+z^2/b^2=1$ on the zx-plane;

$$\therefore\ y'=b\sqrt{a^2-x^2}/a \quad \text{and} \quad z'=c\sqrt{a^2-x^2}/a,$$

so that
$$M=\pi\mu bc/a^2\,.\int_{-a}^{a}(x^{2n}+1)(a^2-x^2)dx$$
$$=\frac{4\mu\pi abc\{3a^{2n}+(2n+1)(2n+3)\}}{3(2n+1)(2n+3)}.$$

Now the volume of the ellipsoidi s $4\pi abc/3$; hence, by division, the mean density is

$$\frac{\mu\{3a^{2n}+(2n+1)(2n+3)\}}{(2n+1)(2n+3)}.$$

126. Change of Variables. It is sometimes necessary in order to evaluate a multiple integral to change the variables, as, for instance, from Cartesian coordinates to polars. When each variable is expressible in terms of a single parameter, the transformation is quite simple, but when each variable is given as a function of two or more new variables, the transformation is more troublesome. Formulae giving the differential relations between the old and new variables in such cases will now be established.

Suppose that to evaluate the double integral $\iint\phi(x, y)dx\,.\,dy$, it is necessary to change the variables x, y into u, v, where $x=P(u, v)$, and $y=Q(u, v)$. From (39),

$$(a)\ dx=\frac{\partial x}{\partial u}\,.\,du+\frac{\partial x}{\partial v}\,.\,dv,$$

$$(b)\ dy=\frac{\partial y}{\partial u}\,.\,du+\frac{\partial y}{\partial v}\,.\,dv.$$

Now, in evaluating the integral, x is considered temporarily constant whilst the y-integration is performed. So in changing the variables x may first be regarded as constant, then $dx=0$, and (a) becomes

$$\frac{\partial x}{\partial u}\,.\,du+\frac{\partial x}{\partial v}\,.\,dv=0.$$

Eliminating du between this relation and (b),

$$dy = J \,.\, dv \Big/ \frac{\partial x}{\partial u},$$

where $J = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$, *i.e.* the Jacobian of x, y with respect to u, v, as defined on p. 87, which by (42) does not vanish.

Now let y be constant whilst x is variable, then from the relation just obtained, $dv = 0$, and from (a) $dx = \frac{\partial x}{\partial u} \,.\, du$.

Hence $$\iint \phi(x, y) \,.\, dx \,.\, dy = \iint F(u, v) \,.\, J \,.\, du \,.\, dv,$$

where $$F(u, v) = \phi\{P(u, v), Q(u, v)\}.$$

Similarly, it may be shewn that

$$\iiint \phi(x, y, z) dx \,.\, dy \,.\, dz = \iiint F(u, v, w) \,.\, J \,.\, du \,.\, dv \,.\, dw.$$

Hence, if $\mathbf{x} = \mathbf{P}(\mathbf{u}, \mathbf{v})$, $\mathbf{y} = \mathbf{Q}(\mathbf{u}, \mathbf{v})$,

$$\iint \phi(\mathbf{x}, \mathbf{y}) \,.\, \mathbf{dx} \,.\, \mathbf{dy} = \iint \phi\{\mathbf{P}(\mathbf{u}, \mathbf{v}),\ \mathbf{Q}(\mathbf{u}, \mathbf{v})\}\ \mathbf{J} \,.\, \mathbf{du} \,.\, \mathbf{dv},$$

where $$\mathbf{J} = \begin{vmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{u}} & \frac{\partial \mathbf{y}}{\partial \mathbf{u}} \\ \frac{\partial \mathbf{x}}{\partial \mathbf{v}} & \frac{\partial \mathbf{y}}{\partial \mathbf{v}} \end{vmatrix},$$

and if $\mathbf{x} = \mathbf{P}(\mathbf{u}, \mathbf{v}, \mathbf{w})$, $\mathbf{y} = \mathbf{Q}(\mathbf{u}, \mathbf{v}, \mathbf{w})$, $\mathbf{z} = \mathbf{R}(\mathbf{u}, \mathbf{v}, \mathbf{w})$, ...**(135)**

$$\iiint \phi(\mathbf{x}, \mathbf{y}, \mathbf{z}) \,.\, \mathbf{dx\, dy\, dz} = \iiint \phi(\mathbf{P}, \mathbf{Q}, \mathbf{R}) \,.\, \mathbf{J} \,.\, \mathbf{du\, dv\, dw},$$

where $$\mathbf{J} = \begin{vmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{u}} & \frac{\partial \mathbf{y}}{\partial \mathbf{u}} & \frac{\partial \mathbf{z}}{\partial \mathbf{u}} \\ \frac{\partial \mathbf{x}}{\partial \mathbf{v}} & \frac{\partial \mathbf{y}}{\partial \mathbf{v}} & \frac{\partial \mathbf{z}}{\partial \mathbf{v}} \\ \frac{\partial \mathbf{x}}{\partial \mathbf{w}} & \frac{\partial \mathbf{y}}{\partial \mathbf{w}} & \frac{\partial \mathbf{z}}{\partial \mathbf{w}} \end{vmatrix}.$$

If J' be the Jacobian of u, v, w with respect to x, y, z, where $u = A(x, y, z)$, $v = B(x, y, z)$, $w = C(x, y, z)$, then $JJ' = 1$, and this is true for any number of variables provided the relations defining the functions are independent. The theorem will be proved for two pairs of variables.

Let $x=P(u, v)$, $y=Q(u, v)$, and suppose these equations solved for u, v, so that $u=R(x, y)$, $v=S(x, y)$; then in the former x and y are independent, and in the latter u and v are independent.

Writing the Jacobians in determinant form and changing columns into rows in J,

$$JJ'=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial x}, & \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial u}\cdot\frac{\partial v}{\partial x}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}, & \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial y} \end{vmatrix}$$

$$=\begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \text{ since } x \text{ and } y \text{ are independent.}$$

Similarly, it may be shewn for any number of variables.

Hence, using the notation of Ex. 3, § 32;

If $y_s=\phi_s(x_1, x_2, \dots x_s)$, be s independent functions in v_s variables, and

$$J=\frac{\partial(y_1, y_2, \dots y_s)}{\partial(x_1, x_2, \dots x_s)},\quad J'=\frac{\partial(x_1, x_2, \dots x_s)}{\partial(y_1, y_2, \dots y_s)},$$

then $$JJ'=1. \qquad \text{.................................(186)}$$

When x_s is expressed in terms of y_s, this rule will facilitate the determination of J, since J' may be found directly.

Ex. 7. Prove that the area in the positive quadrant enclosed between the curves $a^2y=x^3$, $a'^2y=x^3$, $b^2x=y^3$ and $b'^2x=y^3$ is equal to $\frac{1}{2}(a-a')(b-b')$. (L.U., Sc.)

Shew that the locus of the point $x=a \sin\theta\cos\phi$, $y=\sin\theta\sin\phi$, $z=c\cos\theta$ is an ellipsoid, whose volume is

$$8abc\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin\theta \,.\, \cos^2\theta \,.\, d\theta \,.\, d\phi\,;$$

hence evaluate this double integral.

(i) Let $x^3/y=u$, $y^3/x=v$, then

$$J'=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3x^2/y & -y^3/x^2 \\ -x^3/y^2 & 3y^2/x \end{vmatrix} = 8xy = 8\sqrt{uv}\,;$$

$$\therefore\; J=1/J'=1/(8\sqrt{uv}).$$

Now the required area $= \int_{x_1}^{x_2}\int_{y_1}^{y_2} dx \, . \, dy$

$$= \tfrac{1}{8}\int_{b'^2}^{b^2}\int_{a'^2}^{a^2} u^{\frac{1}{2}} \, . \, v^{\frac{1}{2}} \, . \, du \, . \, dv, \text{ by (135)},$$

$$= \tfrac{1}{4}\int_{b'^2}^{b^2} u^{\frac{1}{2}}(a-a')\,du = \tfrac{1}{2}(\mathbf{a}-\mathbf{a}')(\mathbf{b}-\mathbf{b}').$$

(ii) Eliminating θ and ϕ from the parametric equations,

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta$$
$$= \sin^2\theta + \cos^2\theta = 1 \, ;$$

$\therefore$ the locus is an ellipsoid.

The volume of an ellipsoid $= 8\int_0^a\int_0^y z \, . \, dx \, . \, dy.$

Now to change the variables,

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} a\cos\theta\cos\phi & b\cos\theta\sin\phi \\ -a\sin\theta\sin\phi & b\sin\theta\cos\phi \end{vmatrix} = ab\sin\theta\cos\theta \, ;$$

$$\therefore \quad V = 8abc\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta \, . \, d\phi \, . \, d\theta = 8abc\int_0^{\frac{\pi}{2}} \frac{\Gamma(1)\,\Gamma(\frac{3}{2})}{2\Gamma(\frac{5}{2})} \, . \, d\phi$$

$$= \tfrac{8}{3}abc\int_0^{\frac{\pi}{2}} d\phi = \tfrac{4}{3}\pi\mathbf{abc}.$$

127. Intersection of a Plane and an Ellipsoid. The volumes of the two portions of an ellipsoid cut off by a plane may be found by evaluating the triple integral $\iiint dx\,dy\,dz$ between the proper limits. In general, however, the calculation is very long and tedious, and often difficult. This may be obviated by taking a slice parallel to the intersecting plane and integrating along the normal. If, therefore, the area of such a slice is known, the triple integral may be replaced by a single integral.

Now just as an ellipse has conjugate diameters, so an ellipsoid has conjugate planes, passing through the origin, such that their lines of intersection are conjugate diameters of the elliptic sections

cut off respectively by them. From the properties of these conjugate diameters (Ex. 12 (*c*), p. 226), the equation of the ellipsoid referred to its conjugate planes may be written

$$x^2/\alpha^2 + y^2/\beta^2 + z^2/\gamma^2 = 1,$$

where α, β, γ are the semi-lengths of the conjugate diameters.

Let the plane $z = \lambda$, parallel to the conjugate yz-plane, intersect the surface, then the section on it is the ellipse

$$x^2/\alpha^2 + y^2/\beta^2 = 1 - \lambda^2/\gamma^2.$$

If ω be the angle between the coordinate axes of this ellipse, which are a pair of conjugate diameters, and a', b' be the semi-axes, then, by (73*h*),

$$\alpha\beta/(1 - \lambda^2/\gamma^2) = a'b' \operatorname{cosec} \omega,$$

and the area A of the section $= \pi a'b' = \pi\alpha\beta(1 - \lambda^2/\gamma^2) \sin \omega$.

Hence if A_0 be the area of the parallel section through the origin

$$A_0 = \pi\alpha\beta \sin \omega,$$

so that, by division, to eliminate ω,

$$A/A_0 = 1 - \lambda^2/\gamma^2.$$

Further, let p_0, p be the perpendiculars from the origin to the respective planes, $z = \gamma$, $z = \lambda$, then $\lambda/\gamma = p/p_0$.

Hence $$\mathbf{A = A_0(1 - p^2/p_0{}^2)}. \quad \text{.........................(137a)}$$

Ex. 8. (*a*) *Find the area of the elliptic section cut off from an ellipsoid by any plane through the origin.*

(*b*) *Prove that the plane* $x/a + y/b + z/c = 1$ *divides the volume of the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ *in the ratio*

$$(3\sqrt{3} - 4) : (3\sqrt{3} + 4). \qquad \text{(L.U., Sc.)}$$

(*a*) Let $lx + my + nz = 0$ be any plane intersecting the ellipsoid, $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and passing through the origin, then by (111) the section is the ellipse

$$A'x^2 + 2H'xy + B'y^2 = 0,$$

where
$$A' = (m^2/a^2 + l^2/b^2)/(l^2 + m^2),$$
$$B' = (l^2n^2/a^2 + m^2n^2/b^2)/(l^2 + m^2) + (l^2 + m^2)/c^2,$$
$$H' = lmn(1/a^2 - 1/b^2)/(l^2 + m^2).$$

From Ex. 8, § 73, the squares of the reciprocals of the semi-axes of this ellipse are $\{A'+B'\pm\sqrt{4H'^2-(A'-B')^2}\}/2$, the product of which is $A'B'-H'^2$; hence the area of the ellipse is $\pi/\sqrt{A'B'-H'^2}=\pi abc/\sqrt{a^2l^2+b^2m^2+c^2n^2}$, on putting in the values of A', B', H'.

Now the parallel tangent plane will touch the ellipsoid at the extremity of a conjugate diameter; hence its perpendicular distance from the origin is the p_0 of (137a), and by (115),

$$p_0{}^2=a^2l^2+b^2m^2+c^2n^2,$$

so that the area A_0 is given by the simple relation

$$\mathbf{A_0=\pi abc/p_0.} \qquad \textbf{(137b)}$$

(b) With this result the required volumes may be obtained by a single integration. Consider a slice of thickness dz, parallel to and distant z from the given plane, then if A be its area, and v_1 the volume,

$$\begin{aligned} v_1 &= \int_p^{p_0} A\,.\,dz \\ &= A_0\int_p^{p_0}(1-z^2/p_0{}^2)\,.\,dz, \text{ by (137}a\text{)}, \\ &= A_0(p_0-p_0/3-p+p^3/3p_0{}^2) \\ &= \tfrac{1}{3}\pi abc(2-3\mu+\mu^3), \text{ by (137}b\text{)}, \end{aligned}$$

where $\mu=p/p_0$.

Similarly, the volume of the other portion is

$$v_2=\tfrac{1}{3}\pi abc(2+3\mu-\mu^3).$$

Hence to determine the ratio v_1/v_2, the value of μ must be found. Writing the equation of the cutting plane in the form

$$p=lx+my+nz,$$

and observing that it passes through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, successive substitution gives

$$l=p/a, \quad m=p/b, \quad n=p/c,$$

and $p_0{}^2=a^2l^2+b^2m^2+c^2n^2=3p^2$, on inserting the values of l, m, n; hence $\mu=1/\sqrt{3}$, so that

$$v_1 : v_2=(6\sqrt{3}-8):(6\sqrt{3}+8)=\mathbf{(3\sqrt{3}-4):(3\sqrt{3}+4)}$$

128. Plane Section of a Paraboloid. The formulae given in (137) are not applicable to a paraboloid since it is a non-central quadric surface. The area of a plane section may, however, be readily determined as follows.

Let the equation of the paraboloid be $ax^2+by^2=2z$, and suppose it to be intersected by the plane $lx+my+nz=p$, then the projection of the section on the xy-plane is

$$ax^2+by^2=2(p-lx-my)/n,$$

i.e.
$$a\left(x+\frac{l}{na}\right)^2+b\left(y+\frac{m}{nb}\right)^2=\frac{1}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+2pn\right),$$

which is an ellipse if a and b are positive; hence its area is

$$\frac{\pi}{n^2\sqrt{ab}}\left\{\frac{l^2}{a}+\frac{m^2}{b}+2pn\right\}.$$

Now, if this area be divided into triangular elements whose bases are infinitesimal elements of the bounding curve, the area of each triangle is n times the area of the triangle in the plane section of which it is the projection, by (108). Hence summing all these areas,

$$\mathbf{A}=\frac{\pi}{\mathbf{n}^3\sqrt{\mathbf{ab}}}\left\{\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}+2\mathbf{pn}\right\}. \quad\text{.....................(137c)}$$

EXERCISES 16.

1. Evaluate $\int_0^a\int_0^y dy\,.\,dx/(x^2+a^2)$.

2. Explain the meaning of double integration in connection with finding the mass or volume of a solid. The surface

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

encloses a solid whose density in the first octant is hx. Find the mass of the octant and the distance of its mass centre from $y=0$. (L.U.)

3. Find the value of the double integral $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$. (L.U.)

4. Shew that the volume enclosed by the surface $xy=z$ cut off by the planes $x=0$, $y=a$, $x=b$, $y=c$ is $b^2(c^2-a^2)/4$.

5. The equation of a surface is given in the form $z=f(x, y)$, where $f(x, y)$ is a continuous single valued function of x and of y.

Shew that the volume between this surface and the plane $z=0$ cut from a cylinder that has its generating lines parallel to the axis of z, can be written

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\,dx\,.\,dy.$$

Find the volume between the surface $z=xy$, the plane $bx+ay=ab$ and the coordinate planes. (L.U.)

6. Find the value of $\iint(a^2-x^2)dx\,dy$ taken over half the circle $x^2+y^2=a^2$.

A horizontal boiler has a flat bottom, and its ends are plane and semicircular. If it is just full of water, shew that the depth of the centre of pressure of either end is $0{\cdot}7\times$ total depth, very nearly. (L.U.)

7. Explain the use of double integrals in finding centres of pressure of areas under fluid pressure.

A rectangular tank is filled with water; the ends are vertical and of area 4 ft. by 6 ft. each, the 4 ft. edge makes 30° with the upward vertical. Find the total fluid thrust on an end and the centre of pressure. (L.U.)

8. Evaluate $\int_{1/a}^{3/a}\int_0^{y_1}(3y^2+1)dx\,dy/a^2$, where $y_1{}^2=a^2x^2-1$.

9. A right circular cylinder, of radius a, with axis along the axis of z, stands with its base in the plane of xy. If it is cut by a surface with ordinates z given as a function of x and y, shew that the volume of the solid bounded by the surface, the cylinder and the base can be expressed by the integral

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z\,.\,dx\,dy.$$

Find the volume of the solid left when the upper part of the above right circular cylinder is cut away by the parabolic cylinder $z=b+cy^2$. (L.U.)

10. If $z=(1-x^2/a^2-y^2/b^2)^{\frac{1}{2}}-1+x/a+y/b$, change the variables in the double integral $\iint z\,.\,dx\,dy$ from x, y to θ, ϕ, where $x=a\sin\theta\cos\phi$, $y=b\sin\theta\sin\phi$; hence evaluate it between the limits $\pi/2$, $\pi/4$ for θ and $\pi/2$, 0 for ϕ.

11. Find the area of the section of the surface $3(1-z)=2x^2+y^2$ made by the plane $z=c$, where c is less than unity.

Utilise this result to obtain the volume of the solid bounded by the portion of the surface above the plane of xy. (L.U.)

12. Find the value of $\iint xy\,dx\,dy$ taken over the positive quadrant of the ellipse $b^2x^2+a^2y^2=a^2b^2$, and also of $\iiint r\,.\,dV$, taken through the volume of the sphere of radius a, r denoting the distance of a point from a fixed point on the surface of the sphere. (L.U.)

13. Shew that the area bounded by the isothermals $pv=a$, $pv=b$, and the adiabatics $pv^{y}=c$, $pv^{y}=d$, is

$$\frac{b-a}{y-1} \cdot \log \frac{d}{c}.$$

14. P is any point on the catenary $y=c\cosh(x/c)$, whose vertex is A; find the length of the arc AP.

Find the values of

$$\frac{\iint y\,dx\,dy}{\iint dx\,dy} \quad \text{and} \quad \frac{\iint x\,dx\,dy}{\iint dx\,dy}$$

taken over the area bounded by the above catenary, the axis of x and the ordinate $x=a$. (L.U.)

15. Find the value of $\iint r^3 \cos^2\theta \,.\, dr\,d\theta$ taken over the area of the circle $r=2a\cos\theta$. (L.U.)

16. Change the variables in $\iint(1-z^2)^{-\frac{1}{2}}dx\,dy$, where $z^2=x^2+y^2$, from x, y to θ, ϕ, by means of the relations $x=\sin\theta\cos\phi$, $y=\sin\theta\sin\phi$; hence evaluate the integral between the limits ϕ to 0 for θ and $\frac{\pi}{4}$ to 0 for ϕ.

***17.** Shew that $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin\theta \,.\, d\theta \,.\, d\phi}{\lambda} = \frac{\pi}{2abc}$, where

$$\lambda^2=(a^2\sin^2\theta\cos^2\phi+b^2\sin^2\theta\sin^2\phi+c^2\cos^2\theta)^3.$$

18. Find the volume of the portion of the paraboloid $x^2/9+y^2/4=2z$ cut off by the plane which passes through the points (3, 2, 1·5), (0, 6, 0·5) and (1·5, 0, 3).

19. Find the ratios of the volumes cut off from the ellipsoid

$$4x^2+81y^2+576z^2=7056$$

by a plane which contains the points (0, 2, 2·5), (3, 0, 2·5) and (6, 3, 0).

***20.** Shew from (133) that the portion of the surface of the sphere $x=a\sin\theta\cos\phi$, $y=a\sin\theta\sin\phi$, $z=a\cos\theta$, bounded by two meridian arcs and the curve $\theta=f(\phi)$, is given by

$$S=a^2\int\{1-\cos f(\phi)\}d\phi.$$

Hence prove that the area of a spherical triangle bounded by two meridian arcs and a great circle is a^2E, where E is the spherical excess of the triangle.

***21.** Prove that the volume enclosed between the cylinders

$$x^2+y^2-2ax=0 \quad \text{and} \quad z^2=2ax$$

is $128a^3/15$. (B.U., Sc.)

***22.** The height z of a point on the surface of a mound above the plane of xy is given by the equation $a^3z=(a^2-x^2)(a^2-y^2)$, $2a$ being the side of the square base of the mound. Write down the expression for the volume of the mound as a double integral, and find its value. (B.U.)

23. Shew that the value of the double integral

$$\iint \frac{dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}}$$

taken over the area of a square bounded by $x=0$, $x=1$, $y=0$, $y=1$ is $\pi/6$. (S.U., Sc.)

***24.** Taking any point on the edge of the circular base of a right solid cone of height h, base radius r and density ρ, as origin and the axis of z parallel to the axis of the cone, shew that the product of inertia for the zx-plane of the solid is given by

$$2\rho \int_0^h \int_x^{2r-x} xyz\,dz\,.\,dx.$$

Hence evaluate this double integral.

***25.** Shew that the volume of the paraboloid $x^2/a^2+y^2/b^2=2z$ cut off by the plane $lx+my+nz=p$ is

$$\pi ab(a^2l^2+b^2m^2+2pn)^2/4n^4.$$

Hence prove that the distance μ between two parallel planes cutting a slice of volume V from the paraboloid is given by the quadratic

$$\pi ab\mu(a^2l^2+b^2m^2+2pn+\mu n)=Vn^3.$$

***26.** A square lamina in a vertical plane is totally immersed in a homogeneous liquid of density ρ, its centre is at depth d below the free surface, and a side makes an angle θ, less than $\frac{\pi}{4}$, with the horizontal. Prove that the total thrusts on the portions of the lamina above and below a horizontal line through the centre are

$$2g\rho a^2\{d\pm\tfrac{1}{8}a(2\cos\theta+\sec\theta)\},$$

where $2a$ is the length of a side of the square. (L.U., Sc.)

CHAPTER XVII

ELEMENTARY HARMONIC ANALYSIS

129. Expansion of Periodic Functions. The determination of a rational integral function approximating to a given empirical function which is continuous and non-periodic has been dealt with in Chapter IX. The case of a periodic function will now be considered briefly.

The period of such a function may be taken as 2π, for in any particular case an appropriate change in the independent variable can generally be made to effect this. The approximating function may then be taken in the form

$$A_0 + A_1 \cos x + A_2 \cos 2x + \dots + A_r \cos rx + \dots$$
$$+ B_1 \sin x + B_2 \sin 2x + \dots + B_r \sin rx + \dots,$$

where A_r, B_r are constants such that the series will be valid for values of x within the period.

A determination of these constants will now be made.

Ex. 1. *Find the coefficients* A_0, A_r, B_r *if*

$$f(x) = A_0 + A_1 \cos x + \dots + A_r \cos rx + \dots$$
$$+ B_1 \sin x + B_2 \sin 2x + \dots + B_r \sin rx + \dots$$

between the values $x=0$ *and* $x=2\pi$; *and determine their values in the case in which* $f(x)=x$ *between these limits.* (L.U.)

Write $f(x) = A_0 + \sum_{r=1}^{n} A_r \cos rx + \sum_{r=1}^{n} B_r \sin rx$.

Multiply out by $\cos rx$, and integrate both sides from 0 to 2π, then

$$\int_0^{2\pi} f(x) \cos rx \,.\, dx = \int_0^{2\pi} A_0 + \sum_{r=1}^{n} A_r \cos rx + \sum_{r=1}^{n} B_r \sin rx \cos rx \,.\, dx.$$

Now $\int_0^{2\pi} \cos rx \,.\, dx = \frac{1}{r}\Big[\sin rx\Big]_0^{2\pi} = 0$, since r is an integer,

and $\int_0^{2\pi} \cos sx \,.\, \cos rx \,.\, dx = \frac{1}{2}\int_0^{2\pi}\{\cos(r+s)x + \cos(r-s)x\}dx$

$$= \frac{1}{2}\left[\frac{\sin(r+s)x}{r+s} \mp \frac{\sin(r-s)x}{r-s}\right]_0^{2\pi} = 0,$$

as long as s and r are different integers.

When $s = r$, the integral becomes

$$\int_0^{2\pi} \cos^2 rx \,.\, dx = \frac{1}{2}\int_0^{2\pi}(1 + \cos 2rx)\,dx = \frac{1}{2}\Big[x + \sin 2rx/2r\Big]_0^{2\pi} = \pi.$$

Similarly

$$\int_0^{2\pi} \sin rx \,.\, \cos rx \,.\, dx = 0 \quad \text{and} \quad \int_0^{2\pi} \sin sx \,.\, \cos rx \,.\, dx = 0.$$

Hence every term on the right-hand side of the above equation vanishes, except that which contains A_r, so that

$$\int_0^{2\pi} f(x) \,.\, \cos rx \,.\, dx = \pi A_r.$$

This determines the coefficients A_r.

In precisely the same way, by taking $\sin rx$ as the multiplying factor and integrating,

$$\int_0^{2\pi} f(x) \sin rx \,.\, dx = \pi B_r.$$

Finally,

$$\int_0^{2\pi} f(x) \,.\, dx = \int_0^{2\pi}\left(A_0 + \sum_{r=1}^{n} A_r \cos rx + \sum_{r=1}^{n} B_r \sin rx\right)dx$$
$$= 2\pi A_0.$$

Hence all the coefficients have now been found.

When $f(x) = x$, then

$$2\pi A_0 = \int_0^{2\pi} x \,.\, dx = 2\pi^2; \qquad \therefore\ A_0 = \pi,$$

$$\pi A_r = \int_0^{2\pi} x \cos rx \,.\, dx = 0; \qquad \therefore\ A_0 = 0,$$

$$\pi B_r = \int_0^{2\pi} x \sin rx \,.\, dx = -2\pi/r; \qquad \therefore\ B_r = -2/r,$$

and the series becomes

$$x=\pi-2\left(\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{n}\sin nx+\ldots\right).$$

130. Fourier Series. A trigonometrical series whose coefficients are

$$\left.\begin{aligned} &\mathbf{A}_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathbf{f(x)\,.\,dx}, \quad \mathbf{A_r}=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathbf{f(x)\cos rx\,.\,dx}, \\ \textbf{and} \qquad &\mathbf{B_r}=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathbf{f(x)\sin rx\,.\,dx}, \\ \textbf{where} \qquad &\mathbf{f(x)}=\mathbf{A}_0+\sum_{r=}^{n}{}_1\mathbf{A_r}\cos\mathbf{rx}+\sum_{r=1}^{n}\mathbf{B_r}\sin\mathbf{rx} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(138)$$

is called a **Fourier Series**, after the French mathematician Jean Fourier, who first stated the theorem in 1822, that an empirical periodic function defined in the interval $0-2\pi$ could be represented by such a series.

It should be noted that in (138) the range is from $-\pi$ to π, instead of 0 to 2π. This is often more convenient, and really means that the origin is changed to the period $(\pi, 0)$.

The determination of the coefficients when $f(x)$ is given as a series of discrete values through which a graph may be drawn is more difficult, since the integrations cannot now be effected analytically. Several methods have been devised, but space permits only the discussion of one of these, which is a general method in practice.

Ex. 2. *Shew how a Fourier series can be constructed which will represent approximately the equation of a given curve between the values* 0 *and* 2π.

Let $y=f(x)$ represent the approximate equations of the given graph; let the interval $x=0$ to $x=2\pi$ be divided into n equal parts, bounded by $n+1$ ordinates, y_s $(s=0, 1, \ldots n)$.

Let θ be the distance between consecutive ordinates, then

$$\theta=2\pi/n.$$

Assume that y has the form

$$y=A_0+\sum_{r=1}^{n}A_r\cos rx+\sum_{r=1}^{n}B_r\sin rx,$$

then since y_s is the ordinate at $x = s\theta = 2\pi s/n$,

$$y_s = A_0 + \sum_{r=1}^{n} A_r \cos rs\theta + \sum_{r=1}^{n} B_r \sin rs\theta\,;$$

$$\therefore\ \sum_{s=0}^{n-1} y_s = nA_0 + \sum_{s=0}^{n-1}\{A_1 \cos s\theta + A_2 \cos 2s\theta + \ldots + B_1 \sin s\theta + B_2 \sin 2s\theta + \ldots\}.$$

Taking the general term of the series of cosines,

$$\therefore\ \sum_{s=0}^{n-1} \cos ms\theta = 1 + \cos m\theta + \cos 2m\theta + \ldots + \cos (n-1)m\theta$$

$$= \left(\cos \frac{n-1}{2} \,.\, m\theta \,.\, \sin \tfrac{1}{2}nm\theta\right) \Big/ \sin \tfrac{1}{2}m\theta,$$

by Ex. 10 (i) (p. 51), on writing $m\theta$ for θ, and putting $a = 0$,

$$= \left(\cos \frac{n-1}{n} m\pi \,.\, \sin m\pi\right) \sin \frac{m\pi}{n},$$

on putting in the value of θ,

$$= 0, \text{ since } m \text{ is an integer.}$$

Hence the whole of the cosine series vanishes.

Similarly, taking the general term of the sine series,

$$\sum_{s=0}^{n-1} \sin ms\theta = \sin m\theta + \sin 2m\theta + \ldots + \sin (n-1)m\theta$$

$$= \sin \tfrac{1}{2}(n-1)m\theta \,.\, \sin \tfrac{1}{2}nm\theta / \sin \tfrac{1}{2}m\theta,$$

on putting in the value of θ.

Hence the whole of the sine series also vanishes, and

$$nA_0 = \sum_{s=0}^{n-1} y_s,$$

i.e. A_0 is the arithmetic mean of the n ordinates.

The coefficients A_r may now be determined by multiplying the equations for y_s by $\cos rs\theta$, and on taking the sum, all the coefficients except that of A_r will vanish, thus giving an equation for A_r. Similarly, by multiplying the y_s equations by $\sin rs\theta$, and summing, a relation in B_r remains.

In practice, only twelve equidistant ordinates are taken, and the corresponding Fourier series assumed is

$$y = A_0 + \sum_{r=1}^{6} A_r \cos rs\theta + \sum_{r=1}^{5} B_r \sin rs\theta,$$

where $\theta = \pi/6$.

From the above analysis, the coefficients are given by the equations,

$$\mathbf{12A_0} = \sum_{s=0}^{11} \mathbf{y_s}, \quad \mathbf{6A_r} = \sum_{s=0}^{11} \mathbf{y_s \cos rs\theta}, \quad \mathbf{6B_r} = \sum_{s=0}^{11} \mathbf{y_s \sin rs\theta}. \quad \ldots(\mathbf{189a})$$

These may be simplified according as r is odd or even.

Let $r = 2p+1$ $(p=0,\ 1,\ 2)$, then

$$6A_{2p+1} = y_0 - y_6 + \sum_{m=1}^{5} \{y_m \cos(2p+1)m\theta$$
$$+ y_{12-m}\cos(2p+1)(12-m)\theta\}$$
$$= y_0 - y_6 + \sum_{m=1}^{5} (y_m + y_{12-m})\cos(2p+1)m\theta.$$

Since $\cos 3r\theta = 0$, $\cos 4r\theta = -\cos 2r\theta$, and $\cos 5r\theta = -\cos r\theta$, r being an integer, odd or even, and $6\theta = \pi$, the above expression reduces to

$$6A_{2p+1} = y_0 - y_6 + (y_1 + y_{11} - y_5 - y_7)\cos(2p+1)\theta$$
$$+ (y_2 + y_{10} - y_4 - y_8)\cos 2(2p+1)\theta.$$

Similarly, if $r = 2p$ $(p=2,\ 4)$,

$$6A_{2p} = y_0 + y_6 + (y_1 + y_{11} + y_5 + y_7)\cos 2p\theta$$
$$+ (y_2 + y_{10} + y_4 + y_8)\cos 4p\theta + (y_3 + y_9)\cos 6p\theta,$$

and

$$6B_{2p+1} = (y_1 - y_{11} + y_5 - y_7)\sin(2p+1)\theta$$
$$+ (y_2 - y_{10} + y_4 - y_8)\sin 2(2p+1)\theta$$
$$+ (y_3 - y_9)\sin 3(2p+1)\theta,$$

$$6B_{2p} = (y_1 - y_{11} - y_5 + y_7)\sin 2p\theta + (y_2 - y_{10} - y_4 + y_8)\sin 4p\theta.$$

The actual numerical calculation is best carried out by the following simple scheme of tabulation devised by C. Runge.

Ordinates.	a	y_0	y_1	y_2	y_3	y_4	y_5	
	b		y_{11}	y_{10}	y_9	y_8	y_7	y_6
$a+b$	c	y_0	u_1	u_2	u_3	u_4	u_5	y_6
	d	y_6	u_5	u_4				
$c+d$	v	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{u}_3$			
$c-d$	w	$\mathbf{w}_0$	$\mathbf{w}_1$	$\mathbf{w}_2$				
$a-b$	e		a_1	a_2	a_3	a_4	a_5	
	f		a_5	a_4				
$e+f$	g		$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$			
$e-f$	h		$\mathbf{c}_1$	$\mathbf{c}_2$				

The equations giving the coefficients A_r, B_r may now be easily determined from the following table:

	$r=0$ and 6		1 and 5		2 and 4		3
	v_0 v_2	v_1 v_3	w_0 $w_2/2$	$w_1\sqrt{3}/2$	v_0 $-v_2/2$	$v_1/2$ $-v_3$	w_0-w_2
Sums	α_1	α_2	α_3	α_4	α_5	α_6	—
	$12A_0=\alpha_1+\alpha_2$ $12A_6=\alpha_1-\alpha_2$		$6A_1=\alpha_3+\alpha_4$ $6A_5=\alpha_3-\alpha_4$		$6A_2=\alpha_5+\alpha_6$ $6A_4=\alpha_5-\alpha_6$		$6A_3=w_0-w_2$

	1 and 5		2 and 4		3
	$b_1/2$ b_3	$b_2\sqrt{3}/2$	$c_1\sqrt{3}/2$	$c_2\sqrt{3}/2$	b_1-b_3
Sums	β_1	β_2	β_3	β_4	—
	$6B_1=\beta_1+\beta_2$ $6B_5=\beta_1-\beta_2$		$6B_2=\beta_3+\beta_4$ $6B_4=\beta_3-\beta_4$		$6B_3=b_1-b_3$

(139b)

Ex. 3. *An empirical periodic function is defined by the following twelve equidistant ordinates covering the whole period.*

5·0, 8·1, 10·3, 11·5, 11·4, 10·3, 8·2, 6·0, 4·1, 3·8, 2·2, 2·8.

Construct the approximate Fourier series for the function.

Taking the given ordinates, and tabulating them according to the scheme of Ex. 2 above, the numerical calculation appears as follows:

Ordinates.	a	5·0	8·1	10·3	11·5	11·4	10·3	
	b		2·8	2·2	3·8	4·1	6·0	8·2
$a+b$	c	5·0	10·9	12·5	15·3	15·5	16·3	8·2
	d	8·2	16·3	15·5				
$c+d$	v	**13·2**	**27·2**	**28·0**	**15·3**			
$c-d$	w	**−3·2**	**−5·4**	**−3·0**				
$a-b$	e		5·3	8·1	7·7	7·3	4·3	
	f		4·3	7·3				
$e+f$	g		**9·6**	**15·4**	**7·7**			
$e-f$	h		**1·0**	**0·8**				

.Transferring the rows v, w, g and h to the second table :

$r=0$ and 6		1 and 5		2 and 4		3
13·2 28·0	27·2 15·3	$-3·2$ $-1·5$	$-2·7\sqrt{3}$	13·2 -14	13·6 $-15·3$	$-3·2+3$
41·2	42·5	$-4·7$	$-2·7\sqrt{3}$	$-0·8$	$-1·7$	
$12A_0=83·7$ $12A_6=-1·3$		$6A_1=-4·7-2·7\sqrt{3}$ $6A_5=-4·7+2·7\sqrt{3}$		$6A_2=-2·5$ $6A_4=0·9$		$6A_3=-0·2$

1 and 5		2 and 4		3
4·8 7·7	$7·7\sqrt{3}$	$0·5\sqrt{3}$	$0·4\sqrt{3}$	$9·6-7·7$
12·5	$7·7\sqrt{3}$	$0·5\sqrt{3}$	$0·4\sqrt{3}$	
$6B_1=12·5+7·73\sqrt{3}$ $6B_5=12·5-7·73\sqrt{3}$		$6B_2=0·9\sqrt{3}$ $6B_4=0·1\sqrt{3}$		$6B_3=1·9$

Putting $\sqrt{3}=1·732$, the required series becomes

$$y=6·98-1·56\cos x-0·41\cos 2x-0·03\cos 3x+0·15\cos 4x$$
$$-0·003\cos 5x-0·11\cos 6x+4·31\sin x+0·26\sin 2x$$
$$+0·32\sin 3x+0·03\sin 4x-0·14\sin 5x.$$

131. Validity of Fourier Series. It should be carefully observed that the representation of a periodic function by a Fourier series depends upon the assumption that such a series is valid within the period. A rigorous proof of this is by no means easy, and will not be here attempted. It was first given by Dirichlet in 1829, and may be found in the treatises on the Calculus mentioned on p. xii. In the case of a function defined analytically, the infinite series supposed to represent it must be shewn to be convergent for all values of the independent variable within the period. In practical cases, however, it is generally sufficient to see that the function is single-valued, finite and, in general, continuous. It must also be periodic, satisfying the condition $f(x \pm 2n\pi)=f(x)$, and the Fourier series representing it must be tested for the end

values of the period, *i.e.* for $x=0$ and $x=2\pi$, for often the series is true for values within the period but breaks down at the boundaries. Ex. 11 of the following exercises will afford an illustration of this.

Exercises 17.

An empirical periodic function of x is defined by each of the following groups of twelve ordinates covering the period 2π. Express each function as a Fourier series.

1. 30, 40·5, 47, 51·5, 53·5. 53·7, 51·4, 45, 35, 25, 17·5, 17.

2. 20, 39·5, 54, 64, 69·5, 73, 75, 73·5, 70, 63·5, 54, 40·5.

3. 44·2, 82·6, 110·3, 124·6, 113·5, 100·7, 71·2, 43·3, 28·3, 23·7, 22·2, 28·3.

4. 13·6, 18·5, 20·7, 20·2, 17·8, 14·3, 10·1, 5·6, 1·9, 0·5, 2·5, 7·5.

5. 16·4, 16·3, 14·1, 11·7, 9·4, 7·1, 5·7, 5·0, 4·1, 5·1, 8·4, 12·2.

6. For the purposes of an approximation, a half arch of the sine curve $y=a \sin x$ is to be replaced by the straight line $y=mx$. In order to find the best value of m, proceed as follows. Take the square of the difference of the ordinates of the curve and the line, and find the mean value by integration from 0 to $\pi/2$. Then find, in terms of a, the value of m that makes this mean value a minimum. (L.U.)

7. A crank a feet in length rotates uniformly with angular velocity ω; the connecting rod is l feet long; find the distance of the cross-head from the end of the stroke as a function of the time. Shew that the motion is very nearly simple harmonic combined with one of half the period.

8. Give concisely, with proofs, a graphical method for determining the coefficient of a Fourier series valid between $x=0$ and $x=2\pi$ which is approximately the equation of a given graph. (L.U.)

9. Find a Fourier series for x valid between the interval $x=0$, $x=\pi$.

10. Shew that if $\phi(x)$ is an even function, the expansion of $\phi(x)$ consists of cosines alone, when the period ranges from $-\pi$ to π.

11. Find a Fourier series for e^x valid within the interval 0 to 2π, Test the series at $x=0$ and $x=2\pi$.

***12.** Shew that

$$1+2a\cos x+a^2=2\sum_{r=1}^{\infty}(-)^{r+1}a^r\cos rx/r \text{ when } a<1, \text{ and}$$

$$=2\log a+2\sum_{r=1}^{\infty}(-)^{r+1}\cos rx/(ra^r) \text{ when } a>1.$$

***13.** If $\phi(x)=A_0+\sum_{r=1}^{\infty} A_r \cos rx+\sum_{r=1}^{\infty} B_r \sin rx$, and the Fourier series be denoted by S, the coefficients may be derived from the condition that M is a minimum where

$$M=\int_0^{2\pi} \{\phi(x)-S\}^2 dx.$$

This condition will be satisfied if $\frac{\partial M}{\partial A_r}=0$, and $\frac{\partial M}{\partial B_r}=0$; shew that this gives

$$2\pi A_0=\int_0^{2\pi} \phi(x)\,.\,dx, \quad \pi A_r=\int_0^{2\pi} \phi(x) \cos rx\,.\,dx$$

and

$$\pi B_r=\int_0^{2\pi} \phi(x) \sin rx\,.\,dx.$$

14. A function of x is equal to x for values of x between 0 and $\pi/2$ and equal to $\pi/2$ for values between $\pi/2$ and π. Prove that it may be represented by the series

$$(1+2/\pi) \sin x-\tfrac{1}{2} \sin 2x+\tfrac{1}{3}(1-2/3\pi) \sin 3x-\tfrac{1}{4} \sin 4x$$
$$+\tfrac{1}{5}(1+2/5\pi) \sin 5x-\ldots. \qquad \text{(Li.U.)}$$

15. The value of y, a function of x of period 2π, is given for twelve equidistant values of x covering the whole period: 22·8, 20·3, 15·2, 9·4, 5·2, 3·1, 2·9, 4·3, 7·4, 12·1, 17·3, 21·6. Express y in a Fourier series, neglecting terms in $\cos nx$ for $n>3$. (Li.U.)

***16.** Determine the Fourier series of cosines of multiples of x which is equal to $|\sin x|$, *i.e.* $=\sin x$ if $0<x<\pi$, and $=-\sin x$ if $\pi<x<2\pi$, viz.

$$\frac{4}{\pi}\left(\frac{1}{2}-\frac{\cos 2x}{1\,.\,3}-\frac{\cos 4x}{3\,.\,5}-\frac{\cos 6x}{5\,.\,7}-\ldots\right). \qquad \text{(Br.U.)}$$

CHAPTER XVIII

DIFFERENTIAL EQUATIONS

132. Formation of a Differential Equation. Any functional relation between two or more variables usually involves, in its expression, at least one constant. In order to remove this constant it is necessary to differentiate the relationship with respect to one of the variables. The eliminant thus contains a differential coefficient, and is called a **Differential Equation.** In general, one differentiation is required to remove each constant, so that a functional relationship involving n constants will lead to a differential equation of the nth order, and conversely, the functional relationship between the variables in a differential equation of the nth order, will involve n arbitrary constants in its most general expression. This is a fundamental principle in the theory of differential equations which is exceedingly important in practice.

Ex. 1. *Eliminate the constant A from the equation $y = A \tan^2 x$.*

Differentiating with respect to x, $\frac{dy}{dx} = 2A \tan x \cdot \sec^2 x$.

Divide this equation by the given one, and

$$\frac{dy}{dx} = 4y \operatorname{cosec} 2x,$$

which is the eliminant.

This result might have been obtained directly by logarithmic differentiation, thus taking logarithms and differentiating,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2 \sec^2 x}{\tan x} = 4 \operatorname{cosec} 2x;$$

$$\therefore \frac{dy}{dx} = 4y \operatorname{cosec} 2x, \text{ as before.}$$

Ex. 2. *Eliminate the constants A and B from the equation*

$$y = Ax^n + Bx^{-n-1}.$$

Since there are two constants to eliminate, it will be necessary to differentiate twice.

Now $\quad y = Ax^n + Bx^{-n-1}; \quad \ldots\ldots\ldots (a)$

$\therefore \quad y_1 = nAx^{n-1} - (n+1)Bx^{-n-2}, \quad \ldots\ldots\ldots (b)$

and $\quad y_2 = n(n-1)Ax^{n-2} + (n+1)(n+2)Bx^{-n-3}, \quad \ldots\ldots\ldots (c)$

where
$$y_1 \equiv \frac{dy}{dx} \quad \text{and} \quad y_2 \equiv \frac{d^2y}{dx^2}.$$

Solving (a) and (b) for A and B:

$$A = \frac{(n+1)y + xy_1}{(2n+1)x^n}, \quad B = \frac{(ny - xy_1)x^{n+1}}{2n+1}.$$

Substituting these values in (c),

$$(2n+1)x^2y_2 = n(n-1)\{(n+1)y + xy_1\} + (n+1)(n+2)(ny - xy_1)$$
$$= n(n+1)(2n+1)y - 2(2n+1)xy_1;$$
$$\therefore \quad \mathbf{x^2 \cdot \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0.}$$

EXERCISES 18A.

In the following exercises, variables are denoted by small letters and constants by capitals, unless otherwise stated.

Eliminate, by differentiation, the constants in each of the following equations:

1. $y = A \sin x$. **2.** $y = Ae^x$. **3.** $y = A\sqrt{1+x^2}$.

4. $y(x^3 - 6) = x(x + A)$. **5.** $y^2(1 - x^2) = A \sin^2 x$.

6. $e^{xy} = Ax^2$. **7.** $y^2(A^2 - x^2) = A^2 + x^2$.

***8.** $x^2(x^2 + y^2) = A^2(y^2 - x^2)$. ***9.** $3x^2 + 2Axy + 3y^2 = 1$.

10. $x \sin y = Ay \cos x$. **11.** $y = A \sin x + B \cos x$.

12. $xy = Ax^3 + B$. **13.** $y = Ax + Bx^{-2}$.

14. $y^3 = Ax + Bx^2e^x$. ***15.** $y = A \cos(\log x) + B \sin(\log x)$.

***16.** $y + A = \sqrt{x^3 - B}$. **17.** $y + A = \log(x + B)$.

18. $y = A \cos x + B \cos 2x$.

***19.** $6(y + Ax) = B\{2x(x^2 + 3)\tan^{-1} x - 2\log(1 + x^2) - x^2\}$.

***20.** $Ax^2 + 6xy + By^2 = 1$.

21. The displacement x of a body executing simple harmonic motion is, in time t, given by the equation

$$x = A \sin \omega t + B \cos \omega t,$$

where ω is a fixed constant, and A, B are arbitrary constants, depending upon initial conditions.

Shew, by eliminating these arbitrary constants, that the motion is represented by the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

and interpret this equation.

22. The displacement x of a particle executing simple harmonic motion, which is being damped by friction, is in time t given by

$$x = Ae^{-ft} \sin (\omega t + B),$$

where f, ω are fixed constants and A, B the arbitrary constants. Shew by eliminating A and B that the differential equation representing such motion is

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + (f^2 + \omega^2)x = 0.$$

***23.** The space s described in time t by a body falling under gravity in a resisting medium of constant density is given by

$$\mu^2(s + A) = g \log \cosh \mu(t + B),$$

where μ, g are fixed constants depending upon the medium and gravity respectively, and A, B are the arbitrary constants. Prove, by eliminating these constants, that the differential equation representing the fall is

$$\frac{d^2s}{dt^2} + \frac{\mu^2}{g} \cdot \left(\frac{ds}{dt}\right)^2 - g = 0.$$

***24.** The general equation of a conic may be written

$$Ax^2 + 2Hxy + y^2 + 2Gx + 2Fy + C = 0,$$

where A, C, F, G, H are arbitrary constants depending only upon the position of the cutting plane.

Regarding this equation as a quadratic in y, solve it, and then, by differentiation, shew that if $\frac{1}{u}$ be written for $\frac{d^2y}{dx^2}$,

$$u^{\frac{2}{3}} = Ex^2 + 2Mx + N,$$

where $$\frac{H^2 - A}{E} = \frac{HF - G}{M} = \frac{F^2 - C}{N} = \{(F^2 - C)(H^2 - A) - HF - G)^2\}^{\frac{2}{3}}.$$

Finally, by further differentiation, to eliminate the new constants E, M, N, prove that the general differential equation of a conic section is

$$\frac{d^3}{dx^3}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\} = 0.$$

183. Linear Differential Equations of the First Order. A differential equation is an equation involving two or more variables and their derivatives or differential coefficients. When the derivatives present are complete, the equations are called **Ordinary Differential Equations** to distinguish them from **Partial Differential Equations** in which the derivatives present are partial differential coefficients.

The **Order** of an equation is that of the highest derivative present, and when only the first power of each derivative is present, the equation is said to be **linear.** Thus the following are examples of linear equations of the First, Second and Third Orders respectively:

$$(a)\quad (16-y^2)\frac{dy}{dx}=9+x^2,$$

$$(b)\quad \frac{d^2y}{dx^2}-\frac{1}{x}\cdot\frac{dy}{dx}+y=3x\sin x,$$

$$(c)\quad \cos x\,.\,\frac{d^3y}{dx^3}+\sin x\,.\,\frac{d^2y}{dx^2}+y^2+\cos^2 x=0.$$

Only ordinary linear equations in two variables will be considered here.

Ex. 3. *The gradient of a curve at any point is given by the equation*

$$2\sin x\,.\,\frac{dy}{dx}=1-y^2\,;$$

find the equation of curve.

Here the relation between x and y is required.

Separating the variables, the given equation becomes

$$\frac{2\,.\,dy}{1-y^2}=\frac{dx}{\sin x}.$$

Integrating each side and adding on an arbitrary constant,

$$\log\frac{1+y}{1-y}=\log\tan\frac{x}{2}+A'.$$

Since A' is quite arbitrary, put $A'=\log A$, then

$$\log\frac{1+y}{1-y}=\log A\tan\frac{x}{2},$$

or

$$1+y=A(1-y)\tan\frac{x}{2}.$$

Ex. 4. *Obtain the general solution of the equation*

$$A\frac{dy}{dx}+By=f(x),$$

where A and B are constants.

A particle of mass 1 *lb. moves in a medium whose resistance is v/3 lb. weight, where v is the velocity, and is subject to an accelerating force constant in direction which at time t is* $4t^2$ *lb. weight. If the particle starts from rest, find its velocity after two seconds.* (L.U.)

In the given equation the variables cannot be separated directly, hence the term in y must first be removed.

Let $y=uv$, where u and v are functions of x to be determined, then

$$\frac{dy}{dx}=u\,.\,\frac{dv}{dx}+v\,.\,\frac{du}{dx}.$$

Substituting in the given equation, and writing X for $f(x)$,

$$A\left(u\,.\,\frac{dv}{dx}+v\,.\,\frac{du}{dx}\right)+Buv=X,$$

i.e. $$Av\,.\,\frac{du}{dx}+\left(A\frac{dv}{dx}+Bv\right)u=X.$$

Now so far u and v are only subject to the condition that their product must be equal to y; another condition may therefore be arbitrarily imposed upon them. Let this be that the coefficient of u in the last equation should vanish,

then $$A\frac{dv}{dx}+Bv=0,$$

i.e. $$\frac{dv}{v}+c\,.\,dx=0, \text{ where } c=B/A,$$

which, on integration, gives

$$\log v+cx=0,$$

there being no need to introduce an arbitrary constant, since this would disappear later;

$$\therefore\ v=e^{-cx}.$$

Hence on substituting this value for v in the equation in u, v,

$$Ae^{-cx}\frac{du}{dx}=X;$$

$$\therefore\ A\,du=Xe^{cx}dx,$$

which, on integration, gives

$$Au = C + \int Xe^{cx} \,.\, dx,$$

C being an arbitrary constant;

$$\therefore \quad \mathbf{Ay} = \left(\mathbf{C} + \int \mathbf{Xe^{cx}} \,.\, \mathbf{dx}\right)\mathbf{e^{-cx}},$$

which is the general solution of the given equation.

Let s be the displacement of the particle in time t, then the equation of motion is

$$\frac{d^2s}{dt^2} = (4t^2 - \tfrac{1}{3}v)g.$$

But $v = \frac{ds}{dt}$; $\therefore$ $\frac{d^2s}{dt^2} = \frac{dv}{dt}$, and the equation becomes

$$\frac{dv}{dt} + kv = 4gt^2,$$

where $k = g/3$.

Put $v = ue^{-k\int dt} = ue^{-kt}$, this being the substitution necessary to remove the term in v, as shewn above.

Then $$du = 4gt^2e^{kt} \,.\, dt,$$

so that $$u = 4g\int e^{kt}t^2 \,.\, dt + A,$$

which, on integrating by parts, gives

$$u = 4g(k^2t^2 - 2kt + 2)e^{kt}/k^3 + A.$$

Since $v = ue^{-kt}$, the velocity at any time t is given by

$$v = 4g(k^2t^2 - 2kt + 2)/k^3 + Ae^{-kt}.$$

As the particle starts from rest, $v = 0$, when $t = 0$; hence

$$0 = 8g/k^3 + A \quad \text{or} \quad A = -8g/k^3;$$

$$\therefore \quad v = 4g(k^2t^2 - 2kt + 2 - 2e^{-kt})/k^3,$$

i.e. $$vg^2 = 12(g^2t^2 - 6gt + 18 - 18e^{-gt/3}).$$

When $t = 2$, taking $g = 32$,

$$1024v = 12(4096 - 384 + 18 - 9{\cdot}774 \times 10^{-9})$$

$$= 12 \times 3730, \text{ neglecting the last term;}$$

$$\therefore \quad v = \mathbf{43{\cdot}7} \text{ ft. per sec. approx.}$$

134. Integrating Factors. If P, Q denote functions of x, the general linear equation of the first order may be written

$$\frac{dy}{dx}+Py=Q,$$

and the foregoing analysis shews that the term in y may be removed by putting $y=ue^{-\int P\,dx}$. This substitution may, however, be obviated by multiplying the equation throughout by $\int e^{P\,dx}$; the left-hand side then becomes a perfect differential, thus

$$e^{\int P\,dx}\cdot\frac{dy}{dx}+Pe^{\int P\,dx}\cdot y=Qe^{\int P\,dx},$$

or
$$\frac{d}{dx}\left(ye^{\int P\,dx}\right)=Qe^{\int P\,dx},$$

which gives
$$ye^{\int P\,dx}=A+\int Qe^{\int P.\,dx}\,.\,dx.$$

Hence, **the equation $\frac{dy}{dx}+Py=Q$ becomes integrable when it is multiplied throughout by the factor $e^{\int P\,dx}$; the solution then becomes**

$$ye^{\int P\,dx}=A+\int Qe^{\int P\,dx}dx. \qquad (140)$$

The factor $e^{\int P\,dx}$ is called an **integrating factor**.

A similar method may be applied to equations of the form $\frac{dy}{dx}+Py=Qy^n$, as the following example will illustrate.

Ex. 5. *Solve the equation* $3\frac{dy}{dx}-y\cot x+2e^xy^4=0$.

This contains y^4 as well as y; to remove the former divide by $-y^4$;

$$-\frac{3}{y^4}\cdot\frac{dy}{dx}+\frac{\cot x}{y^3}=2e^x.$$

Now put $z=\frac{1}{y^3}$, then, by substitution, the equation becomes

$$\frac{dz}{dx}+z\cot x=2e^x.$$

The integrating factor of this equation is

$$e^{\int\cot x\,.\,dx}=e^{\log\sin x}=\sin x.$$

Multiply out by this factor,

$$\sin x \cdot \frac{dz}{dx} + z\cos x = 2e^x \sin x;$$

$$\therefore \frac{d}{dx}(z\sin x) = 2e^x \sin x.$$

Hence, by integration,

$$z \cdot \sin x = 2\int e^x \sin x \cdot dx + A = e^x(\sin x - \cos x) + A;$$

$$\therefore \mathbf{\sin x = y^2\{e^x(\sin x - \cos x) + A\}}.$$

EXERCISES 18B.

In each of the following exercises, the gradient $y_1 \equiv \dfrac{dy}{dx}$ is given for the curve; find the equation of the curve in its simplest form.

1. $xy_1 + y = 0$. **2.** $axy_1 + y = 0$. **3.** $y_1(a+x)^2 = 2a$.

4. $y_1(x^2 - 7x + 12) = 2$. **5.** $y_1(2x^2 - 4x + 7) = 4(x-1)y$.

6. $y_1 = y\cot x$. **7.** $y_1 = e^x(1 + \tan x + \tan^2 x)$.

8. $y_1(x^2 - 4x + 3) = y(2x-3)$. **9.** $y_1(1-x)^2 = y\log x$.

10. $y_1 = 2x\tan^{-1} x$.

11. $xy_1 - y = 2x^2 \operatorname{cosec} 2x$. **12.** $y_1 \sin x - y\cos x = 2e^x \sin^3 x$.

13. $(x^2-1)(x-2)y_1 + 2(x-2)y = (x+1)(x^2+x-3)$.

14. $y_1 + ay - \cos bx = 0$.

15. $(x-2)(x+1)y_1 - 3xy - 3x(x-2)^2(x+1) = 0$.

***16.** $(3 - \cos x)(y_1 + y\operatorname{cosec} x) = 2(1 + \cos x)$.

***17.** $6y_1\cos^2 x - y\sin 2x + 2y^4 \sin^3 x = 0$.

***18.** $(1+x)^2(y_1 - y) + xy^2 = 0$. ***19.** $2(1+x^2)y_1 - y + y^3 = 0$.

***20.** $2x(x+1)(x+2)y_1 - 2(x+1)y + (x+2)^{\frac{3}{2}}y^3 = 0$.

21. A pulley ABC having its centre at O is driven by a belt connecting it with a driving pulley. The belt first makes contact with ABC at A and leaves it at B. The tension T at any point on this arc of contact is given by the equation

$$\frac{dT}{d\theta} = \mu T,$$

where θ is the angular distance of the point from A measured at O, and μ is a constant depending upon friction. If T_1, T_2 are the tensions at B and A respectively, and $\angle BOA = \phi$, shew that

$$T_2 = T_1 e^{\mu\phi}.$$

22. The general equation for the growth of an electric current in a circuit having inductance and resistance only is

$$l\frac{di}{dt}+ri=E,$$

where l is the coefficient of self-inductance, r is the resistance, E the applied E.M.F., all of which quantities may be regarded as constants, whilst i is the current produced in time t. Shew that

$$ir=E\left(1-e^{-\frac{rt}{l}}\right).$$

If, when the current has reached a steady state, the applied E.M.F. is withdrawn, so that the above equation becomes

$$l\cdot\frac{di}{dt}+ri=0,$$

shew that the corresponding decay of the current is given by

$$i=i_0e^{-\frac{rt}{l}},$$

where i_0 is the value of i when $t=0$.

23. Find the equation of the curve for which the gradient is given by the equation

$$(x^2-3x+2)dy=y(2x-3)dx.$$

To determine the constant, it is known that the curve passes through the point (3, 1).

***24.** The displacement x, in time t, of a particle moving along a straight line is given by the equation

$$\left(\frac{dx}{dt}\right)^2=v^2+\mu^2(a^2-x^2),$$

where a, v, μ are constants. If $x=a$, when $t=0$, shew that

$$\mu t=\cos^{-1}\frac{a}{b}-\cos^{-1}\frac{x}{b},$$

where $b=\sqrt{\frac{v^2}{\mu^2}+a^2}$.

***25.** A thick-walled uniform cylindrical pipe of external and internal radii, a, b respectively, is subjected to a constant internal pressure P lb. per sq. inch acting normally to the inside surface. If this produces a pressure p on a layer of material distant x from the axis, where $a>x>b$, the equation giving p in terms of x is

$$\frac{dp}{dx}+\frac{2(p-c)}{x}=0,$$

where $c(b^2-a^2)=b^2P$, assuming there to be no external pressure. Find the precise relation between p and x, and shew that it may be written in the form

$$\frac{a^2-b^2}{b^2}\cdot p=\frac{a^2-x^2}{x^2}\cdot P.$$

***26.** The gradient of a curve which passes through the point $(-1, 0)$ is defined by the equation

$$\frac{dy}{dx} - \frac{y}{x} - \frac{5x}{(2+x)(3-x)} = 0.$$

Find the complete equation of the curve, and from it find the value of y when $x=2$.

***27.** The extension z of a long uniformly tapering tie-rod, fixed vertically at one end and carrying a load of w tons at the other, is given by the equation

$$AE\frac{dz}{dx} = w,$$

where E is the modulus of elasticity and A the area of cross-section at a depth x feet from the fixed end. If the radii of the circular cross-sections at the fixed and loaded ends are a and b inches respectively, a being greater than b, shew that

$$A = \pi\left\{a - \frac{a-b}{l} \cdot x\right\}^2,$$

where l is the length of the rod in feet.

Hence, prove that

$$\pi a\left(a - \frac{a-b}{l} \cdot x\right)Ez = wx,$$

and calculate z when $x=l=30$, $a=3$, $b=1$, $w=13$ and $E=1{\cdot}3\times 10^4$.

***28.** A heavy uniform chain weighing w lb. per foot is lying on a rough vertical circle of radius r. The chain is just on the point of motion, and its tension T at any point where the slope of the circle is θ, is given by the equation

$$\frac{dT}{d\theta} - \mu T = wr(\sin\theta - \mu\cos\theta),$$

μ being the constant of friction. If $T=0$ where $\theta=\pi/2$, find the complete expression for the tension, and from it, calculate T, when $\theta=0{\cdot}75\pi$, $w=5$, $r=8{\cdot}32$ and $\mu=0{\cdot}2$.

***29.** A particle is moving on an ellipse whose major axis is $2a$, and eccentricity e. The time which it takes to reach a point on the ellipse where its distance from the focus is r, is given by the equation

$$a\omega \cdot \frac{dt}{dr} = \frac{r}{\sqrt{a^2e^2 - (a-r)^2}},$$

where ω is a constant. Change the variable r into ϕ by the substitution $r=a(1-e\cos\phi)$, and assuming that $t=0$ when $r=a(1-e)$, obtain, by integration, Kepler's equation

$$\omega t = \phi - e\sin\phi.$$

***30.** Solve completely the equation

$$\frac{dy}{dx} + 2y = e^{-2x}(1 - x^2) \sin^{-1}x,$$

it being given that x and y are zero simultaneously.

***31.** Solve completely the equation

$$\frac{dy}{dx} + y \tan x + \sin x \cos^4 x = 0,$$

having given that $y = 0$ when $x = 0$.

32. An important equation in the theory of the stability of an aeroplane is

$$\frac{dv}{dt} = g \cos \alpha - kv,$$

where g, α, k are constants. Solve the equation completely having given that $v = 0$ when $t = 0$.

33. Solve $x(x-2)\dfrac{dy}{dx} - 2(x-1)y - x^3(x-2) = 0$, given that $x = 3$ when $y = 9$.

34. The equation giving the current y in a conductor at any time t is of the form

$$a\frac{dy}{dt} + by = c \sin pt.$$

Find the general value of y and determine the constant of integration if $y = 0$ when $t = 0$. (L.U.)

35. Solve $$a\frac{dy}{dt} + by = ce^{-bt/a}.$$ (L.U.)

***36.** If $v \cdot \dfrac{dv}{dx} = \dfrac{dv}{dt} = g\left(1 - \dfrac{v^2}{V^2}\right)$, and when $t = 0$, $x = 0$, $v = 0$, prove that

$$e^{gx/V^2} = \cosh (gt/V).$$

A particle falls freely from rest in a slightly resisting medium in which the resistance varies as the square of the velocity. Shew that so long as the velocity of the particle is small compared with the terminal velocity, the distance fallen in any time is approximately $x - x^2/6h$, where x is the distance through which it would have fallen freely in the time and h is the distance through which it would have to fall freely to acquire the terminal velocity. (L.U.)

37. The equation giving the horizontal oscillation of a compass needle is of the form

$$\frac{d\omega^2}{d\theta} = -a \sin \theta - b\omega^2.$$

Find ω in any position if it is zero when $\theta = \pi/3$. (L.U.)

38. Solve the equation

$$\frac{dx}{dt}+3x=e^{-2t}+t^2. \qquad \text{(L.U.)}$$

***39.** Solve $x(x^2+y^2)\dfrac{dy}{dx}=2y^3$. (Br.U.)

40. Solve $\sin x \,.\, \dfrac{dy}{dx} - y\cos x = \sin^2 x \cos^2 x$. (Li.U.)

135. Linear Differential Equations with Constant Coefficients of a Higher Order than the First. The general linear equation of the nth order, with constant coefficients, may be written in the form

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_n y=X,$$

where $a_1, a_2, a_3, \ldots, a_n$ are constants, and X is a function of x.

In this section only equations where $X=0$ will be considered.

Ex. 6. *Solve the equation*

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+b^2y=0,$$

where a and b are constants, and work out the solutions in suitable form in each of the following cases : (i) *when* $a=2{\cdot}9$ *and* $b=2{\cdot}1$, (ii) *when* $a=3{\cdot}5$ *and* $b=3{\cdot}7$, (iii) *when* $a=b$.

Assume as a trial solution, $y=e^{\alpha x}$, where α is some constant which remains to be determined.

Substituting in the given equation

$$\alpha^2e^{\alpha x}+2a\alpha e^{\alpha x}+b^2e^{\alpha x}=0,$$

or, dividing out by $e^{\alpha x}$, which is not zero for finite values of x,

$$\alpha^2+2a\alpha+b^2=0.$$

This is called the **Auxiliary Equation**, and is of the same degree as the order of the given differential equation. The order, therefore, indicates the number of roots of the auxiliary equation, in this case, two.

Solving the above auxiliary equation, and denoting its roots by α_1 and α_2 :

$$\alpha_1=-a+\sqrt{a^2-b^2}, \quad \text{and} \quad \alpha_2=-a-\sqrt{a^2-b^2}.$$

Hence two particular solutions of the given differential equation are $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$; these are particular solutions because they

involve no arbitrary constants. Since, however, the given differential equation is of the second order, the complete solution must involve two arbitrary constants, so that the general solution may be written,

$$y = Ae^{a_1 x} + Be^{a_2 x},$$

or, putting in the values of a_1 and a_2,

$$y = (Ae^{x\sqrt{a^2-b^2}} + Be^{-x\sqrt{a^2-b^2}})\,.\,e^{-ax}.$$

This may assume several forms, according as $a >, =$ or $< b$, as the following will illustrate.

(i) When $a = 2{\cdot}9$ and $b = 2{\cdot}1$, $\sqrt{a^2 - b^2} = 2$;

$$\therefore\; y = (Ae^{2x} + Be^{-2x})e^{-2{\cdot}9x}.$$

But since $\qquad e^{2x} = \cosh 2x + \sinh 2x,$

and $\qquad e^{-2x} = \cosh 2x - \sinh 2x,$

by substituting these values and writing L for $A + B$ and M for $A - B$, the solution may also be written

$$y = (L \cosh 2x + M \sinh 2x)e^{-2{\cdot}9x}.$$

(ii) When $a = 3{\cdot}5$ and $b = 3{\cdot}7$,

$$\sqrt{a^2 - b^2} = \sqrt{-1{\cdot}44} = 1{\cdot}2\sqrt{-1} = 1{\cdot}2i, \text{ where } i = \sqrt{-1}\,;$$

$$\therefore\; y = (Ae^{1{\cdot}2ix} + Be^{-1{\cdot}2ix})e^{-3{\cdot}5x}.$$

But $\qquad e^{1{\cdot}2ix} = \cos 1{\cdot}2x + i \sin 1{\cdot}2x,$

and $\qquad e^{-1{\cdot}2ix} = \cos 1{\cdot}2x - i \sin 1{\cdot}2x.$

Hence, by substitution, and writing L for $A + B$ and M for $i(A - B)$, the solution becomes

$$y = (L \cos 1{\cdot}2x + M \sin 1{\cdot}2x)e^{-3{\cdot}5x}.$$

(iii) Finally, when $a = b$, the solution becomes

$$y = (A + B)e^{-ax} \quad\text{or}\quad y = Ce^{-ax},$$

where $C = A + B$. There is thus only one arbitrary constant, instead of two, and the solution is therefore incomplete.

To determine the second part, let $y = ze^{-ax}$, where z is a function of x to be determined; then

$$\frac{dy}{dx} = e^{-ax}\left(\frac{dz}{dx} - az\right) \quad\text{and}\quad \frac{d^2y}{dx^2} = e^{-ax}\left(\frac{d^2z}{dx^2} - 2a\frac{dz}{dx} + a^2z\right).$$

Substituting in the given equation,

$$\frac{d^2z}{dx^2}=0.$$

Integrating $$\frac{dz}{dx}=E, \text{ an arbitrary constant,}$$

or $$dz=E\,.\,dx.$$

Integrating again, $$z=Ex+F\,;$$

$$\therefore\ y=(\mathbf{Ex}+\mathbf{F})\mathbf{e}^{-\mathbf{ax}}.$$

This is the general solution, because it contains two independent arbitrary constants.

Summing up the results, the solution of the equation

$$\mathbf{\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+b^2y=0}$$

is (a) $\mathbf{y=(L\cosh x\sqrt{a^2-b^2}+M\sinh x\sqrt{a^2-b^2})e^{-ax}}$, **when** $\mathbf{a>b}$, ...(141)

(b) $\mathbf{y=(L\cos x\sqrt{b^2-a^2}+M\sin x\sqrt{b^2-a^2})e^{-ax}}$, **when** $\mathbf{a<b}$, **or**

(c) $\mathbf{y=(Ex+F)e^{-ax}}$, **when** $\mathbf{a=b}$.

The conditions giving rise to these three forms of the solution may be expressed as follows in terms of the roots of the auxiliary equation.

If $\mathbf{f(\alpha)=0}$ be the auxiliary equation, the solution of the given differential equation is periodic when the roots of $\mathbf{f(\alpha)=0}$ are unreal, and non-periodic when they are real.

Ex. 7. *If* $a\frac{d\theta}{dt}+2b\theta+u=0$, *and* $\theta=\frac{dx}{dt}$ *and* $x=ku$, *find the differential equation connecting* x, t *and the constants* a, b, *and* k. *Express the relation between* a, b, *and* k *so that if* x *denote a displacement and* t *the time, the motion may be just non-oscillatory. On this supposition shew that*

$$x=e^{-bt/a}(x_0+bx_0t/a+\theta_0t),$$

where x_0 *and* θ_0 *are the values of* x *and* θ *when* $t=0$. (L.U.)

Since $\theta=\frac{dx}{dt}$, $\therefore\ \frac{d\theta}{dt}=\frac{d^2x}{dt^2}$, so that the differential equation becomes

$$a\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\frac{x}{k}=0.$$

The auxiliary equation is thus

$$ak\alpha^2+2bk\alpha+1=0,$$

the roots of which are

$$\alpha = \{-bk \pm \sqrt{b^2k^2 - ak}\}/ak.$$

For the solution to be just non-oscillatory, *i.e.* non-periodic, $b^2k^2 - ak$ must be zero, *i.e.*

$$\mathbf{b^2k = a.}$$

With this condition $\alpha = -b/a$, so that the roots are equal; hence, by (141), the solution of the differential equation is

$$x = e^{-bt/a}(Et + F).$$

To determine the constants E and F, put $x = x_0$ and $t = 0$, then

$$F = x_0.$$

Also, by differentiation,

$$\theta = \frac{dx}{dt} = e^{-bt/a}\{E(1 - bt/a) - bF/a\}.$$

Putting $\theta = \theta_0$, $t = 0$, $F = x_0$,

$$E = \theta_0 + bx_0/a,$$

so that the solution becomes finally,

$$x = e^{-bt/a}\{x_0 + (\theta_0 + bx_0/a)t\},$$

which agrees with that given.

Ex. 8. *Find the complete solution of the equation*

$$5\frac{d^2y}{dx^2} + 2 + 5y = 0,$$

having given that $\frac{dy}{dx} = 0$ *when* $y = 1{\cdot}4$, *and* $x = \pi/6$ *when* $y = 0{\cdot}5$.

This equation is characterised by the absence of a function of x other than the differential coefficient. When this is the case the equation can be integrated directly, as the following solution will shew.

Multiply throughout by $2 \, . \, \frac{dy}{dx}$, then

$$10 \cdot \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + 2(2 + 5y)\frac{dy}{dx} = 0,$$

which, on integration, gives

$$5\left(\frac{dy}{dx}\right)^2 + 4y + 5y^2 = A.$$

The given conditions serve to determine the arbitrary constants; thus, since $\frac{dy}{dx}=0$ when $y=1\cdot4$,

$$A=5\cdot6+9\cdot8=15\cdot4;$$

$$\therefore\ 5\,.\left(\frac{dy}{dx}\right)^2=15\cdot4-4y-5y^2;$$

$$\therefore\ \left(\frac{dy}{dx}\right)^2=3\cdot08-0\cdot8y-y^2$$

$$=1\cdot8^2-(0\cdot4+y)^2;$$

$$\therefore\ dx=\frac{dy}{\sqrt{1\cdot8^2-(0\cdot4+y)^2}}.$$

Hence, by integration,

$$x=\sin^{-1}\frac{0\cdot4+y}{1\cdot8}+B=\sin^{-1}\frac{2+5y}{9}+B.$$

Since $x=\frac{\pi}{6}$ when $y=\frac{1}{2}$, $B=0$;

$$\therefore\ x=\sin^{-1}\frac{2+5y}{9},$$

or

$$5y+2=9\sin x.$$

Exercises 18c.

The constants of integration should be determined where the necessary conditions are given.

Solve the following equations:

1. $\frac{d^2x}{dt^2}+\omega^2x=0$, given that $\frac{dx}{dt}=0$ and $x=a$ when $t=0$.

2. $\frac{d^2s}{dt^2}=a$, given that $\frac{ds}{dt}=u$ and $s=0$, when $t=0$.

3. $\frac{d^2y}{dx^2}-y=2$, given that $\frac{dy}{dx}=3$ when $y=1$, and $x=2$ when $y=-1$.

4. $\frac{d^2y}{dx^2}+n^2y=0$.

5. $\frac{d^2u}{dt^2}+u=3$, given that $\frac{du}{dt}=0$ and $t=1$ when $u=-1$.

6. $2\cdot\frac{d^2s}{dt^2}=\frac{a^2}{s^3}$, given that $\frac{ds}{dt}=0$ and $t=0$ when $s=a$.

7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0.$ 8. $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 10y = 0.$

9. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 0.$

10. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 29y = 0$, given that $\dfrac{dy}{dx} = 15$ and $y = 0$ when $x = 0$.

11. $4\dfrac{d^2x}{dt^2} - 20\dfrac{dx}{dt} + 25x = 0.$ 12. $5\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 3y = 0.$

13. $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 9y = 0.$

*14. $4(1+z^2)\dfrac{d^2z}{dx^2} - 8z \cdot \left(\dfrac{dz}{dx}\right)^2 - 8(1+z^2)\dfrac{dz}{dx} + 3(1+z^2)^2 \tan^{-1} z = 0,$ by putting $z = \tan y$.

Solve the following equations by changing x into t by the substitution $x = e^t$:

15. $10x^2\dfrac{d^2y}{dx^2} - 13x\dfrac{dy}{dx} + 12y = 0.$ 16. $2x^2\dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 3y = 0.$

17. $4x^2\dfrac{d^2y}{dx^2} - 8x\dfrac{dy}{dx} + 9y = 0.$ *18. $2x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 0.$

*19. $3x^2\dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 5y = 0.$

20. If $c\dfrac{d^2y}{dx^2} = M$, $\dfrac{dM}{dx} = F$, $\dfrac{dF}{dx} = w$, where w and c are constants, find the precise relation between x, y, c and w, having given that both M and y vanish at each of the points where $x = 0$ and $x = l$.

*21. A uniform vertical strut OA, whose length is l, is loaded at its upper extremity A with a weight w, with the result that the strut is deflected from the vertical between its extremities. If A remains vertically above O, and at a height x from O, the deflection of the strut from OA is y, measured horizontally, then x and y are connected by the equation

$$\frac{d^2y}{dx^2} + cwy = 0,$$

where c is a constant depending upon the cross-section and the material of the strut. Solve the equation, having given that d is the maximum deflection, *i.e.* the value of y when $\dfrac{dy}{dx} = 0$.

Assuming that the maximum deflection takes place at the middle of the strut, shew that

$$cl^2w = \pi^2,$$

nearly.

***22.** If the upper end A of the strut in the previous question be free to move laterally, whilst O remains fixed in both direction and position, and that under a load w, A moves to a position h horizontally from the vertical through O, then x and y are connected by the equation

$$\frac{d^2y}{dx^2}=cw(h-y).$$

Solve this equation completely, and shew that the solution may be expressed in the form

$$y=2h\sin^2(\tfrac{1}{2}x\sqrt{cw}).$$

Find also the value of w producing the maximum deflection.

***23.** In the case of a uniformly loaded beam supported at each end, it is found that, at a distance x from its middle point, the deflection y from the horizontal is given by

$$c\frac{d^2y}{dx^2}=\tfrac{1}{2}w(x^2-\tfrac{1}{4}l^2),$$

where l=length, w=load per unit length of beam, and c is a constant depending upon its section and material.

Find the actual relation between x and y.

Assuming that y is a maximum when $x=0$, and that its value there is h, prove that $5l^4w=384ch$.

24. The vertical deflection y at a horizontal distance x from A of a loaded horizontal cantilever AB, having the end A fixed, is given by the general equation

$$c\frac{d^2y}{dx^2}=\alpha(l-x)^n,$$

where c is a constant, l is the length from A to B, and α, n are constants depending upon the distribution of the load.

Determine the precise relation between x and y in each of the following cases:

(i) A load W is placed at B, in which case $\alpha=W$, $n=1$.

(ii) A load is distributed uniformly along AB, in which case $2\alpha=w$ =load per unit length, and $n=2$.

25. The deflection of a loaded beam AB of length l is y, measured vertically downwards from the horizontal line AB, at a distance x from A along AB. The general equation connecting x and y is then

$$c\frac{d^2y}{dx^2}=\alpha x(ax-b)+\beta,$$

when the ends A, B are fixed. c is a constant depending upon the section and material of the beam, and α, β, a, b are constants depending upon the distribution of the load.

Assuming the greatest deflection to be at the middle point, except

in (iii), determine the precise relation between x and y in each of the following cases:

(i) A load W is placed mid-way between A and B, in which case

$$2a=W, \quad \beta=\tfrac{1}{8}Wl, \quad a=0, \quad b=1.$$

(ii) The load is uniformly distributed along AB, in which case $2a=w=$load per unit length, $\beta=\frac{1}{12}wl^2$, $a=1$, $b=l$.

(iii) The load W is placed at C, where $AC=s$, s being greater than $\frac{1}{2}l$, in which case $a=W$, $a=0$, $\beta=0$, $b=\dfrac{l-s}{l}$, and it is known that when $x=s$, then

$$\frac{dy}{dx}=\frac{Ws(l-s)(l-2s)}{3cl}.$$

Determine the deflection at C.

***26.** The distance s passed through in time t by a body falling from rest under gravity in a medium of constant density, satisfies the equation

$$\frac{d^2s}{dt^2}+\mu^2\left(\frac{ds}{dt}\right)^2=c^2,$$

where μ and c are constants. Find, by integration, the equation connecting s and t.

If v be the velocity of the body in time t, shew by means of the identity

$$\frac{d^2s}{dt^2}=\frac{dv}{dt}=\frac{dv}{ds}\cdot\frac{ds}{dt}=v\cdot\frac{dv}{ds},$$

that
$$2\mu^2s+\log(c^2-\mu^2v^2)=0.$$

If $\mu=0{\cdot}03$ and $c=5{\cdot}66$, find how long it will take for the body to fall 982 feet.

27. Solve the equation $\dfrac{d^4y}{dx^4}=c$, with the conditions (i) $\dfrac{d^3y}{dx^3}=a$ when $x=0$, (ii) $\dfrac{d^2y}{dx^2}=0$ when $x=0$, (iii) $\dfrac{dy}{dx}=0$ when $x=l$, and (iv) $y=0$ when $x=l$. (B.U.)

28. Shew that the solution of

$$\frac{d^2y}{dx^2}-2a\cdot\frac{dy}{dx}+(a^2+b^2)y=0,$$

where a and b are constants, is given by

$$y=Ae^{ax}\cos bx+Be^{ax}\sin bx,$$

where A and B are arbitrary constants. (M.U.)

29. Shew that the solution of

$$10\frac{d^2r}{d\theta^2}+7\frac{dr}{d\theta}-12r=0,$$

given that $r=23$ and $\dfrac{dr}{d\theta}=0$, when $\theta=0$, is $re^{1{\cdot}5\theta}=15e^{2{\cdot}3\theta}+8$.

30. A curve passing through the point (0, 1·5), is given by the differential equation

$$2 \cdot \frac{d^2y}{dx^2} = 3 - 2y.$$

Its gradient is zero when $y = 4$, find the equation of the curve, and find the value of x when $y = 2·75$.

***31.** A body oscillating in a medium of constant density has its motion retarded during each swing, and the displacement x in time t satisfies the equation

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = 0,$$

provided $f < n$, f and n being constants. Having given that $\frac{dx}{dt} = 0$ and $x = a$ when $t = 0$, find the exact equation connecting x and t, and shew that it may be written in the form

$$\mu x = ane^{-ft} \sin(\mu t + \epsilon),$$

where $\mu^2 = n^2 - f^2$ and $\tan \epsilon = \mu/f$.

***32.** If the constants f and n in the equation

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = 0$$

are such that $f > n$, what form does the solution take? Is the motion oscillatory in this case?

Examine also the case in which $f = n$.

33. Obtain the general solution of the differential equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0,$$

distinguishing between the cases when the roots of the auxiliary quadratic are (1) real and unequal, (2) imaginary, (3) equal.

A condenser of capacity C is discharged through a circuit of resistance R and inductance L. Prove that the charge Q at any time t is given by

$$L \cdot \frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0;$$

hence shew that if R is sufficiently small, the discharge is oscillatory, and determine the period of oscillation. Calculate the frequency if the capacity is 0·02 microfarad, the inductance 0·0003 henry and the resistance negligible. (L.U.)

136. Solution of the Linear Differential Equation with Constant Coefficients by Operators. Let D denote the **operator** $\frac{d}{dx}$, so that

$$Dy \equiv \frac{dy}{dx}, \quad D^2y \equiv \frac{d^2y}{dx^2},$$

and so on ; and let $f(D)$ denote the operator

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n,$$

where $a_1, a_2, a_3, \dots a_n$ are constants, then the linear equation of the nth order may be briefly written

$$f(D)y = X,$$

where X is a function of x.

Let u be a particular solution of this equation, so that

$$f(D)u = X\,;$$

then, if the complete solution be $y = u + v$, by substitution,

$$f(D)u + f(D)v = X\,;$$

hence
$$f(D)v = 0.$$

v may therefore be found by the method illustrated in Ex. 1 of § 135 (page 440).

The solution of the equation

$$f(D)y = X$$

therefore consists of two parts :

(i) The complete solution v, of $f(D)y = 0$, containing n arbitrary constants, and called the **complementary function** (C.F.)

(ii) A particular solution u, of $f(D)y = X$, which, since it contains no arbitrary constants, is called the **Particular Integral** (P.I.).

Since
$$f(D)u = X, \quad \therefore\ u = \frac{X}{f(D)},$$

and the evaluation of u may be effected by the following important results of operation :

$$\left.\begin{array}{l}
\text{(a)}\ \dfrac{1}{\mathbf{D}}\cdot\mathbf{X} = \mathbf{D}^{-1}\cdot\mathbf{X} = \displaystyle\int \mathbf{X}\,d\mathbf{x}, \\[2ex]
\text{(b)}\ \mathbf{f}(\mathbf{D})\cdot\mathbf{e}^{\mathbf{ax}} = \mathbf{f}(\mathbf{a})\cdot\mathbf{e}^{\mathbf{ax}},\quad \dfrac{1}{\mathbf{f}(\mathbf{D})}\cdot\mathbf{e}^{\mathbf{ax}} = \dfrac{\mathbf{e}^{\mathbf{ax}}}{\mathbf{f}(\mathbf{a})}, \\[2ex]
\text{(c)}\ \mathbf{f}(\mathbf{D})\mathbf{e}^{\mathbf{ax}}\cdot\mathbf{X} = \mathbf{e}^{\mathbf{ax}}\cdot\mathbf{f}(\mathbf{D}+\mathbf{a})\cdot\mathbf{X}, \\[1ex]
\quad \dfrac{1}{\mathbf{f}(\mathbf{D})}\cdot\mathbf{e}^{\mathbf{ax}}\cdot\mathbf{X} = \mathbf{e}^{\mathbf{ax}}\cdot\dfrac{1}{\mathbf{f}(\mathbf{D}+\mathbf{a})}\cdot\mathbf{X}, \\[2ex]
\text{(d)}\ \mathbf{f}(\mathbf{D}^2)\dfrac{\sin \mathbf{bx}}{\cos \mathbf{bx}} = \mathbf{f}(-\mathbf{b}^2)\dfrac{\sin \mathbf{bx}}{\cos \mathbf{bx}}, \\[2ex]
\quad \dfrac{1}{\mathbf{f}(\mathbf{D}^2)}\cdot\dfrac{\sin \mathbf{bx}}{\cos \mathbf{bx}} = \dfrac{1}{\mathbf{f}(-\mathbf{b}^2)}\cdot\dfrac{\sin \mathbf{bx}}{\cos \mathbf{bx}}.
\end{array}\right\}\ \dots\dots\dots\dots\ \textbf{(142)}$$

If, after applying these results, an operation $\frac{X}{F(D)}$ remains, this may readily be evaluated when $F(D)$ is capable of valid expansion; thus, assuming this to be the case,

$$\frac{1}{\mathbf{F(D)}}\cdot\mathbf{X}=\{\mathbf{F(D)}\}^{-1}\mathbf{X}=(1+c_1\mathbf{D}+c_2\mathbf{D}^2+\ldots)\cdot\mathbf{X}.\ \ldots\ldots\ldots(142e)$$

Ex. 9. *Find u when* $(D^2-3D+4)u=e^{3x}$.

Here $u=\frac{1}{D^2-3D+4}\cdot e^{3x}=\frac{e^{3x}}{3^2-3\cdot3+4}=\frac{1}{4}e^{3x}$ by (b).

Ex. 10. *Find y when* $(D^2-3D+3)y=x^3e^{2x}$.

Here, applying (142c),

$$y=\frac{1}{D^2-3D+3}\cdot e^{2x}x^3=e^{2x}\frac{1}{(D+2)^2-3(D+2)+3}\cdot x^3$$

$$=e^{2x}\frac{1}{D^2+D+1}\cdot x^3=e^{2x}(1+D+D^2)^{-1}x^3$$

$$=e^{2x}\{1-(D+D^2)+(D+D^2)^2-(D+D^2)^3+\ldots\}x^3, \text{ by } (142e),$$

$$=e^{2x}(1-D+D^3-D^4+\ldots)x^3$$

$$=\mathbf{e^{2x}(x^3-3x^2+6)}.$$

Note that in this case there is no need to expand the operator beyond D^3, since $D^4(x^3)=0$.

Ex. 11. *Perform the operation*

$$\frac{1}{D^4+D^3+D^2-D-2}\cdot \sin 2x.$$

By (142d), the result of the operation is

$$\frac{1}{(-4)^2+D^3-4-D-2}\sin 2x=\frac{1}{10+D(D^2-1)}\sin 2x$$

$$=\frac{1}{10+D(-4-1)}\sin 2x=\frac{1}{5(2-D)}\sin 2x.$$

To render the denominator a function of D^2, multiply numerator and denominator by $2+D$, then the operation becomes

$$\frac{2+D}{5(4-D^2)}\sin 2x=\frac{2+D}{5(4+4)}\sin 2x=\frac{1}{40}(2+D)\sin 2x$$

$$=\frac{1}{20}(\mathbf{\sin 2x+\cos 2x}).$$

Ex. 12. *Evaluate the integral*

$$I=\int e^{4x}\cos 3x\,.\,dx$$

by operators.

By integrating by parts, we obtain the standard result

$$I=\int e^{4x}\cos 3x\,.\,dx=\tfrac{1}{25}e^{4x}(4\cos 3x+3\sin 3x).$$

By operators, we have

$$I=\frac{1}{D}\,.\,e^{4x}\cos 3x,\text{ by }(a),=e^{4x}\,\frac{1}{D+4}\,.\cos 3x,\text{ by }(142c),$$

$$=e^{4x}\,\frac{4-D}{16-D^2}\,.\cos 3x=e^{4x}\,\frac{4-D}{16-(-9)}\,.\cos 3x,\text{ by }(142d),$$

$$=\tfrac{1}{25}\,.\,e^{4x}(4-D)\cos 3x$$

$$=\tfrac{1}{25}\,.\,e^{4x}(4\cos 3x+3\sin 3x),\text{ as before.}$$

137. Solution of Equations by Operators. The particular integral of the equation $f(D)y=X$ may readily be determined in many cases by the application of (142), as the following examples will shew.

Ex. 13. *Solve the equation*

$$2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+3y=17e^x\sin 2x.$$

By (141), the complementary function is $Ae^{1\cdot 5x}+Be^x$.
Writing the equation in the form

$$(2D^2-5D+3)y=17e^x\sin 2x,$$

the particular integral is

$$y=\frac{17}{2D^2-5D+3}\,.\,e^x\sin 2x=e^x\,.\,\frac{17}{2(D+1)^2-5(D+1)+3}\,.\sin 2x$$

$$=e^x\frac{17}{2D^2-D}\,.\sin 2x=-e^x\frac{17}{8+D}\,.\sin 2x=-e^x\frac{17(8-D)}{64-D^2}\,.\sin 2x$$

$$=-e^x\frac{17(8-D)}{68}\,.\sin 2x=-\tfrac{1}{4}e^x(8\sin 2x-2\cos 2x)$$

$$=\tfrac{1}{2}e^x(\cos 2x-4\sin 2x).$$

Hence, the complete solution is

$$2y = \mathbf{E}\mathbf{e}^{1\cdot 5\mathbf{x}} + \mathbf{e}^{\mathbf{x}}(\mathbf{F} + \cos 2\mathbf{x} - 4 \sin 2\mathbf{x}),$$

writing $2A = E$ and $2B = F$.

Ex. 14. *Solve the equation*

$$t^3 \frac{d^3y}{dt^3} + 2t^2 \frac{d^2y}{dt^2} - 9t \frac{dy}{dt} + 9y = 24t \,(\log t)^2$$

by putting $t = e^x$.

Since $t = e^x$, $\quad \dfrac{dt}{dx} = e^x = t \quad$ and $\quad \dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = e^{-x}\dfrac{dy}{dx} = e^{-x}Dy$;

$$\therefore\ t\frac{dy}{dt} = Dy,$$

$$\frac{d^2y}{dt^2} = \frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = e^{-2x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) = e^{-2x}D(D-1)y\,;$$

$$\therefore\ t^2\frac{d^2y}{dt^2} = D(D-1)y.$$

Similarly,

$$t^3 \cdot \frac{d^3y}{dt^3} = D(D-1)(D-2)y.$$

Substituting in the given equation,

$$\{D(D-1)(D-2) + 2D(D-1) - 9D + 9\}y = 24e^x x^2,$$

or

$$(D-1)(D-3)(D+3)y = 24e^x x^2.$$

Hence, the C.F. is $\mathbf{A}'\mathbf{e}^{\mathbf{x}} + \mathbf{B}\mathbf{e}^{3\mathbf{x}} + \mathbf{C}\mathbf{e}^{-3\mathbf{x}}$,

and the P.I. is given by

$$y = \frac{24}{(D-1)(D-3)(D+3)} \cdot e^x x^2 = e^x \cdot \frac{24}{D(D-2)(D+4)} x^2, \text{ by (142}c\text{)},$$

$$= -e^x\left\{\frac{3}{D} + \frac{2}{2-D} - \frac{1}{4+D}\right\}x^2$$

$$= -e^x\{3D^{-1} + (1 - \tfrac{1}{2}D)^{-1} - \tfrac{1}{4}(1 + \tfrac{1}{4}D)^{-1}\}x^2$$

$$= -e^x\{3D^{-1} + 1 + \tfrac{1}{2}D + \tfrac{1}{4}D^2 + \ldots - \tfrac{1}{4} + \tfrac{1}{16}D - \tfrac{1}{64}D^2 + \ldots\}x^2$$

$$= -e^x\{3D^{-1} + \tfrac{3}{4} + \tfrac{9}{16}D + \tfrac{15}{64}D^2 + \ldots\}x^2$$

$$= -e^x(x^3 + \tfrac{3}{4}x^2 + \tfrac{9}{8}x + \tfrac{15}{32}).$$

Hence the complete solution of the equation is

$$y = \mathbf{e}^{\mathbf{x}}(\mathbf{A} - \mathbf{x}^3 - \tfrac{3}{4}\mathbf{x}^2 - \tfrac{9}{8}\mathbf{x} + \mathbf{B}\mathbf{e}^{2\mathbf{x}} + \mathbf{C}\mathbf{e}^{-2\mathbf{x}}),$$

where A is written for $A' - \frac{15}{32}$.

Ex. 15. *Find the particular integrals of* (*a*) $(D-3)y=e^{3x}$, (*b*) $(D^2+4)y=2\sin 2x$. *Hence solve* (*b*) *completely.*

(*a*) Applying (142*b*),

$$y=\frac{e^{3x}}{D-3}=\frac{e^{3x}}{3-3}=\frac{e^{3x}}{0}=\infty.$$

This difficulty may be overcome by considering unity as a function multiplying e^{3x}; then applying (142*c*),

$$y=e^{3x}\frac{1}{D+3-3}=e^{3x}\frac{1}{D}=\mathbf{e^{3x}\,.\,x}.$$

(*b*) Applying (46*d*) to this equation,

$$y=\frac{2}{-4+4}\,.\,\sin 2x=\infty.$$

This difficulty may be overcome by writing the exponential expression for $\sin 2x$ and proceeding as in (*a*): thus,

$$y=\frac{1}{D^2+4}\cdot\frac{1}{i}(e^{2ix}-e^{-2ix}),\text{ where } i=\sqrt{-1}$$

$$=\frac{e^{2ix}}{i}\,\frac{1}{(D+2i)^2+4}-\frac{e^{-2ix}}{i}\,\frac{1}{(D-2i)^2+4},\text{ by (142}c\text{)},$$

$$=\frac{e^{2ix}}{i}\cdot\frac{1}{D^2+4iD}-\frac{e^{-2ix}}{i}\,\frac{1}{D^2-4iD}$$

$$=-\frac{e^{2ix}}{4}\,\frac{1}{D}\left(1+\frac{D}{4i}\right)^{-1}-\frac{e^{-2ix}}{4}\cdot\frac{1}{D}\left(1-\frac{D}{4i}\right)^{-1}$$

$$=-\frac{e^{2ix}}{4}\left\{\frac{1}{D}-\frac{1}{4i}-\frac{D}{16}+\dots\right\}-\frac{e^{-2ix}}{4}\left\{\frac{1}{D}+\frac{1}{4i}-\frac{D}{16}+\dots\right\}$$

$$=-\frac{e^{2ix}}{4}\left(x-\frac{1}{4i}\right)-\frac{e^{-2ix}}{4}\left(x+\frac{1}{4i}\right)$$

$$=-\tfrac{1}{4}x(e^{2ix}+e^{-2ix})+\frac{1}{16i}(e^{2ix}-e^{-2ix})$$

$$=-\tfrac{1}{2}\mathbf{x}\cos 2\mathbf{x}+\tfrac{1}{8}\sin 2\mathbf{x}.$$

It is readily seen that the complementary function of the equation is

$$A'\sin 2x+B\cos 2x;$$

hence, writing A for $A'+\frac{1}{8}$, the complete solution is

$$y=\mathbf{A}\sin 2\mathbf{x}+(\mathbf{B}-\tfrac{1}{2}\mathbf{x})\cos 2\mathbf{x}.$$

Exercises 18d.

Establish by means of the operator $\frac{1}{D}$, the following results :

1. $\int e^{ax} \sin bx \, . \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$.

2. $\int x \cosh x \, . \, dx = x \sinh x - \cosh x$.

3. $128\int e^{4x} x^3 \, . \, dx = (32x^3 - 24x^2 + 12x - 3)e^{4x}$.

4. $9\int z^2 \log z \, . \, dz = z^3(3 \log z - 1)$, by putting $z = e^t$.

Find the particular integrals of the following equations :

5. $(D^3 - 6D^2 + 11D - 6)y = 2 \sinh 5x$.
6. $(6D^2 - 5D + 1)y = 3x^2 + 2x + 1$.
7. $(D^3 - 3D^2 + 5D - 45)y = 26 \cos 5x$.
8. $(D^2 - 5D + 4)y = e^{4x} \sin x$
9. $(D^2 - 1)y = 2x \cosh x$.
10. $(D^2 - 5D + 6)y = e^{3x}$.
11. $(D^2 - 4D + 4)y = 2e^{2x}$.
*12. $(D^4 + 7D^2 + 6)y = 5 \cos x$.

Solve completely each of the following equations :

13. $(D^2 - 7D + 12)y = 3x^2 e^{3x}$.
*14. $(D^2 + 12D + 11)y = 2 \sin 2x$.
*15. $(D^2 + D + 5)y = 17 \sin 5x$.
*16. $(D^2 - 5D - 6)y = 3x + 5 \sin x$.
17. $(D^2 + 4D + 6)y = 5e^{-2x}$.
18. $(D^3 - D^2 - 12D)y = 5e^{2x}$.
19. $(D^2 + 4D + 4)y = ae^{-3x}$.
*20. $(D^2 - 4D + 4)y = 2 \sin 3x$.
21. $(D^2 + 2D - 3)y = 3x^2 + 4x$.
22. $D(D^2 + 6D + 13)y = x^3 + a^3 e^{2x}$.
*23. $(D^2 + 6D - 72)y = 3e^{2x} + 4 \sin^2 x$.
24. $(D^2 + 2D + 1)y = 2e^x + 5x^2$.
*25. $(D^3 - D^2 - 6D)y = x^2 + a^2 \sin x$.
*26. $(D^2 + 1)y = 4 \sin x$.
27. $(D^2 + 6D - 7)y = e^x + 4x^2$.
*28. $(D^2 - 2mD + m^2)y = x^2 e^{mx}$.
*29. $(D^3 - 3D + 2)y = (ax + b)e^x + ce^{-2x}$.
30. $(D^3 - 2D + 4)y = x + e^{-2x}$.
*31. $(D^4 + 3D^2 - 4)y = 4 \cos x + 3e^x$.

32. $(D^4 - 16)y = 6x + 7$; determine also the arbitrary constants when x, y, Dy, D^2y and D^3y vanish simultaneously.

33. $(4D^2 + 16D + 15)y = 4e^{-\frac{3}{2}x}$; determine also the arbitrary constants when $y = 3$, and $Dy = -5{\cdot}5$ for the zero value of x.

34. $(D - 4)^2 y = x \cosh 2x$.

Solve completely each of the following equations by changing z into x by means of the substitution $z = e^x$:

35. $z^2 \dfrac{d^2y}{dz^2} - 4z\dfrac{dy}{dz} + 6y = z.$

36. $z^2 \dfrac{d^2y}{dz^2} - 3z\dfrac{dy}{dz} + 13y = 9z^2 \log z.$

37. $z^3 \dfrac{d^3y}{dz^3} - z^2 \dfrac{d^2y}{dz^2} - 12y = z \sin (2 \log z).$

***38.** $z^3 \dfrac{d^3y}{dz^3} + 4z^2 \dfrac{d^2y}{dz^2} - 2z\dfrac{dy}{dz} - 4y = z^3 + z^{-3}.$

***39.** $z^3 \dfrac{d^3y}{dz^3} - 3z^2 \dfrac{d^2y}{dz^2} + 6z\dfrac{dy}{dz} - 6y = 2(\log z)^2.$

***40.** $z^3 \dfrac{d^3y}{dz^3} + 4z^2 \dfrac{d^2y}{dz^2} - 3z\dfrac{dy}{dz} + 3y = 24z \log z + 5z^2.$

41. The vertical deflection y at a distance x along a horizontal strut subjected to a constant thrust at each end, is given by the equation

$$c\frac{d^2y}{dx^2} + Py + \frac{w}{2}(\tfrac{1}{2}l^2 - x^2) = 0,$$

where l is the length, w the transverse load per unit length, c and P are constants.

Solve the equation completely having given that $\dfrac{dy}{dx} = 0$ for $x = 0$, and $y = 0$ for $x = \tfrac{1}{2}l$.

42. If the strut in Ex. 41 were a tie-rod, the same equation applies, provided the sign of P is reversed. Solve the equation in this case.

43. An important equation in the theory of strains set up in rotating discs and cylinders is

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - y = cz^3,$$

where c is a constant.

Solve the equation completely.

***44.** A body is vibrating at the end of a vertical spring, and the point of support of the spring is also made to vibrate. If the frictional resistance of the medium be neglected, the equation of motion is

$$\frac{d^2x}{dt^2} + n^2x = n^2a \sin \omega t,$$

where a, n, w are constants. Solve the equation completely.

***45.** If, in Ex. 44, the frictional resistance of the medium is considered, the equation of motion becomes

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = n^2a \sin \omega t.$$

Solve this equation when $n > f$.

***46.** Obtain the solution to the equation of Ex. 45, when $n < f$, and interpret it.

***47.** The equation of motion of the body hung on a vertical spring whose point of support is executing a damped vibration is

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + (f^2 + n^2)x = e^{-ft} \sin \omega t,$$

where n, f, w are constants. Solve the equation completely, having given that both x and $\frac{dx}{dt}$ are zero when $t = 0$.

48. Solve the differential equation

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 50y = \sin \omega t.$$

A spring is loaded with 32 lb. weight, its point of support has a motion given by $\sin \omega t$ feet, the resistance to the motion is measured by 10 times the speed in feet per second. If the spring extends 1/50th foot per lb. load and its mass may be neglected, give, when the oscillations have become steady, and $\omega^2 = 50$, the amplitude of the oscillations. (L.U.)

49. The motion of a weight at the lower end of a spring is given by $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = \sin 2t$, the other end of the spring having a simple harmonic motion. Solve this equation, and point out the part of it which gives the steady motion when t is large. (L.U.)

50. Solve $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = e^{ax}$. (L.U.)

138. Simultaneous Linear Differential Equations. In many practical problems differential equations occur involving several dependent variables. In such cases there are as many equations as there are dependent variables, and the methods of solution are illustrated in the following examples.

Ex. 16. *Solve the simultaneous equations*

$$\frac{dx}{dt} + kx + ly = E \cos pt, \quad \frac{dy}{dt} - ky - lx = F \cos pt,$$

where E, F, p, k and l are constants. (L.U., Sc.)

The given equations are of great practical importance, especially in Electricity.

Write them in the form

$$(D + k)x + ly = E \cos pt,$$
$$-lx + (D - k)y = F \cos pt.$$

Eliminate x, then

$$(D^2 - k^2 + l^2)y = \{(D+k)F + lE\} \cos pt$$
$$= -pF \sin pt + (kF + lE) \cos pt.$$

Now the C.F. is clearly

$$A'e^{bt} + B'e^{-bt} = A \cosh bt + B \sinh bt,$$

where $\quad b = \sqrt{k^2 - l^2}$, assuming that $k > l$.

$$\text{And the P.I.} = \frac{-pF \sin pt + (kF + lE) \cos pt}{D^2 - k^2 + l^2}$$

$$= \frac{pF \sin pt - (kF + lE) \cos pt}{p^2 + k^2 - l^2}.$$

Hence the complete integral for y is

$$\mathbf{y = A \cosh bt + B \sinh bt + \{pF \sin pt - (kF + lE) \cos pt\}/(p^2 + b^2),}$$

where $b^2 = k^2 - l^2$.

Similarly, by eliminating y, the complete integral for x becomes

$$\mathbf{x = L \cosh bt + M \sinh bt + \{pE \sin pt + (kE + lF) \cos pt\}/(p^2 + b^2).}$$

If A and B be arbitrary constants, then L and M will not be independent constants: for by substituting the values of x and y in the second differential equation, the relations

$$lL = bB - kA, \quad lM = bA - kB,$$

must be satisfied.

The same relations must hold for the first differential equation to be satisfied by the values of x and y.

It is left as an exercise for the student to work out the case when $k < l$.

Ex. 17. *Solve the equations*

$$\frac{dx}{dt} - y - z = e^{3t}, \quad \frac{dy}{dt} + 2x + 4y - z = e^{-4t}, \quad \frac{dz}{dt} - 2x - z = t.$$

Here the number of dependent variables is greater than 2, and in such cases it is better to employ the following method rather than that of elimination.

Multiply the second equation by a and the third by β, and add all the equations together, then

$$\frac{d}{dt}(x+ay+\beta z)+2(a-\beta)x+(4a-1)y-(1+a+\beta)z=e^{3t}+ae^{-4t}+\beta t.$$

Denote the function $x+ay+\beta z$ by ω, and suppose a, β, λ to be three constants so chosen that

$$\lambda\omega=2(a-\beta)x+(4a-1)y-(1+a+\beta)z,$$

then
$$\frac{d\omega}{dt}+\lambda\omega=e^{3t}+ae^{-4t}+\beta t.$$

Hence, by integration,

$$\omega e^{\lambda t}=\int(e^{3t}+ae^{-3t}+\beta t)e^{\lambda t}\,.\,dt+A$$
$$=\frac{e^{(\lambda+3)t}}{\lambda+3}+\frac{ae^{(\lambda-4)t}}{\lambda-4}+\frac{\beta e^{\lambda t}}{\lambda^2}(\lambda t-1)+A.$$

It therefore remains to determine a, β and λ.

Since $\lambda\omega$ is to be identically equal to

$$2(a-\beta)x+(4a-1)y-(1+a+\beta)z,$$

by equating corresponding coefficients of x, y, z,

$$2a-2\beta-\lambda=0,$$
$$(4-\lambda)a \qquad -1=0,$$
$$a+(1+\lambda)\beta+1=0.$$

Eliminate a and β from these equations, and

$$\begin{vmatrix} 2 & -2 & -\lambda \\ 4-\lambda & 0 & -1 \\ 1 & 1+\lambda & 1 \end{vmatrix}=0,$$

i.e.
$$\lambda^3-3\lambda^2-4\lambda+12=0,$$

or
$$(\lambda+2)(\lambda-2)(\lambda-3)=0\,;$$
$$\therefore\ \lambda=2,\ -2,\text{ or }3,$$

and the corresponding values of a, β become $a=\frac{1}{2}$, $\frac{1}{6}$, 1 and $\beta=-\frac{1}{2}$, $\frac{7}{6}$, $-\frac{1}{2}$.

Hence the three integrals are

$$\mathbf{x}+\tfrac{1}{2}(\mathbf{y}-\mathbf{z})=\tfrac{1}{5}\mathbf{e}^{3t}-\tfrac{1}{4}\mathbf{e}^{-4t}-\tfrac{1}{8}(2t-1)+\mathbf{A}_1\mathbf{e}^{-2t},$$
$$\mathbf{x}+\tfrac{1}{6}(\mathbf{y}+7\mathbf{x})=\mathbf{e}^{3t}-\tfrac{1}{36}\mathbf{e}^{-4t}-\tfrac{7}{24}(2t+1)+\mathbf{A}_2\mathbf{e}^{2t},$$
$$\mathbf{x}+\mathbf{y}-\tfrac{1}{2}\mathbf{z}=\tfrac{1}{6}\mathbf{e}^{3t}-\mathbf{e}^{-4t}-\tfrac{1}{18}(3t-1)+\mathbf{A}_3\mathbf{e}^{-3t}.$$

These equations may now be solved to give x, y, z if necessary.

EXERCISES 18 E.

Solve each of the following systems of simultaneous equations :

1. $\dfrac{dx}{dt}=x+15y, \quad \dfrac{dy}{dt}=4(x+2y).$

2. $\dfrac{dx}{dt}+7x-y=0, \quad \dfrac{dy}{dt}+2x+5y=0.$ (Li.U.)

3. $\dfrac{dx}{dt}=ax+by, \quad \dfrac{dy}{dt}=bx-ay.$ (Br.U.)

4. $\dfrac{dy}{dx}+y=z+e^x, \quad \dfrac{dz}{dx}+z=y+e^x.$ (B.U.)

5. $\dfrac{dx}{dt}+4x+5y=7\cos 2t, \quad \dfrac{dy}{dt}-4y-5x=8\cos 2t.$

6. $\dfrac{dx}{dt}=4x+6y+2z, \quad \dfrac{dy}{dt}=3(x+y), \quad \dfrac{dz}{dt}+16x+24y+5z=0.$

7. $\dfrac{dx}{dt}=cz-ax, \quad \dfrac{dy}{dt}=ax-by, \quad \dfrac{dz}{dt}=by-cz.$

8. $\dfrac{dx}{dt}+2x-y+z=t, \quad \dfrac{dy}{dt}+4x+3y+2z=0, \quad \dfrac{dz}{dt}+4x+2y+3z=e^t.$

9. $\dfrac{dx}{dt}+10x-2y+9z=0, \quad \dfrac{dy}{dt}+4x+4y+6z=0,$
$$\dfrac{dz}{dt}+14x+22y+25z=0.$$

10. $\dfrac{dx}{dt}+4x+3y+2z=2\sin 2t, \quad \dfrac{dy}{dt}+4x+8y+4z=0,$
$$\dfrac{dz}{dt}+22x+9y+8z=14\cos 2t.$$

11. $5\dfrac{dx}{dt}+3\dfrac{dy}{dt}+4y=0, \quad 4\dfrac{dx}{dt}+5\dfrac{dy}{dt}+2x=0.$

Apply the method of Ex. 17 (p. 457) to solve:

*12. $\dfrac{d^2x}{dt^2}=7x-8y, \quad \dfrac{d^2y}{dt^2}=3x-4y.$

*13. $\dfrac{d^2x}{dt^2}=8x-7y, \quad \dfrac{d^2y}{dt^2}=4x-3y.$

*14. Solve, by the method of Ex. 16 (p. 456),
$$\frac{d^2x}{dt^2}-\frac{dy}{dt}+6x=0, \quad \frac{d^2y}{dt^2}+\frac{dx}{dt}+6y=0.$$

15. Solve $\frac{du}{dx}+v=\sin x$, $\frac{dv}{dx}+u=\cos x$, given that when $x=0$, $u=1$ and $v=0$. Hence shew that $u^2-v^2=\cos^2 x+2\sin x.\sinh x$.

***16.** The following equations occur in problems on small oscillations:

$$a\frac{d^2x}{dt^2}+bx+cy=0, \quad p\frac{d^2x}{dt^2}+q\frac{d^2y}{dt^2}+rx+sy=0,$$

a, b, c, p, q, r, s, being constants. By assuming that $y=mx$, where m is constant, shew that m is given by the quadratic

$$cqm^2+(cp+bq-as)m+bp-ar=0.$$

Hence solve the equations for the case in which $a=1$, $b=5$, $c=2$, $p=3$, $q=1$, $r=27$, $s=9$, it being given that $x=3$ and $\frac{dy}{dx}=0$ when $t=0$.

***17.** If $\frac{dx}{dt}=63y-72z$, $\frac{dy}{dt}=20z-28x$, $\frac{dz}{dt}=8x-5y$, shew that

$$4x^2+9y^2+36z^2 \quad \text{and} \quad 5x+18y+63z$$

are constant.

***18.** Solve

$$2\frac{dx}{dt}+3\frac{dy}{dt}-3{\cdot}2x=4\cos t,$$

$$3\frac{dx}{dt}+5\frac{dy}{dt}-12{\cdot}5y=0.$$

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ANSWERS TO THE EXERCISES

Exercises 1. (Page 13.)

1. 19. 17. 13. 11. 3^{10}. 2^{12}. **2.** $1+16x+116x^2+504x^3+1463x^4+\ldots$.

3. 2472. **4.** 12.

5. $a_1=\frac{1}{2}p$, $a_2=-(4q-p^2)/8$, $a_3=p(p^2-4q)/16$,
$a_4=-(16q^2+24p^2q-5p^4)/128$.

6. 0·99890717.

10. 564·99646 ; error less than 0·000001.

13. $a=b=4$, $c=-2$. **14.** $1/\sqrt{2}=0{\cdot}7071$.

16. Fifth term $=-5\times6144$; cube root $=0{\cdot}855$.

17. $N+a/5N^4-2a^2/25N^9+6a^3/125N^{14}-\ldots$.

18. $(1-x/3)^{-\frac{1}{2}}$, 243/32. **19.** $(8/9)^{-\frac{7}{2}}=1{\cdot}51$.

24. $x+x^2+7x^3+13x^4+\ldots$; $\{3^{n+2}-(-2)^{n+2}-5\}/30$.

25. $2/(1-2x)+(x-1)/(x^2+1)$; $3+5x+7x^2+15x^3+33x^4+\ldots$.

27. $8/(4x-3)-(6x+1)/(3x^2+2)$; $-19/6-59x/9-431x^2/108-\ldots$.

28. $\{(4x-3)/(1+2x)^2+3/(1-3x)\}/25$; $x-x^2+9x^3-5x^4\ldots$.

29. $\{(x-14)/(4-x)^2-(x-6)/(4+x^2)\}/20$; -2469×2^{-25}.

32. $n(n+1)(2n+1)/6$; 4740

33. $n(n+1)(2n+1)(3n^2+3n-1)/30$. **34.** $n(n+1)(n+2)(3n+5)/12$.

36. $n(n+1)(n+2)(n+3)/4$. **37.** $n/(2n+1)$. **39.** $\frac{1}{2}n/(3n+2)$.

40. $1/(1-x)^2(1+x)$.

Exercises 2. (Page 31.)

1. -123. **2.** 4 **3.** 0. **4.** 5040. **5.** -204. **6.** -195.

7. -972. **8.** 12. **9.** 234. **10.** 8100. **11.** 0. **12.** 0.

13. $x=22/13$, $y=157/117$, $z=203/117$.

14. $x=865/203$, $y=79/203$, $z=-311/203$.

15. $x=-12$, $y=9$, $z=-2$. **16.** $x=22/41$, $y=18/41$, $z=1/41$.

17. $x=1$, $y=3$, $z=5$. **18.** $x=4$, $y=5$, $z=2$.

19. $x=2$, $y=-2$, $z=3$, $w=5$. **20.** $x=1$, $y=3$, $z=-2$, $w=7$.

22. $x=0$, 2 or -2. **23.** $x=0$ or $\frac{1}{2}$. **24.** $x=5/2$.

25. $x=-2$ or -3. **26.** $x=2abc/(a^2+b^2+c^2-2ab-2bc-2ca)$.

27. $x=\frac{1}{2}, -\frac{1}{2}$ or $\frac{1}{3}$.

28. $x=c+a\pm(b+d)$, or $\pm\sqrt{a^2+b^2+c^2+d^2-2ca-2bd}$.

29. $(c+a)^2=b^2$.

30. $\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ a & b & c \end{vmatrix}=0.\quad a=-1.$

32. $(a^2+b^2+c^2+d^2)^2$.

35. $\begin{vmatrix} a_1^2+a_2^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1^2+b_2^2 \end{vmatrix}$.

36. $\begin{vmatrix} 181 & 182 \\ 182 & 185 \end{vmatrix}=361.$

37. $\begin{vmatrix} 47 & 46 \\ 69 & 68 \end{vmatrix}=22.$

38. $\begin{vmatrix} 7 & 2 & 8 \\ 9 & 4 & 9 \\ 10 & 12 & 11 \end{vmatrix}=78.$

39. $2\begin{vmatrix} 93 & -18 & 70 \\ -9 & 29 & 23 \\ 35 & 23 & 59 \end{vmatrix}=676.$

40. $\begin{vmatrix} 1+3i & 5+7i \\ -5+7i & 1-3i \end{vmatrix}$.

42. $(a^3+b^3+c^3-3abc)^2$.

44. $4(x+y+z)(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2$. 0.

48. $\lambda=\sin x\cos x$.

49. -2.

50. $a=0$, or 2, $x=0$ or 2, $y=0$ or 10.

Exercises 3. (Page 53.)

2. $A=5$, $\alpha=53°\ 8'=0\ 9274$ radian.

3. $s=5{\cdot}5$.

10. $\sin^2 B=c(a-c)/\{b(a+b-2c)\}$.

13. $\theta=\frac{1}{3}n\pi$ or $\frac{1}{6}(6n\pm1)\pi$.

14. $\theta=n\pi+1{\cdot}1071$, $n\pi+2{\cdot}0345$, $\frac{1}{6}(6n+1)\pi$, or $\frac{1}{6}(6n+5)\pi$.

15. $\theta=\frac{1}{4}(4n\pm1)\pi$, or $\frac{2}{5}(\pi+0{\cdot}7470)$.

16. $\theta=\frac{1}{2}n\pi$, or $\{2n+(-1)^n\}\pi/16$.

17. $\theta=\frac{1}{2}n\pi$, $2(n\pi\pm0{\cdot}6590)$, or $\frac{1}{3}(6n\pm1)\pi$.

18. $x=3$.

21. If CO be produced to meet the circle again in C', this point represents the second value.

22. $\frac{1}{2}+\sqrt{3}+\frac{3}{2}\sqrt{3}i$, $\frac{7}{2}+\sqrt{3}+(\frac{3}{2}\sqrt{3}-2)i$, $5-3i$, $\frac{7}{2}-\sqrt{3}-(\frac{3}{2}\sqrt{3}+2)i$, $\frac{1}{2}-\sqrt{3}-\frac{3}{2}\sqrt{3}\,.\,i$.

23. (i) $2(p^2-q^2)/(p^2+q^2)^2$;
(ii) $(\tan x\operatorname{sech}^2 y+i\tanh y\sec^2 x)/(1+\tan^2 x\tanh^2 y)$.

24. $6{\cdot}403(\cos 38°\ 40'+i\sin 38°\ 40')$; Sq. rt. $=\pm(2{\cdot}388+0{\cdot}8377i)$.

25. $3{\cdot}060(\cos 140°\ 10'+i\sin 140°\ 10')$; Cube roots $=0{\cdot}9954+1{\cdot}057i$, $-1{\cdot}413+0{\cdot}3335i$, and $0{\cdot}4176-1{\cdot}391i$.

26. $\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}b/a$.

27. (i) $r=113$, $\theta=-7°\ 38'$, Product $=112-15i$.
(ii) $r=2641$, $\theta=-122°\ 43'$, „ $=-1792-1940i$.
„ $\theta=-\ 12°\ 43'$, „ $=\ 2576-581{\cdot}5i$.
„ $\theta=\ 107°\ 17'$, „ $=-784{\cdot}7+2522i$.

28. $P=\frac{1}{2}\log\dfrac{x^2-2ax+a^2+y^2}{x^2+2ax+a^2+y^2}$; $Q=\tan^{-1}\dfrac{2ay}{x^2+y^2-a^2}$.

30. $x=(u^2+v^2-u)/(v^2+u^2-2u+1)$, $y=-v/(v^2+u^2-2u+1)$.

31. $A=\sqrt{\dfrac{1}{R^2}+\left(Cp-\dfrac{1}{Lp}\right)^2}$; $\theta=\tan^{-1}R(Cp-1/Lp)$.

32. $\cosh 1\cos 1+i\sinh 1\sin 1$, $\sqrt{\cos^2 1+\sinh^2 1}(\cos\theta+i\sin\theta)$, where

$$\theta=\tan^{-1}(\tanh 1\tan 1).$$

$$a=\sqrt{1/(\cos^2 p+\sinh^2 p)},\quad b=\tan^{-1}(\cot p/\tanh p).$$

34. $(a^2+b^2)^{\frac{1}{2}p}e^{-q\tan^{-1}b/a}(\cos\phi+i\sin\phi)$, where $\phi=\frac{1}{2}q\log(a^2+b^2)+p\tan^{-1}\dfrac{b}{a}$.

Expression will be real when $p\tan^{-1}\dfrac{b}{a}=-\frac{1}{2}q\log(a^2+b^2)$.

38. 0·52110, 1·12763. **40.** 0·4055.

42. $2\tanh x/(1+\tanh^2 x)$, $A=\frac{1}{2}(5+7i)$, $B=\frac{1}{2}(5-7i)$. **43.** $x=2{\cdot}23$.

44. $x=0{\cdot}4055$ or $0{\cdot}2231$. **45.** $x=1{\cdot}85$. **46.** $x=0{\cdot}1$. **47.** $x=-2{\cdot}0795$.

48. $x=0{\cdot}47$ or $-0{\cdot}47$. **49.** $x=0{\cdot}9163$. **50.** $x=0{\cdot}6931$. **51.** $x=0{\cdot}5101$.

52. $(1-ae^{i\theta})^{-\frac{1}{2}}$; $\frac{1}{2}\{(1-ae^{i\theta})^{-\frac{1}{2}}+(1-ae^{-i\theta})^{-\frac{1}{2}}\}$.

53. $\sin\left\{\alpha+\dfrac{1}{2}\cdot\dfrac{n-1}{n}\cdot\pi\right\}\Big/\sin\dfrac{\pi}{2n}$.

54. $\{n\sin x\cos x+\cos(n+2)x\,.\,\sin nx\}/(2\sin x)$. **55.** e.

56. $\sin x+x\cos x$. **57.** $\sin\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta/\sin\frac{1}{2}\beta$.

58. $\frac{1}{2}$. **61.** $\frac{1}{2}a\operatorname{cosec}^2\dfrac{\pi}{2n}$, where a is the side.

62. $\frac{1}{2}\log(1-2x\cos\alpha+x^2)$. **63.** $2r\cot\dfrac{\pi}{2n}$.

Exercises 4. (Page 68.)

1. $A=52°\ 46'$; $B=60°\ 1'$; $c=64°$.
2. $a=57°\ 46'$; $b=39°\ 9'$, $c=65°\ 34'$.
4. $A=44°\ 58'$; $b=61°\ 46'$; $C=64°\ 42'$.
5. $A=74°\ 50'$; $B=53°\ 32'$; $c=78°\ 26'$.
6. $a=67°\ 58'$; $b=78°\ 12'$; $c=56°\ 45'$.
7. $A=52°\ 29'$; $B=71°\ 22'$; $b=66°\ 15'$.
8. $a=36°\ 36'$; $b=77°\ 8'$; $B=82°\ 14'$.
9. $A=51°\ 16'$; $b=47°\ 32'$; $c=60°\ 11'$.
10. $b=c=B=90°$.
11. $A=82°\ 56'$; $b=51°\ 21'$; $c=84°\ 24'$.
12. $a=64°\ 24'$; $b=39°\ 30'$; $c=70°\ 32'$.
13. $c=86°\ 30'$; $B=64°\ 12'$; $A=82°\ 48'$.
15. $A=88°\ 11'$; $B=51°\ 12'$; $C=62°\ 27'$.
16. $A=58°\ 24'$; $B=46°\ 30'$; $C=94°$.
17. $A=56°\ 20'$; $B=89°\ 46'$; $C=48°\ 48'$.
18. $A=62°\ 28'$; $B=57°\ 38'$; $C=100°\ 5'$.
19. $A=36°\ 40'$; $B=38°\ 24'$; $C=120°\ 46'$.
20. $A=63°\ 51'$; $B=60°\ 45'$; $c=54°\ 18'$.
21. $A=66°\ 25'$; $b=104°\ 54'$; $C=79°\ 54'$.

22. $a=56°\ 36'$; $B=69°\ 29'$; $C=100°\ 36'$.
23. $A=117°\ 37'$; $b=32°\ 37'$; $C=53°\ 56'$.
24. $A=87°\ 30'$; $B=63°\ 14'$; $c=25°\ 48'$.
26. $A=49°\ 6'$.
30. $A=B=C=75°$; $a=b=c=69°\ 34'$.
33. $A=64°\ 27'$; $B=56°\ 58'$; $c=88°\ 38'$.
34. 3316 miles. **35.** 2941 miles. **36.** 3308 miles. **37.** 1091 miles.
38. 3518 miles. **39.** 5971 miles. **40.** 6352 miles. **41.** 5953 miles.
42. 4007 miles. **43.** 85° 10′. **44.** 248·7 miles.
45. 47° 33′ or 3287 miles.

Exercises 5a. (Page 79.)

1. $20x-17$. **2.** $18x^2+46x-1$. **3.** $-8/(3x^3)$.
4. $-1/(4x^{\frac{3}{2}})-6/x^4$. **5.** $-6/(3x+2)^2$. **6.** $-28/(2x+3)^3$.
7. $10^x \log_e 10$. **8.** ace^{cx}. **9.** $a^{ax+b+1}\log_e a$.
10. $4/(4x-9)$. **11.** $4\sec^2 4x$. **12.** $5a\cos(5x+3)$.
13. $1/\sqrt{4a^2-x^2}$. **14.** $\sqrt{3}/\sqrt{8+2x}-3x^2$. **15.** $3a/(a^2+9x^2)$.
16. $1/(2x^2-2x+1)$. **17.** $2\cosh(2x+3)$. **18.** $4/\sqrt{16x^2-25}$.
19. $5x^4+1/(5x)$. **20.** $\tan^2 x$.
21. $10(6x-1)(6x^2-2x+7)^4$. **22.** $3x\sqrt{(a^2+x^2)}$.
23. $-4x(b^2-x^2)^{\frac{1}{3}}/3$. **24.** $-\frac{1}{2}(8x+3)/(4x^2+3x+1)^{\frac{1}{2}}$.
25. $1/\sqrt{x^2-1}$. **26.** $4(1+\tan^2 2x)/(1-\tan^2 2x)$.
27. $-6\sin 2x\,.\,\cos^2 2x$. **28.** $-4\sin\frac{x}{a}\,.\,\cos^3\frac{x}{a}$. **29.** $\cot x$.
30. $9\sec^2 3x\tan^2 3x$. **31.** $(2x^2+3x+4)/\sqrt{x^2+2x+4}$.
32. $x(1+2\log 4x)$. **33.** $(x-a)(x-b)^2(2x-2b+3)$.
34. $2x\log(x+3)+x^2/(x+3)$. **35.** $\log\cot x - x\operatorname{cosec}^2 x\,.\,\tan x$.
36. $4\sin x+4x\cos x$. **37.** $4\sin 3x+3(4x-3)\cos 3x$.
38. $\sec^2(x+5)\log(x+9)+\dfrac{\tan(x+5)}{x+9}$.
39. $4x\sin(3x+2)+6x^2\cos(3x+2)$. **40.** $4ab\sin^3(bx+c)\cos(bx+c)$.
41. $\sin(\log x)+\cos(\log x)$. **42.** $e^x(\sin x+\cos x)$.
43. $2e^{\sin 2x}\cos 2x$. **44.** $e^{-2x}(3\cos 3x-2\sin 3x)$.
45. $2(1+x^2)/(1-x^2)^2$. **46.** $2(x^2-x-3)/(3+x^2)^2$.
47. $-3/(2x-1)^2$. **48.** $-55(4x-9)^2$.
49. $a^2(a^2+x^2)^{\frac{3}{2}}$. **50.** $4ab(a+bx)/(a-bx)^3$.
51. $(2+2x-x^2)/(1-x)^2$. **52.** $(5x^2-6x-2)/(x^2-x+1)^2$.
53. $-x(x^2-3)/(1+x^2)^{\frac{3}{2}}$. **54.** $-(a^2+2x)/(a^2+x^2)^{\frac{1}{2}}$.
55. $2x(3x-10)/(3x-5)^2$. **56.** $-7(3x^2-10x+2)/(5-3x)^2$.
57. $(39x^2-2x+69)/(5x^2+6x-9)^2$. **58.** $-5a/\{2(a-2x)^{\frac{1}{2}}(a+3x)^{\frac{3}{2}}\}$.

59. $30x(5+x)^2/(3+x)^6$.

60. $28x/(2x^2+5)(4x^2+3)$.

61. $2(1-x^2)/(x^2+x+1)(x^2-x+1)$.

62. $2x\log\dfrac{5x-9}{6x+11}+\dfrac{109x^2}{(5x-9)(6x+11)}$.

63. $2(\sin x+x\cos x)/(1-x^2\sin^2 x)$.

64. $\{1-x(\log x)^2\}/\{x(1+x\log x)^2\}$.

65. $a^{2\sin x}\left\{2\cos x\log a\log\dfrac{\sqrt{a^2-x^2}}{a+2x}-\dfrac{a(x+2a)}{(a^2-x^2)(a+2x)}\right\}$.

66. $2(b^2-x^2)^{2x}\{\log(b^2-x^2)-2x^2/(b^2-x^2)\}$.

67. $e^{ax}\{a\cos(bx+c)-b\sin(bx+c)\}$.

68. $1/(1+\cos x)$.

69. $-(b+a\sin x)/(a+b\sin x)^2$.

70. $2x\tan^{-1}\dfrac{2x-3}{3x-2}+\dfrac{5x^2}{13x^2-24x+13}$.

71. $3/(1+x^2)$.

72. $1/2$.

73. $\dfrac{4\sqrt{1-x}}{3(2-x)\sqrt{x}}-\dfrac{4}{3(2-x)^2}\sin^{-1}(2x-1)$.

74. $\dfrac{e^x}{a}\cos\dfrac{e^x}{a}\log\dfrac{1}{x}-\dfrac{1}{x}\sin\dfrac{e^x}{a}$.

75. $(\sin x)^x(\log\sin x+x\cot x)$.

76. $e^x(\sin^2 x+\sin 2x)$.

77. $\dfrac{y(x\operatorname{cosec}2x-2\log\tan x)}{x(x\operatorname{cosec}2y-2\log\tan x)}$.

78. $-(ax+hy)/(hx+by)$.

79. $y^2/\{x(1-y\log x)\}$.

80. $e^{x^x}x^x\log ex$.

81. $x^{x^x+x}\{(\log x)^2+\log x+1/x\}$.

82. $-\sqrt{b^2-a^2}/(b+a\cos x)$.

83. $x^x(1+\log x)+x^{\frac{1}{x}-2}(1-\log x)$.

84. $1/(x\log x)$.

85. $2(1-\log x)/x^2$.

86. $(\tan x\log\sin x+\cot x\log\cos x)/(\log\cos x)^2$.

87. $(y\sec^2 x-\sin y)/(x\cos y-\tan x)$.

88. $(1-x)/x$.

89. $(a^4+2a^2x^2-x^4)/\{(a^2+x^2)^{\frac{1}{2}}(a^2-x^2)^{\frac{3}{2}}\}$.

90. $2/(1-x^4)$.

92. $a/(a^2+b^2+2bx+x^2)$.

93. (i) $-5\,.\,10^{-5x}\log 10$. (ii) $\sec x$. (iii) $-6/\sqrt{3(2-3x^2)}$.

96. $2x\cot x^2$.

98. $-\cos x\operatorname{cosec}^2 x$; $ab/(a^2\cos^2 x+b^2\sin^2 x)$;

$$(-1)^n\frac{|\underline{n}}{3}\left\{\frac{1}{(x-1)^{n+1}}-\frac{2^{n+1}}{(2x+1)^{n+1}}\right\}.$$

101. $y=24{\cdot}336$ ft.

Exercises 5b. (Page 90.)

1. $3(x^2+ay)$, $3(y^2+ax)$.

2. $2(ax+hy+g)$, $2(hx+by+f)$.

3. $2x/a^2$, $-2y/b^2$.

4. $4x/(x^2+y^2+a^2)$, $4y/(x^2+y^2+a^2)$.

5. $(x^2+2xy-y^2)/2(x^4+2x^2y^2+y^4+x^2+2xy+y^2)$,
$(y^2+2xy-x^2)/2(x^4+2x^2y^2+y^4+x^2+2xy+y^2)$.

6. $ay+2bx+3x^2$, $2y+ax-3cy^2$.

7. (1) $6x+4y-3z=24$, (2) $4x/a^2+3y/b^2+3z/c^2=1$.

9. v.

18. $2xy$, x^2; $x(4y^2+xy-2x^2)/(x+2y)$.

28. (i) $2x$; (ii) $2x+v^2/2x^3$.

Exercises 5c. (Page 100.)

M. = maximum, m. = minimum, I. = point of inflexion.

1. M. $x=-1$, $y=2$; m. $x=5$, $y=-106$; I. $x=2$
2. M. $x=2\cdot5$, $y=88$; m. $x=3$, $y=87\cdot75$; I. $x=2\cdot75$.
3. M. $x=1$, $y=14$; m. $x=1\cdot5$, $y=13\cdot75$; I. $x=1\cdot25$.
4. m. $x=\pm3$, $y=8$.
5. m. $x=0$, $y=-4$; I. $x=5$, $5/3$.
6. M. $x=1$, $y=14$; m. $x=2$, $y=-9$; I. $x=0$, $0\cdot61$, $1\cdot64$.
7. M. $x=-0\cdot5$, $y=-0\cdot25$; m. $x=2$, $y=1$.
8. M. $x=4$, $y=1$; m. $x=6$, $y=9$.
9. M. $x=4/3$, $y=4/27$; m $x=2$, $y=0$; I. $x=5/3$.
10. M. $x=0$, $y=8$; m. $x=\pm\sqrt{2}$, $y=4$; I. $x=\pm\sqrt{1\cdot2}$.
11. M. $x=0\cdot5282$, $y=0\cdot0669$.
12. M. $2n\pi+\tan^{-1}a/b$, $y=\sqrt{a^2+b^2}$;
 m. $x=(2n-1)\pi+\tan^{-1}a/b$, $y=-\sqrt{a^2+b^2}$.
13. M. $x=-2$, $y=0$; m. $x=4/3$, $y=-500/27$.
14. $n/4$.
15. $r=4\cdot5$ in., $l=9$ in.
16. radius $=5$ ft., side $=10$ ft.
17. $l=20$ ft., $r=10$ ft..
18. 100.
19. $9/(4\pi+18)=0\cdot294$, $9/(2\pi+9)=0\cdot589$, radius $=3/(2\pi+18)=0\cdot124$.
20. Side $=6\sqrt{2}$ ft., volume $=648$ cu. ft.
21. $66°\ 4'=1\cdot153$ radian, $V=2\pi r^3\sqrt{3}/27$.
22. $h:r=1:\sqrt{2}$; $0\cdot5773$.
23. $b:d=1:3$.
24. $r=6$ in.; $z=6\sqrt{3}$ in.
25. $r_1=2r/3$; $h_1=h/3$.
26. $r=10\sqrt{6}/3$, $l=20\sqrt{3}/3$.
27. $r_1=r\sqrt{6}/3$; $l_1=2r\sqrt{3}/3$.
28. $r=5$ ft., $h=5$ ft.
29. $\theta=\pi/4=45°$.
30. Radius $=6$ in., Arc $=12$ in..
32. $x=23$, Expenditure $=$ £325 18s. 10d.
34. M. $x=(2n\pi+\tan^{-1}\beta/a)/\beta$; m. $x=\{(2n-1)\pi+\tan^{-1}\beta/a\}/\beta$, where n is a positive integer.
35. 24 in. × 12 in. × 4 in.
36. $8r^2=h^2$.
37. $r=7$, side $=14$.
38. Side $=$ diameter $=2b/(n\pi+4)$; Each $=4$.
39. 16·9 feet.
40. $CL\omega^2=1$, giving $M^2\omega^2/R^2$, $C\omega^2(2L-R^2C)=2$, giving $4M^2/\{R^2C(4L-R^2C)\}$.
42. M. $x=-5+\sqrt{21}=-0\cdot417$, $y=-0\cdot1966$;
 m. $x=-5-\sqrt{21}=-9\cdot583$, $y=0\cdot948$.
45. $x=2l/9$, $t=\frac{2}{3}\pi\sqrt{13l/g}$.

Exercises 6a. (Page 109.)

1. $\frac{6}{5}x^{\frac{5}{2}}$.
2. $\frac{1}{18}(2x-5)^9$.
3. $\frac{1}{5}(4-x)^{-5}$.
4. $\frac{3}{5}x^{\frac{5}{3}}-2x^{-\frac{1}{2}}$.
5. $\frac{1}{3}\log(3x+7)$.
6. $\frac{9}{4}x^4+8x^2(x+1)$.
7. $\frac{1}{3}x^3-\frac{1}{2}x^2-2x$.
8. $\frac{2}{3}x^3-5x+\frac{3}{x}$.
9. $\frac{4}{5}\log(5x+3)$.
10. $\frac{1}{27}(3x-7)^9$.
11. $x-\frac{2}{5}\log(5x+2)$.
12. $\frac{1}{3}(x^3+44)-x^2+4x-13\log(x+2)$.
13. $\frac{1}{5}\sqrt{x}(x^2-25)$.

14. $\frac{2}{3}(2a+x)^{\frac{3}{2}}$. **15.** $\frac{1}{6}(x^3+3a^3\log x)$. **16.** $-\frac{3}{10}(3-2x)^{\frac{5}{3}}$.

17. $\frac{3}{16}(4x^2-4x-3)+\frac{7}{8}\log(2x+1)$.

18. $\frac{5}{6}\{11-6x-3x^2-2x^3-6\log(1-x)\}$. **19.** $\frac{1}{2}(x-\sin x\cos x)$.

20. $\frac{1}{12}(\sin 3x+9\sin x)$. **21.** $-\frac{1}{5}\cos 5x$. **22.** $\frac{1}{2}\log\sec(2x+3)$.

23. $-\frac{1}{4}\cos 2x$. **24.** $\log\sin x$. **25.** $4+10\log 5=20\cdot094$.

26. $\frac{2}{3}(36-\log 13)=22\cdot29$. **27.** $\frac{1}{3}(18-\log 5\cdot5)=5\cdot4318$.

28. $\frac{4}{3}(31-18\log 5)=2\cdot7077$. **29.** $2(\log 35-\log 19)=1\cdot2218$.

30. 111 **31.** $\frac{5}{2}\log 3-2=0\cdot7465$. **32.** $0\cdot2$.

33. $\frac{1}{2}\pi$. **34.** 1. **35.** $\frac{5}{24}$.

36. $1-\frac{1}{4}\pi=0\cdot2146$. **37.** $\log 9=2\cdot1972$. **38.** $10\frac{2}{3}$.

40. $(x^2-4x)/2+4\log(x+2)$. **41.** $\frac{1}{2}\log(a^2+x^2)$, $\frac{1}{2}\log 2=0\cdot3466$.

43. $\frac{1}{2}(1+v^2)-\frac{1}{2}\log(1+v^2)$. **44.** $\frac{\pi}{2}$. **45.** $y^2=x-1$.

Exercises 6b. (Page 114.)

1. $\frac{1}{x+2}-\frac{1}{x+3}$; $\log\frac{x+2}{x+3}$. **2.** $\frac{1}{x-2}+\frac{1}{x-3}$; $\log(x-2)(x-3)$.

3. $\frac{3}{x+2}-\frac{1}{2x-3}$; $\log\frac{(x+2)^3}{(2x-3)^{\frac{1}{2}}}$. **4.** $\frac{2}{2x-3}-\frac{7}{7x+2}$; $\log\frac{2x-3}{7x+2}$.

5. $\frac{3}{3x-4}-\frac{4}{4x-3}$; $\log\frac{3x-4}{4x-3}$. **6.** $\frac{5}{x-3}+\frac{3}{x+4}$; $\log(x-3)^5(x+4)^3$.

7. $\frac{3}{x+7}+\frac{7}{x-3}$; $\log(x+7)^3(x-3)^7$.

8. $\frac{5}{3-2x}-\frac{2}{4-3x}$; $\frac{2}{3}\log(4-3x)-\frac{5}{2}\log(3-2x)$.

9. $\frac{2}{8-3x}+\frac{3}{7-x}$; $-\frac{2}{3}\log(8-3x)-3\log(7-x)$.

10. $\frac{8}{4x+1}+\frac{6}{2-3x}$; $2\log\frac{4x+1}{2-3x}$.

11. $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$; $\log(x-1)(x-2)^2(x-3)^3$.

12. $\frac{2}{2x-3}+\frac{3}{3x-1}-\frac{2}{x-2}$; $\log\frac{(2x-3)(3x-1)}{(x-2)^2}$.

13. $\frac{1}{x}+\frac{1}{x+3}-\frac{2x}{x^2+4}$; $\log\frac{x(x+3)}{x^2+4}$.

14. $\frac{1}{1-x}-\frac{1}{1+x}+\frac{2x}{1+x^2}$; $\log\frac{1+x^2}{1-x^2}$.

15. $\frac{1}{2-x}+\frac{1}{2+x}$; $\log\frac{2+x}{2-x}$.

16. $\dfrac{2}{x-3}+\dfrac{2x}{x^2+2}-\dfrac{4x}{x^2+3}$; $\log\dfrac{(x-3)^2(x^2+2)}{(x^2+3)^2}$.

17. $\dfrac{3x^2}{x^3-1}-\dfrac{3x^2}{x^3+1}$; $\log\dfrac{x^3-1}{x^3+1}$.

18. $-\dfrac{4x+1}{4x^2+2x+1}+\dfrac{2}{2x-1}$; $\log(2x-1)-\dfrac{1}{2}\log(4x^2+2x+1)$.

19. $\dfrac{1}{a^2-x^2}+\dfrac{1}{a^2+x^2}+\dfrac{1}{b^2-x^2}$; $\dfrac{1}{2a}\log\dfrac{a+x}{a-x}+\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+\dfrac{1}{2b}\log\dfrac{b+x}{b-x}$.

20. $\dfrac{2x-1}{x^2-x+1}-\dfrac{2x+1}{x^2+x+1}$; $\log\dfrac{x^2-x+1}{x^2+x+1}$.

21. $\log 2\cdot5=0\cdot9163$. **22.** $\log 8=2\cdot0794$. **23.** $\log 15-\log 11=0\cdot3102$.

24. $\log 2\cdot86=1\cdot0508$. **25.** $\log 2\cdot5=0\cdot9163$. **26.** $\log 5-\log 3=0\cdot5108$.

27. $\log 4=1\cdot3863$. **28.** $\log 7+3\log 2=4\cdot0252$.

29. $3\log 5-\log 7-\log 3=1\cdot7837$. **30.** $\log 3=1\cdot0986$.

31. $9\log 3-4\log 2=7\cdot115$. **32.** $\dfrac{\pi}{16}$. **33.** $\log 2-\dfrac{\pi}{4}=-0\cdot0923$.

34. $A=2,\ B=0,\ C=-1,\ D=1$; $0\cdot5108$.

35. $8\cdot8+\log 2=9\cdot4931$.

36. $\dfrac{1}{2}\log\tan\dfrac{x}{2}-\dfrac{1}{3}\log\left(1+\tan\dfrac{x}{2}\right)-\dfrac{5}{12}\log\left(1-2\tan\dfrac{x}{2}\right)$.

37. $\log 1\cdot025=0\cdot0247$

38. $2\log 1\cdot25-0\cdot5+\dfrac{\pi}{2}-2\tan^{-1}0\cdot5=0\cdot5896$. **40.** $0\cdot0965$.

Exercises 6c. (Page 118.)

1. $\dfrac{1}{8}\log\dfrac{4+x}{4-x}$. **2.** $\dfrac{1}{12}\log\dfrac{x-6}{x+6}$. **3.** $\dfrac{1}{9}\tan^{-1}\dfrac{x}{9}$.

4. $\dfrac{1}{10}\log\dfrac{x-7}{x+3}$. **5.** $\dfrac{1}{16}\log\dfrac{11+x}{5-x}$. **6.** $\dfrac{1}{9}\tan^{-1}\dfrac{x+5}{9}$.

7. $\dfrac{1}{72}\log\dfrac{3x-17}{3x+7}$. **8.** $\dfrac{1}{24}\log\dfrac{9+2x}{3-2x}$. **9.** $\dfrac{1}{5}\tan^{-1}(5x+1)$.

10. $\dfrac{1}{20}\log\dfrac{x-5}{x}$. **11.** $\dfrac{1}{42}\log\dfrac{7x}{6-7x}$. **12.** $\dfrac{1}{15}\tan^{-1}\dfrac{5x-1}{3}$.

13. $\dfrac{1}{2ac}\log\dfrac{ax+b-c}{ax+b+c}$. **14.** $\dfrac{1}{ac}\tan^{-1}\dfrac{ax+c}{c}$. **15.** $\dfrac{1}{2ac}\log\dfrac{ax}{2c-ax}$.

16. $\dfrac{1}{2a(2+b^2)}\cdot\log\dfrac{ax-2-2b-b^2}{ax+2-2b+b^2}$. **17.** $\dfrac{5}{2}\log 2+\dfrac{1}{8}\pi=2\cdot1255$.

18. $0\cdot646$ **19.** $6-\frac{3}{2}\pi=1\cdot2876$. **20.** $\frac{3}{2}\log 170+1\cdot4940=9\cdot1977$.

21. $\pi/20=0\cdot15708$. **22.** $0\cdot446$. **23.** $\log 2\cdot08=0\cdot7324$.

24. $0\cdot0004$. **25.** $\frac{1}{10}\log 6=0\cdot17918$.

26. $(a^2-x^2)^{-\frac{1}{2}}$; $x/\{a^2(a^2+x^2)^{\frac{1}{2}}\}$. **28.** $\frac{2}{3}\tan^{-1}\frac{1}{3}=0\cdot2145$.

29. $\frac{1}{15}\log 6=0{\cdot}1194$. **30.** $\frac14(\log 3-2)=-0{\cdot}22535$.

31. $\frac{1}{\sqrt5}\log\frac{\sqrt5+1}{\sqrt5-1}=0{\cdot}861$. **34.** $\pi\operatorname{cosec}\alpha$.

39. $A=3$, $B=-2$, $C=1$; $3\pi/2+\log 2=5{\cdot}4055$.

Exercises 6d. (Page 125.)

1. $8\sin^{-1}\frac{x}{4}+\frac12 x\sqrt{16-x^2}$. **2.** $\sin^{-1}\frac{x}{4}$.

3. $\frac12 x\sqrt{x^2-36}-18\cosh^{-1}\frac{x}{6}$. **4.** $\cosh^{-1}\frac{x}{6}$.

5. $\frac12 x\sqrt{x^2+81}+\frac{81}{2}\sinh^{-1}\frac{x}{9}$. **6.** $\sinh^{-1}\frac{x}{9}$.

7. $\frac12(x-2)\sqrt{x^2-4x-21}-\frac{25}{2}\cosh^{-1}\frac{x-2}{5}$. **8.** $\cosh^{-1}\frac{x-2}{5}$.

9. $32\sin^{-1}\frac{x+3}{8}+\frac12(x+3)\sqrt{55-6x-x^2}$. **10.** $\sin^{-1}\frac{x+3}{8}$.

11. $\frac12(x+5)\sqrt{x^2+10x+106}+\frac{81}{2}\sinh^{-1}\frac{x+5}{9}$. **12.** $\sinh^{-1}\frac{x+5}{9}$.

13. $\frac16(3x-5)\sqrt{9x^2-30x-119}-24\cosh^{-1}\frac{3x-5}{12}$. **14.** $\frac12\sin^{-1}\frac{2x+3}{6}$.

15. $\frac{1}{2a}(ax+b)\sqrt{a^2x^2+2abx+b^2-c^2}-\frac{c^2}{2a}\cosh^{-1}\frac{ax+b}{c}$.

16. $\frac1a\sinh^{-1}\frac{ax+c}{c}$. **17.** $\frac{c^2}{2a}\sin^{-1}\frac{ax-c}{c}+\frac{1}{2a}(ax-c)\sqrt{ax(2c-ax)}$.

18. $\sin^{-1}\frac{x-c}{c}$. **19.** $0{\cdot}5955$. **20.** $\frac{\pi}{2}$. **21.** $0{\cdot}8473$.

22. $\log 3{\cdot}5=1{\cdot}2528$. **23.** $\frac{\pi}{6}$. **24.** $\log 2=0{\cdot}6931$. **25.** $\log 2$.

26. $\frac{\pi}{3}$. **27.** π. **28.** $\log 2$. **29.** $81\pi/16$.

30. $9{\cdot}226$. **31.** $0{\cdot}3107$. **32.** $9\pi/4$. **33.** 4π.

34. π/a. **35.** $119{\cdot}6792$. **36.** $394{\cdot}535$. **38.** 97π.

39. $y=a\sinh x$. **41.** $0{\cdot}9707$. **42.** $y=\log 3x-\log 1+\sqrt{1+x^2}$.

Exercises 6e. (Page 132.)

1. $(nx\sin nx+\cos nx)/n^2$. **2.** $\frac12 e^x(\cos x+\sin x)$.

3. $x(\log x-1)=x\log\frac{x}{e}$. **4.** $x\tan^{-1}x-\frac12\log(1+x^2)$.

5. $\frac12(x^2\tan^{-1}x+\tan^{-1}x-x)$. **6.** $\frac14 x^2(2\log x-1)=\frac14 x^2\log\frac{x^2}{e}$.

7. $\frac{1}{27}\{(2-9x^2)\cos 3x+6x\sin 3x\}$. **8.** $\frac38(2x^2-1)\cos 2x+\frac14 x(2x^2-3)\sin 2x$.

9. $\{x\log x+(1-x)\log(1-x)\}/(1-x)$. **11.** $e^x(4x^2+3)$.

12. $e^x x(x^2 - 3x + 6)$. **13.** $e^x \sin x$. **14.** $e^{\tan^{-1} t} t$.
15. $e^x \sec x$. **16.** $e^x \log x$. **17.** $(ea)^x/\log ae$.
18. $25 - 19e = -26·64$. **19.** $(3e - 4)e^3 = 83·45$.
20. $e^{\sin^{-1} 0·6} \log 2 = 1·319$. **21.** $8/81$.
22. $S_3 = -\frac{1}{3}(2 + \sin^2 x) \cos x$; $S_5 = -\frac{1}{15}(8 + 4 \sin^2 x + 3 \sin^4 x) \cos x$.
23. $C_3 = \frac{1}{3}(2 + \cos^2 x) . \sin x$; $C_5 = \frac{1}{15}(8 + 4 \cos^2 x + 3 \cos^4 x) \sin x$.
24. $T_3 = \frac{1}{2} \tan^2 x + \log \cos x$. **25.** $4/3$.
26. $I_3 = 3(x^2 - 2) \sin x - x(x^2 - 6) \cos x$. **27.** $0·5$. **36.** $\pi - 2$.

Exercises 6f. (Page 138.)

1. $8/315 = 0·0254$. **2.** $16/35 = 0·4571$. **3.** $3\pi/16 = 0·5889$.
4. $1/120 = 0·0083$. **5.** $\pi/4$. **6.** $2/35 = 0·05714$.
7. $2/35 = 0·05714$. **10.** $4/35 = 0·11428$. **11.** $4/63 = 0·06351$.
12. $8/15 = 0·5333$. **13.** $-1/6 = -0·1667$. **14.** $3/5 = 0·6$.
15. $\frac{1}{4}\pi - \frac{76}{105} = 0·2105$. **16.** $3\pi/256 = 0·03682$. **17.** $1/60 = 0·0167$.
18. $107\sqrt{2}/10080 = 0·0015$. **19.** $2·1972$. **20.** $0·1567$.
21. $0·4003$. **22.** $\frac{1}{3} \log (\sqrt{8} + 3) = 0·5876$. **23.** $\pi/6$.
24. $\tan^{-1} 2\sqrt{\frac{9-x}{4x-9}} - 2\tan^{-1}\sqrt{\frac{9-x}{4x-9}}$.
25. $A = 6/17$, $B = -3/17$, $C = 4/17$.

$$\text{Integral} = \frac{1}{34}\left\{3 \log \frac{(2x-3)^2}{x^2+2} + 4\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}}\right\}.$$

26. 2π. **27.** $0·04 (\log 28 - \log 3) = 0·089$.
28. $64/45 = 1·42$. **29.** $\frac{5}{2}\pi$. **31.** $\frac{1}{a^2+b^2} \cdot \sqrt{\frac{b^2+x^2}{a^2-x^2}}$; 3.
33. (a) $\log 2$; (b) $7 5 + 8 \log 2 = 1·955$. **34.** π/ab.
35. $a^2\left(2 - \frac{\pi}{2}\right)$. **36.** $\frac{5}{2}a^2\pi$. **40.** $3\pi/128$.
41. $\frac{2}{\sqrt{n}} \tan^{-1}\sqrt{\frac{n(a-x)}{nx-a}} - \tan^{-1}\sqrt{\frac{a-x}{nx-a}}$.
42. $-\frac{\sqrt{2}}{2} \sin^{-1}\sqrt{\frac{1-x^2}{1+x^2}}$. **44.** $\frac{\pi}{\sqrt{8}} = 1·11$.
45. $\frac{\sqrt{3}}{12} \log (2 - \sqrt{3})$. **46.** $0·4236$.

Exercises 7. (Page 151.)

1. $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \dots$. **2.** $\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \dots$.
3. $\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \dots$. **4.** $e^x \sin x = x + x^2 + \frac{2x^3}{\underline{|3}} - \frac{4x^5}{\underline{|5}} - \dots$.

5. $e^x \cos x = 1 + x - \frac{2x^3}{\lfloor 3} - \frac{4x^4}{\lfloor 4} - \ldots$ **6.** $\sin x \cosh x = x + \frac{x^3}{3} - \frac{x^4}{30} - \frac{11x^6}{5040} + \ldots$

12. $\log(1+x) = x - x^2/2 + x^3/3 - \ldots$, $\log 2 = 0{\cdot}6931$.

13. $\log \frac{4}{3} = 0{\cdot}2877$. **16.** $\cos^{-1}x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112} - \ldots$

17. $\sinh^{-1}x = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} - \ldots$

18. $\tanh^{-1}x = x + x^3/3 + x^5/5 + \ldots$ **19.** $(\sin^{-1}x)^2 = x^2 + \frac{x^4}{3} + \frac{8x^6}{45} + \ldots$

20. $\sin(x+h) = \sin x + h\cos x - \frac{1}{2}h^2 \sin x - \frac{1}{6}h^3 \cos x + \ldots$

36. $A_0 = 1$, $A_2 = 1/12$, $A_4 = -1/720$.

Exercises 8. (Page 175.)

1. $x = 1{\cdot}501$. **2.** $x = 1{\cdot}184$. **3.** $x = 3{\cdot}5, 2{\cdot}5, -6$.

4. $x = 1 + \sqrt{13} = 4{\cdot}6056$, or $\frac{1}{2}(-1 - \sqrt{13} \pm \sqrt{\sqrt{13} - 3}) = -1{\cdot}9137$, or $-2{\cdot}6919$.

5. $x = 0{\cdot}2502, 1{\cdot}343, -1{\cdot}471$. **6.** $x = 2{\cdot}35y$; $x = 96{\cdot}35$, $y = 41$.

7. $y = -1{\cdot}195, 0{\cdot}0895, 0{\cdot}3522$; $x = 2{\cdot}34$.

8. $f(x+2) = x^4 - 25x^2 + 144$, $x = -1, -2, 5, 6$.

9. $x = -1, 2, -2 \pm \iota$. **10.** $x = 0{\cdot}8284, -4{\cdot}827$.

11. $x = 1, 3$.

12. $x = 128° 41' = 2{\cdot}246$ radians, or $201° 38' = 3{\cdot}5192$ radians.

13. $x = 0{\cdot}8826$ radian. **14.** $x = 6{\cdot}186$.

15. $x = 0{\cdot}200064, 1{\cdot}44036, -1{\cdot}5419$. **16.** $c = 4$.

17. $x = 1{\cdot}042$. **18.** $x = 1{\cdot}508$. **19.** $x = 1{\cdot}44$.

20. $\theta = 65° 30' = 1{\cdot}1432$ radians; Chord $= 5{\cdot}409$ in.

21. $\theta = 48° 55'$. **22.** $x = 7{\cdot}08$. **23.** $x = 0{\cdot}1047$ radian.

24. $x = 0{\cdot}11$ radian. **25.** $\beta = 54° 35{\cdot}5'$.

26. $\angle EOF$ is the solution of $\theta + \sin\theta + 2\cos\theta = 2$, which is $118° 9'$; Radius $= 5{\cdot}829$ in.

27. $x = 2{\cdot}84$.

28. If θ be the angle subtended at the centre of the circular end by a horizontal chord drawn through the reading "n-hundred gallons," then $\theta - \sin\theta = 0{\cdot}08n$, from which the graph may be drawn. For the reading x ft. from the lower extremity of the vertical diameter, $x = 5\left(1 - \cos\frac{\theta}{2}\right)$. For 500 gallons, $\theta = 79° 12'$ and $x = 1{\cdot}1475$ ft.

29. $x = 1{\cdot}4429$. **30.** $x = 0{\cdot}4578$ or $3{\cdot}314$.

31. $x = 1 \pm \sqrt{6} = 1{\cdot}449$, or $-3{\cdot}449$. **32.** $x = 0{\cdot}259$ or $3{\cdot}26$.

33. $x = 3{\cdot}4563$.

34. $x = -0{\cdot}5858$ or $-3{\cdot}4142$; $x = -0{\cdot}7913$ or $3{\cdot}7913$.

35. $x = 2, -1 \pm \sqrt{3} = 0{\cdot}732$ or $-2{\cdot}732$.

36. $x = 0$ or $0{\cdot}63$. **37.** $x = 0{\cdot}125$.

Exercises 9. (Page 191.)

Equational laws are only approximate.

1. $K_p = 0{\cdot}124 + 0{\cdot}00005T$. 2. $T = 40h^{1{\cdot}46}$, $V = 1400h^{1{\cdot}46}$.
3. $P = 31v^{1{\cdot}77}$. 4. $n = 1{\cdot}065$, $c = 480$. 5. $W = 16K + 4300$.
6. $n = 1{\cdot}473$; $C = 1{\cdot}115$. 7. $A = 10$, $b = 0{\cdot}273$. 8. $a = 2{\cdot}6$, $b = 30{\cdot}2$.
9. $a = 1{\cdot}758 \times 10^{-4}$, $b = -0{\cdot}18$.
10. $a = 3{\cdot}11$, $b = 1{\cdot}064 \times 10^{-2}$, $c = -2{\cdot}75 \times 10^{-6}$.
11. $Q = 8{\cdot}3t^2 + 2{\cdot}64 \times 10^c$ 12. 1532 lb.; $\pm 20{\cdot}3$, $\pm 1{\cdot}325$. 13. 4·47%.
14. $\phi = 0{\cdot}1779$ or $0{\cdot}1774$ by approx. formula; Percentage error $= 0{\cdot}281$.
15. $\partial k = 5{\cdot}084 \times 10^{-3}$; k reliable to nearest tenth.
16. $(2 \sin A \sin C \,.\, \partial b + a \sin A \,.\, \partial C + c \sin C \,.\, \partial A)/(b \sin A \sin C)$.
17. 0·13464.
18. 329·8 sq. in.; True area $= 329{\cdot}5$ sq. in.; Percentage error $= +0{\cdot}09105$.
20. $-0{\cdot}04527$.

Exercises 10. (Page 233.)

1. 70. 2. $\frac{1}{2}a(2b - a)$; Points are collinear.
3. $2y = 5x + 7$. 4. $a = 0$ or 3; $x = 0$ or $5x - 3y + 9 = 0$.
5. $PQ = QT = 25\sqrt{2}/2$; $\tan PQT = 4/3$.
6. $x = (lx_2 + mx_1)/(l + m)$, $y = (ly_2 + my_1)/(l + m)$; 17 : 49; 20 : 13.
7. $-3/2$. 9. $x = \alpha + r \cos \theta$, $y = \beta + r \sin \theta$.
10. $\tan^{-1}(167/12) = \tan^{-1} 13{\cdot}92 = 85° \; 53'$ approx.
11. $\tan^{-1}\{2\sqrt{h^2 - ab}/(a + b)\}$.
12. Two lines are $x - 7y + 2 = 0$, $2x + y - 3 = 0$; Angle between them $= \tan^{-1} 3 = 71° \; 34'$; Parallel lines through (1, 2) are
$$2x^2 - 13xy - 7y^2 + 30x - 15y + 52 = (x - 7y + 13)(2x + y + 4) = 0.$$
13. $12x - y - 31 = 0$. 14. $a(y - k) = b(x - h)$.
19. Two straight lines, $3x = 2$, $3y = -7$.
20. An ellipse; centre at origin, semi-axes 1·6, 1·2.
21. Two straight lines, $3x + y - 1 = 0$, $2x - y + 3 = 0$.
22. Two parallel straight lines, $3x + 4y + 2 = 0$, $3x + 4y + 1 = 0$.
23. A parabola whose axes are $x + 3y - 4 = 0$, $3x - y - 2 = 0$, and latus rectum 2.
24. A parabola whose axes are $3x + 4y - 25 = 0$, $4x - 3y = 0$, and latus rectum 1.
25. A parabola whose axes are $15y - 8x + 3 = 0$, $15x + 8y + 2 = 0$, and latus rectum 4/17.
26. Two straight lines, $2x + y - 10 = 0$, $2x + y = 0$.
27. A hyperbola; centre $(-2, -6)$, squares of semi-axes, $48(\sqrt{5} + 2)$, $48(\sqrt{5} - 2)$.
28. A hyperbola; centre $(1, -3)$, semi-axes, $\sqrt{2}$, $3\sqrt{2}$.
29. (*a*) Two parallel straight lines, $x + 2y - 1 = 0$, $x + 2y + 3 = 0$.
(*b*) An ellipse; centre $(-1, -0{\cdot}5)$, semi-axes 1, 1·5.

31. $5x^2 - 2xy + 5y^2 - 20x - 20y + 15 = 0.$ **35.** $y = 2x.$

36. If a, b be the radii of large and small circles respectively, the ellipse referred to the centre of the larger circle as origin and the line of centres as x-axis, is $4x^2/(a+b)^2 + 4y^2/\{(a+b)^2 - d^2\} = 1$, where d is the distance between the centres. Its eccentricity $= d/(a+b)$.

39. $10x^2 + 3y^2 = 187$; $\{0, \pm\sqrt{(1309/30)}\}$; $20x + 21y = 0.$

42. $x^2 + 9y^2 = r^2$; $e = 2\sqrt{2}/3 = 0{\cdot}9428.$ **43.** $e = \sqrt{3}/2.$

44. $x^2/a^2 + (2xy/ab)\cos\epsilon + y^2/b^2 = \sin^2\epsilon.$

(*a*) Two straight lines, (*b*) Circle of radius a.

45. $ay = (2x^2 - a^2)\cos\epsilon - 2x\sqrt{a^2 - x^2}\,.\,\sin\epsilon.$

47. Locus is $4x^2/b^2 - 4y^2/(a^2 - b^2) = 1$ referred to BD as x-axis and its mid-point as origin.

49. $1/r = k + A\cos(\theta + a)$; $A < k$.

51. $V = 170$ ft. per sec.

Exercises 11. (Page 261.)

1. $(x - 3)^2 + (y - 4)^2 = 10.$

2. $g^2 + f^2 > c$; $\sqrt{g^2 + f^2 - c}$; $(-g, -f)$; $3x^2 + 12xy + 8y^2 - 24x - 24y + 12 = 0$; $2\sqrt{3}.$

5. $y = mx + a/m$; $y = mx - 2am - am^3.$

12. $a = 10$, $b = 8$; $(-150/17, 64/17).$

15. $101x^2 + 48xy + 81y^2 - 330x - 324y + 441 = 0$; $m = 4(-2 \pm \sqrt{13})/27 = 0{\cdot}2379$ or $-0{\cdot}8305$; $\{1, 2(23 \pm 2\sqrt{13})/27\}.$

24. $(0{\cdot}2, 0{\cdot}4).$

26. $x\eta^{\frac{1}{3}} + y\xi^{\frac{1}{3}} = (a^2\xi\eta)^{\frac{1}{3}}$; $(x - \xi)\xi^{\frac{1}{3}} - (y - \eta)\eta^{\frac{1}{3}} = 0.$

28. $2y = m(3x - am^2)$; $m = 2\sqrt{2/3}.$

32. When $ab - h^2$ is not zero, the locus is the director circle of the conic. When $ab - h^2 = 0$, the conic is a parabola and the locus is its directrix.

33. $(1 - 2t^3)y = t(2 - t^3)x - 3at^2$;

$(1 - 2t^3)x + t(2 - t^3)y + 3at(1 - t)(1 + t + 3t^2 + t^3 + t^4)/(1 + t^3) = 0.$

34. $\theta = \tan^{-1}(\pm 2\sqrt{6}) = 78^\circ\,30'$ or $101^\circ\,30'$ approx.

Length of common chord $= 8\sqrt{6}/5 = 3{\cdot}9192.$

35. $(0, 0)$; $(2a, 4a)$; 90° ; $30^\circ\,58'$. **36.** $16^\circ\,19'$.

37. $(0, 0)$, 90° ; $(2a^{\frac{1}{2}}c^{\frac{1}{2}}, \pm 8^{\frac{1}{2}}a^{\frac{1}{4}}c^{\frac{1}{4}})$, $\tan^{-1}\{8^{\frac{1}{2}}a^{\frac{1}{4}}c^{\frac{1}{4}}/(2c^{\frac{1}{2}} + 3a^{\frac{1}{2}})\}.$

39. Tangents at $(1{\cdot}8, 2{\cdot}4)$ are : $3x + 4y = 15$, $3y - 4x = 0.$

41. At $(\pm 8, \pm 3)$, $2x + 3y = \pm 25$, $3x + 8y = \pm 48.$

At $(\pm 6, \pm 4)$, $3x + 8y = \pm 50$, $2x + 3y = \pm 24.$

Angle of intersection $= 13^\circ\,8'$.

43. $c = 0$ or 4.

44. (i) $t=200 \sin 65°/g=5{\cdot}665$ sec.

(ii) Taking horizontal and vertical lines through point of projection as axes, $x=2{\cdot}555$ ft. and $954{\cdot}8$ ft., $y=4{\cdot}018$ ft.

(iii) Range $=957{\cdot}4$ ft.

45. Common point $=(2, -1)$; Tangent, $11y=4x-19$.

Exercises 12. (Page 296.)

1. $\rho=a$. **2.** $\rho=5\sqrt{5p}/2$. **3.** $\rho=\operatorname{cosec} x$.

4. $\rho=x^{\frac{1}{3}}(9x^{\frac{4}{3}}+4a^{\frac{4}{3}})/54a^{\frac{1}{3}}$.

5. $\rho=3(axy)^{\frac{1}{3}}$. **6.** $\rho=\frac{1}{6}(a^4+9x^4)^{\frac{3}{2}}/a^4x$. **7.** $\rho=\frac{1}{6}x^{\frac{1}{2}}(4a+9x)^{\frac{3}{2}}/a$.

8. $\rho=\frac{1}{2}(x+a)^2/a$. **9.** $\rho=ax^{\frac{1}{2}}(8a-3x)^{\frac{3}{2}}/\{3(2a-x)^2\}$.

10. $\rho=a(4+m^2)^{\frac{3}{2}}/2m^3$. **11.** $\rho=a\cot\theta$.

12. $70° \; 32'$ and $109° \; 28'$; $\rho=a/4$.

15. (i) $b^2\pm ap\sqrt{3}=0$.

(ii) $\left\{\dfrac{p(b^2-2a^2)+2ac}{b^6+2ap(b^2-a^2)(ap-c)}\right\}^3=\dfrac{p}{a^2b^{14}}$, where $c^2=a^2p^2+b^4$.

16. $P(\pm\sqrt{6{\cdot}4}, \pm\sqrt{0{\cdot}6})$; $\rho_1=4\sqrt{1{\cdot}5}/3=1{\cdot}633$, $\rho_2=8{\cdot}8\sqrt{2{\cdot}2}=13{\cdot}05$.

17. $a^2=b^2$; $\rho_1=-(a^4+4c^4)^{\frac{3}{4}}/a^2$; $\rho_2=(a^4+4c^4)^{\frac{3}{4}}/2c^2$; $a^2+2c^2=0$.

18. $\xi=5a$, $\eta=-2a$.

19. At $(0, 0)$, $\rho=\frac{1}{2}c$; at $(3c, 0)$, $\rho=\frac{1}{2}c(9+c^2)^{\frac{3}{2}}/(3c-c^2-3)$.

21. The length of the normal is $y\sqrt{1+y_1^2}$, and $\rho=(1+y_1^2)^{\frac{3}{2}}/y_2$; equating these and multiplying by $2y_1$: $2y_1y_2/(1+y_1^2)=2y_1/y$, the integral of which is $1+y_1^2=a^2y^2$. This on second integration gives

$$ay=\cosh(ax+b),$$

the equation of the required curve.

22. $\rho=-2\sqrt{2}/3=-0{\cdot}9428$. **23.** $\rho=\frac{1}{2}a$.

26. $\rho=2\sqrt{ar}/3$. **27.** $\rho=r\operatorname{cosec}\alpha$. **28.** $\rho=a\sqrt{2}\,.\operatorname{cosec}^{\frac{3}{2}}2\theta$.

29. $A=a^2$, $B=-4b(a-b)/(a-2b)^2$. **30.** $r^3=4ap^2$.

47. 62 ft. 4 in.

Exercises 13. (Page 337.)

1. $(x+2)/5=(y-1)/0=(z-3)/(-5)$. $y=1$.

2. $x+2y=0$. $7\sqrt{5}/5$.

3. $l_1l_2+m_1m_2+n_1n_2=0$. $x/(-7)=y/11=z/(-10)$.

4. $l^2+m^2+n^2=1$. $x+y\sqrt{2}+z=0$.

6. $x+2y+4z-7=0$. P is $(3, -4, 3)$. $\cos QPR=0{\cdot}7044$.

7. $\cos\theta=l_1l_2+m_1m_2+n_1n_2$. Equations of faces are $h\sqrt{2}(x\pm y)\pm sz=0$. Edges are $x\sqrt{2}/(\pm s)=y/0=z/(-h)$, and $x/0=y\sqrt{2}/(\pm s)=z/(-h)$. $\theta=\cos^{-1}\{(2h^2-s^2)/(2h^2+s^2)\}$, where $s=$ side of square base and $h=$ height $=\frac{1}{2}s\sqrt{\cot^2\frac{1}{2}\alpha-1}$.

8. $x-4y+2z+4=0$.

9. $12x-4y+3z=0$. $PN=13$. $-2/ON$, $-1/ON$, $20/(3ON)$, where $ON=\sqrt{445}/3$.

10. $a^2+b^2+c^2-(al+bm+cn)^2$. **11.** $(x-1)/8=(y-1)/12=(z-1)/13$.

13. 2.

14. (i) $3x-5y-7z+23=0$. (ii) $23/\sqrt{83}$. (iii) $3/\sqrt{83}$, $-5/\sqrt{83}$, $-7/\sqrt{83}$. (iv) $-23/3$, $23/5$, $23/7$.

15. $\sqrt{6}/27$, $-19\sqrt{6}/54$, $11\sqrt{6}/54$. $\sin^{-1}\sqrt{11/29}=38°\ 1'$. $\sin^{-1}\sqrt{8/11}=46°\ 39'$.

18. $(2, 1, -3)$. $x+y+z=0$.

19. $4/13$, $-3/13$, $-12/13$. $(7, 4, -10)$, $(-1, 10, 14)$.

21. $2\sqrt{5}/5$. $126°\ 36'$.

22. Taking $(x-\alpha_1)/l_1=(y-\beta_1)/m_1=(z-\gamma_1)/n_1$; $(x-\alpha_2)/l_2=(y-\beta_2)/m_2=(z-\gamma_2)/n_2$ as the equations of the lines, the condition that they should intersect is

$$\begin{vmatrix} l_1 & l_2 & \alpha_1-\alpha_2 \\ m_1 & m_2 & \beta_1-\beta_2 \\ n_1 & n_2 & \gamma_1-\gamma_2 \end{vmatrix}=0.$$

The surface is $$\begin{vmatrix} 2z-y & x-a & 2(a-x) \\ 2(y-a) & z-2x & a-y \\ z-a & 2(a-z) & 2y-x \end{vmatrix}=0,$$

i.e. $2a(x^2+y^2+z^2)+9a(xy+yz+zx)-9a^2(x+y+z)+7a^3=14xyz$.

23. $6\frac{1}{3}$. **24.** 28.

25. $x-\alpha=y-\beta=z-\gamma$, where (α, β, γ) is any arbitrary point on the line.

26. $6\sqrt{5}$.

27. $x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+x_1x_2+y_1y_2=0$,
$(fn-hl)^2+(gn-hm)^2-n(fn-hl)(x_1+x_2)-n(gn-hm)(y_1+y_2)+(x_1x_2+y_1y_2)n^2=0$.

28. The line is $(x-1)/-3=(y+1)/2=z/13$; its direction cosines are $-3/\sqrt{182}$, $2/\sqrt{182}$, $\sqrt{182}/14$; $1°\ 8'$.

29. $AD=32{\cdot}09$ ft., $CD=27{\cdot}48$ ft., $BD=29{\cdot}32$ ft.
$\angle ADB=34°\ 17'$, $\angle ADC=41°\ 33'$, $\angle CDB=41°\ 16'$.

30. D is 24/7 in. from vertex.

31. Take A as $(1, 2, 3)$, B, $(0, 0, 2)$, C, $(0, 0, 1)$, D, $(1, 0, 0)$, then
$AB=\sqrt{6}$, direction cosines are $1/\sqrt{6}$, $2/\sqrt{6}$, $1/\sqrt{6}$.
$AC=3$, „ „ „ $1/3$, $2/3$, $2/3$.
$AD=\sqrt{13}$, „ „ „ 0, $2/\sqrt{13}$, $3/\sqrt{13}$.
$CD=\sqrt{2}$, „ „ „ $-1/\sqrt{2}$, 0, $1/\sqrt{2}$.
$BC=1$, „ „ „ $0, 0, 1$.
$BD=\sqrt{5}$, „ „ „ $1/\sqrt{5}$, 0, $2/\sqrt{5}$.
The edges are: ABC, $2x-y$; ABD, $4x-3y+2z=4$; ACD, $2x-3y+2z=2$; BCD, $y=0$.
Volume $=1/3$.

32. $82° 48'$. **34.** $\lambda = 3$.

35. $x^2+y^2+z^2-2r(x+y+z)+2r^2=0$.
$(2-r)x+(1-r)y+(3-r)z=14-6r$, where $r=3\pm\sqrt{2}$.

36. $317\pi/24$.

37. $r=\sqrt{14}=3{\cdot}742$, $u=\sqrt{10}=3{\cdot}162$, $\theta=\tan^{-1}\sqrt{10}/2=57° 42'$,
$\phi=\tan^{-1}3=71° 34'$. $x^2+y^2+z^2-4y-2z=4$.
$r^2-4r\sin\theta\sin\phi-2r\cos\theta=4$. $u^2+z^2-2z-4u\sin\phi=4$.
$(2x-7)^2+4y^2+(2z-1)^2=50$.

38. (1) Ellipsoid, (2) Elliptic cylinder, (3) Hyperboloid of one sheet, (4) Rectangular hyperboloid.

39. $(x+19)/2=(y+56)/5=z$.
$$\sqrt{\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-l(x-\alpha)-m(y-\beta)-n(z-\gamma)\}}.$$
The cone is $(x-1)^2+(y-3)^2+(z-2)^2-12(x-1)-16(y-3)-2(x-2)=0$.

40. An ellipsoid. $p^2=5(4l^2+3m^2+12n^2)/3$.
Points of contact $(\pm 2, \pm 1, \pm 2)$.

42. $afl+bgm+chn=(al^2+bm^2+cn^2)(af^2+bg^2+ch^2-1)$;
$$(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)+(alx+bmy+cnz)^2=0.$$

43. Let $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ be the given axis, and a the radius of the cylinder. The sphere whose centre is (α, β, γ) and radius a has for its equation $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=a^2$. Let (x', y', z') be any point on a line parallel to the axis, then any other point on this line is given by $x=x'+lr$, $y=y'+mr$, $z=z'+nr$. This will meet the sphere where $(x'+lr-\alpha)^2+(y'+mr-\beta)^2+(z'+nr-\gamma)^2=a^2$, *i.e.*
$$r^2+2r(lx'+my'+nz'-\alpha l-\beta m-\gamma n)+(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2-a^2=0.$$
The line will therefore be tangent to the sphere if the roots of this quadratic are equal, *i.e.* if
$$(lx'+my'+nz'-\alpha l-\beta m-\gamma n)^2=(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2-a^2.$$
This equation will therefore represent the locus of all tangent lines to the sphere parallel to the given line, *i.e.* it is the enveloping cylinder.

(1) For the axis given in the question $\alpha=1$, $\beta=0$, $\gamma=3$, $l=2/\sqrt{6}$, $m=1/\sqrt{6}$, $n=-1/\sqrt{6}$; hence the equation of the cylinder becomes
$$(2x+y-z+1)^2/6=(x-1)^2+y^2+(z-3)^2-4,$$
or $$2x^2+5y^2+5z^2-4xy+2yz+4zx-16x-2y-34z+35=0.$$
(2) $2\sqrt{4\pm\sqrt{10}}=5{\cdot}35$ and $1{\cdot}83$.

44. $(5x+4y+2z-15)^2/45=(x-3)^2+(y-1)^2+(z+2)^2-4$.
Tangent plane is $lx+my+nz=0$, where $l=10(2\alpha-2\beta-\gamma-6)$,
$m=-20\alpha+29\beta-8\gamma+15$, $n=2(-10\alpha-8\beta+41\gamma+120)$, (α, β, γ) being the point of contact.

46. $4\pi\sqrt{33}/19$. **47.** $107 : 125$.

48. Semi-axes of ellipse are $\sqrt{(4\pm\sqrt{3})/13}$; eccentricity $=0{\cdot}7775$.

49. $xx'/a^2+yy'/b^2+zz'/c^2=1$. To solve the second part of the question, let $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n=r$ be any chord of the quadric $ax^2+by^2+cz^2=1$, then the chord intersects the quadric where

$$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1,$$

i.e. $r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)+a\alpha^2+b\beta^2+c\gamma^2-1=0.$

The roots of this quadratic will be equal and opposite if

$$a\alpha l+b\beta m+c\gamma n=0,$$

in which case (α, β, γ) will be the middle point of the chord. Eliminating l, m, n by means of the equations of the line,

$$a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)=0,$$

which is the plane containing all chords bisected at (α, β, γ). In the question $\alpha=1$, $\beta=1$, $\gamma=1$, $a=1/4$, $b=1/9$, $c=1/16$, so that the plane required becomes $36x+16y+9z=61$.

50. $12x+3y+8z=\pm 36$. $32x^2+8xy+5y^2=64$.

51. $(5\sqrt{6}/3, \pm\sqrt{3}/3, 0)$; Diameter $=4{\cdot}8$.

53. $3x+y+z=\sqrt{22}-4$. $3x+y+z-3\sqrt{11}+15=0$.

54. $lx/a^2+my/b^2+nz/c^2=0$; $x/3=-y/4=z/0$; $x/16=y/4=z/19$.

Exercises 14a. (Page 352.)

1. $s=98{\cdot}12$, $A=1440$. **2.** $609/240=2{\cdot}554$, $343/160$.
3. $s=10{\cdot}33$. **5.** $A=16\pi$. **6.** $A=9\pi$. **8.** $A=100$.
9. $A=3$. **10.** $A=57{\cdot}16$. **11.** $s=8r$, $A=3\pi r^2$.
12. $s=8b(a+b)/a$. **13.** $s=8a$. **14.** $s=1{\cdot}5a$. **15.** $s=24$.
16. $s=2a(3\sqrt{3}-1)$. **17.** $s=(a^3-b^3)/ab$. **19.** $s=\log 3{\cdot}2=1{\cdot}1632$.
20. $s=12{\cdot}21$. **22.** $s=2a\{\sqrt{2}+\log(\sqrt{2}+1)\}$; $22{\cdot}95$.
23. $s=2{\cdot}4141$. **24.** $A=62{\cdot}8$ sq. in.
25. (1) $1{\cdot}435\times 10^4$; (2) $1{\cdot}65\times 10^4$. **26.** $a=-3$, $b=9$.
27. $A=\pi ab$, $b=7$. **28.** $\omega=\pi/4b$, $A=0{\cdot}109$. **29.** $s=49\pi=153{\cdot}9$.
30. $s=1{\cdot}015$. **31.** $A=\frac{1}{2}a^2$. **32.** $A=\frac{1}{12}\pi a^2$ **33.** $A=208$.
34. $A=14\sqrt{3}/5$. **37.** $3a^4\pi/16b^2$. **40.** $128a^2/15$. **41.** $A=20$.

Exercises 14b. (Page 362.)

1. $a=3{\cdot}036$, $b=0{\cdot}1423$, $V=1617$. **2.** $16\pi=50{\cdot}24$.
3. $61\pi/1728=0{\cdot}1109$. **4.** $a=1{\cdot}32$, $b=0{\cdot}5$, $V=37{\cdot}5327\pi$.
5. $V=2\pi ah^2$. **7.** $V=2\pi(15-16\log 2)$.
8. $V=\frac{1}{6}h(2ab+a\beta+\alpha b+2\alpha\beta)$, 4750 gallons. **9.** $5180\pi/3$.
10. $S=\frac{64}{3}\pi r^2$, $V=5\pi^2r^3$.
11. $S=4\pi r(r\sin\alpha-r\alpha\cos\alpha+d\alpha)$, (*a*) $S=4\pi r^2(\sin\alpha-\alpha\cos\alpha)$,
(*b*) $S=4\pi r^2\sin\alpha$. (*c*) $S=4\pi r^2$.
12. $S=2\pi r^2(\pi-2)$. **13.** $V=1642$ cu. ft. **14.** $V=334\pi/3$.
15. $r=12{\cdot}5$ in.; $V=8{\cdot}746$ gallons.

16. $V=\frac{1}{3}\pi(2r^3-3r^2d+d^3)$, (a) $V=\frac{28}{81}\pi r^3$, (b) $V=\frac{1}{6}\pi h(h^2+3a^2)$.
14·675 gallons.

17. $V_1=3889\frac{2}{7}$ cu. ft. $r=10{\cdot}5$ ft.

20. $S_1=74{\cdot}9\pi$, $S_2=86{\cdot}968\pi$.

21. $V=216\pi$.

22. Area $=384{\cdot}8$ sq. ft.

24. $V=\frac{2}{3}\pi a^3(14+3\pi)$. Side $=3$.

25. $V=2304\pi=7234{\cdot}56$.

26. (a) $AC=11{\cdot}25$ ft., $CD=7$ ft., $AB=13{\cdot}25$ ft. ; (b) $S=599{\cdot}5$ sq. ft.

27. $V=\frac{1}{2}\pi r^2(h+c)$.

28. $\frac{1}{2}\pi r\{\sqrt{4r^2+(h-c)^2}+2(r+h+c)\}$. Area $=2{\cdot}5\pi$.

29. $\frac{1}{3}\sqrt{\frac{r+b}{r-b}}\{(2r^2+b^2)(h-a)-3ab(r-b)\}+\frac{r^2(ar-bh)}{r-b}\sin^{-1}\frac{\sqrt{r^2-b^2}}{r}$,
2·552 cu. ft.

30. $r=5{\cdot}5$ in.

31. $S=2\pi c^2\left(1-\frac{1}{e}\right)$, $V=\frac{1}{2}\pi c^3\left(e+\frac{5}{e}-4\right)$.

32. $S=16r^2$, $V=\frac{16}{3}r^3$.

33. $V=\frac{4}{3}\pi abc$; radius $=6$.

34. $V=216\pi$.

35. $S=128(\sqrt{2}+1)\pi a^2/1215$, $V=128\pi a^3/5103$.

36. $V=\frac{1}{4}\pi a^3$.

37. 3·3 cu. in.

Exercises 14c. (Page 372.)

1. 1070.
2. 96 cu. ft.
3. 179·5.
4. 3·65 sq. chains.
5. 756 sq. ft.
6. 1·94.
7. 972 sq. ft.
8. 33,420 cu. in.
9. 174,498 cu. ft.
10. 20,448 cu. in.
11. 32,857 cu. in.
12. 1105·12 cu. ft.
14. 1·000043.
15. 1360 cu. ft.
16. (a) 341·48 ; 341·3. (b) 341·33.

Exercises 15a. (Page 381.)

1. $x=2{\cdot}44$, $y=1$.
2. $\bar{x}=1$, $\bar{y}=1$.
3. $x=2{\cdot}56$, $y=2{\cdot}2$.
4. $\bar{x}=1{\cdot}12$, $y=1{\cdot}6$.
5. $x=2{\cdot}4$, $y=0{\cdot}05$; $(-48, -1)$.
6. $x=1\frac{1}{4}$ ft., $\bar{y}=\frac{15}{16}$ ft.
7. At the centre of hexagon.
8. $x=9$ in., $y=6$ in.
9. $\bar{x}=\frac{1}{14}(8+7\sqrt{2})b$, $\bar{y}=\frac{3}{7}(1+\sqrt{2})b$, where b is the side.
10. 3·5 lb.
11. Mid-point of AD.
12. $DG=2\frac{1}{3}$ in.
13. 3·5 in.
14. $PX=44$ in., $y=25{\cdot}12$.
16. $y=\frac{3}{32}\pi a$.
17. $x=\frac{2r\sin\frac{1}{2}\theta}{\theta}$.
18. $y=\frac{4r}{3\pi}$.
19. $\bar{y}=\frac{1}{2}r$.
20. $\bar{y}=\frac{3}{8}r$.
21. $x=y=\frac{4r}{3\pi}$.
22. $\bar{x}=4a/3\pi$.
23. $x=3h/5$.
24. $y=\frac{1}{4}h$ (from base).
25. $y=\frac{a+2b}{3(a+b)}\cdot h$.
26. 0·1 in. from centre.
27. 0·45 in. along axis from flat end of cylinder.
28. $\bar{y}=\frac{2}{3}r\cdot\frac{\sin\phi}{\phi}$, $\bar{y}=\frac{4}{3}r\cdot\frac{\sin^3\phi}{2\phi-\sin 2\phi}$, $y=\frac{4r}{3\pi}$.
29. $y=\frac{5}{6}r$.
30. $y=\frac{a^2}{2h}$, $\bar{y}=\frac{r}{2}$.
31. $x=\frac{46}{99}a=0{\cdot}4647a$, $\bar{y}=\frac{35}{66}a=0{\cdot}5304a$.

32. $r=7$ in. **34.** $\bar{x}=\frac{3}{8}a$, $\bar{y}=\frac{3}{8}b$, $\bar{z}=\frac{3}{8}c$. **35.** $\bar{y}=\frac{(e^4+4e^2-1)c}{4e(e^2-1)}$.

36. (1) $\bar{y}=\frac{1}{3}h$. (2) $\bar{y}=\frac{sh}{3(s+r)}$, where $s^2=r^2+h^2$. **37.** 0·11 cm.

38. $y=45$ in., $\bar{x}=14$ in., $\bar{y}=16{\cdot}5$ in. Length of strip $=27$ in. **39.** 20·5 in.

40. (*a*) $\frac{1}{4}$ height, (*b*) $\frac{1}{4}$ height. **41.** $x=20$ in., $\bar{y}=4{\cdot}8$ in.

42. $x=16/15\pi$, $y=128a^3/105b^2\pi$.

43. Radius of section $=\{(a-b)y+bh\}/h$, when not bearing load or $e^{\frac{1}{2}wy/f}\cdot\sqrt{W/\pi f}$ when bearing load W; $y=\frac{1}{4}(3a^2+2ab+b^2)h/(a^2+ab+b^2)$, where a, b are the radii of the larger and smaller ends respectively.

44. $x=\frac{1}{4}m/h$, $y=0$, $\bar{z}=\frac{1}{8}(m^2a^2+4h^2)/h$, where $m=\tan\theta$; 1·76 ft.

45. $A=2a^{\frac{1}{2}}b^{\frac{3}{2}}/3$; $\bar{x}=3b/5$, $y=3a^{\frac{1}{2}}b^{\frac{1}{2}}/8$; $V=4\pi a^{\frac{1}{2}}b^{\frac{5}{2}}/5$.

48. $A=0{\cdot}8459$; $x=0{\cdot}5126$. **49.** 109·2 lb.

51. $x=\frac{2}{3}a/(\pi-2)$, $\bar{y}=\frac{2}{3}b/(\pi-2)$.

Exercises 15b. (Page 395.)

1. (1) $\frac{1}{12}ml^2$, (2) $\frac{1}{3}ml^2$, 8 in. **2.** $\frac{1}{48}ml^2$, $l=2$ ft.

3. (1) $\frac{1}{2}mr^2$, (2) mr^2; $\frac{3\pi+2}{3(\pi+2)}\cdot r^2$, $r=14{\cdot}75$ in.

4. (1) $\frac{1}{12}Al^2$, (2) $\frac{1}{3}Al^2$, (3) $\frac{1}{12}A(l^2+b^2)$, (4) $\frac{1}{3}A(l^2+b^2)$; $\frac{21}{4}\sqrt{3}$, $21\sqrt{3}$, $\frac{29}{4}\sqrt{3}$, $29\sqrt{3}$.

5. (1) $\frac{1}{6}Ah^2$, (2) $\frac{1}{18}Ah^2$, (3) $\frac{1}{2}Ah^2$; 37044, 12348, 111132.

6. (1) $\frac{1}{4}A(a^2+b^2)$, (2) $\frac{1}{2}A(a^2+b^2)$; $b=8$.

7. (1) $\frac{5}{4}Ar^2$, (2) $\frac{3}{2}Ar^2$; $x=2$ in.

8. (1) $\frac{1}{4}ma^2$, (2) $\frac{1}{4}m(a^2+b^2)$; $b=16{\cdot}5$. **9.** $\frac{8}{35}mh^2$.

10. $\frac{m(64+17\pi)a^2}{4(8+\pi)}$; $k=26{\cdot}36$. **11.** $\frac{1}{2}M(a^2+b^2)$, $\frac{1}{2}Ma^2$; $b=4$ ft. 8 in.

12. $\frac{2}{5}Mr^2$, $k=56{\cdot}2$; Angle decreases by 45° 16′ approx.

13. (1) $\frac{3}{10}Mr^2$, (2) $\frac{1}{20}M(3r^2+2h^2)$.

14. $I_a=\frac{1}{5}M(b^2+c^2)$; $I_b=\frac{1}{5}M(c^2+a^2)$; $I_c=\frac{1}{5}M(a^2+b^2)$; $k_a=8{\cdot}64$, $k_b=5\sqrt{5}=11{\cdot}18$, $k_c=6\sqrt{5}=13{\cdot}42$.

15. $k^2=\frac{1}{6}(p^2+pq+q^2)$. **16.** $\frac{1}{36}M(a^2+b^2+c^2)$, $k=8{\cdot}5$.

17. $k^2=\frac{a^4-3\pi r^4}{12(a^2-\pi r^2)}$, $k=7{\cdot}31$. **18.** 15 in. or 16 in.

19. $k^2=\frac{1}{2}(a^4+b^4+2b^2c^2)/(a^2+b^2)$; $b=1$ ft. or $\sqrt{5{\cdot}4}=2{\cdot}324$ ft.

21. $k^2=\frac{1}{20}(5l^3+20l^2r+45lr^2+32r^3)/(3l+4r)$.

22. $k^2=\frac{1}{20}(976+285\pi)a^2/(14+3\pi)$.

23. (1) $k^2=\frac{1}{6}h^2(a+3b)/(a+b)$, (2) $k=0{\cdot}996$, (3) $k=0{\cdot}516$.

24. $p^2 = \frac{2}{5}(a^5 - b^5)/(a^3 - b^3)$, $q^2 = \frac{3}{5}(a^5 - b^5)/(a^3 - b^3)$; $a = 19 \cdot 3$ in.

25. $\frac{1}{6} M \frac{a^2b^2}{a^2+b^2}$. **26.** $r = 5\sqrt{3}$ feet. **27.** $\frac{2}{5} M \cdot \frac{a^5 - b^5}{a^3 - b^3}$.

28. 6 inches. **29.** $k^2 = R^2 + \frac{3}{4}r^2$; $k = 37$. **30.** $k = 3(3\pi + 8)$.

31. 20·656 ft. per sec. **37.** 19·53 ft. per sec.

39. $6r^2x^2 + 3(4h^2 + r^2)(y^2 + z^2) = 20\mu^2$, where μ is constant.

40. $k = 5 \cdot 45$.

Exercises 16. (Page 416.)

1. $\frac{\pi}{4} - \frac{1}{2} \log 2$. **2.** $\pi a^2 bch/16$; $\bar{y} = 16b/15\pi$.

3. $a^4/8$. **5.** $a^2b^2/24$. **6.** $\pi a^4/8$.

7. Let $YOXA$ be a vertical rectangular end having $OY = 4$ ft. and inclined to the vertical through O at 30°, then total fluid thrust on this end is $750(2\sqrt{3} + 3) = 4849$ lb. Taking OX, OY as axes of x and y respectively, the centre of pressure is

$$\bar{x} = 2\sqrt{3} = 3 \cdot 464 \text{ ft.}, \quad \bar{y} = 2(7 - 2\sqrt{3})/3 = 2 \cdot 357 \text{ ft.}$$

8. $\{102\sqrt{2} - \log(3 + 2\sqrt{2})\}/(8a^3)$. **9.** $V = \frac{1}{4}a^2\pi(4b + a^2c)$.

10. $(4 - \sqrt{2})/6 + (\sqrt{2} - 3)\pi ab/24$.

11. $A = 3\pi\sqrt{2}(1 - c)/2$. $V = \frac{3}{4}\pi c\sqrt{2}(2 - c)$. **12.** $ab^2/6$; $8\pi a^4/5$.

14. $AP = c \sinh(x/c)$; $\frac{1}{8}c\{c \sinh(2a/c) - 2a\}/\{c \sinh(a/c) - a\}$;
$c\{a \sinh(a/c) - c \cosh(a/c) + c\}/\{c \sinh(a/c) - a\}$.

15. $64a^4/15$. **16.** $\log(1 + \sqrt{2}) - \pi/4$. **18.** $121 \cdot 4\pi = 381 \cdot 5$.

19. 725 : 1472. **22.** $\int_{-a}^{a}\int_{-a}^{a} z\,dx\,dy$; $16a^3/9$. **24.** $\pi a^3 h\rho/12$.

Exercises 17. (Page 427.)

	$r=$	0	1	2	3	4	5	6
1.	A_r	38·92.	−11·51.	1·09.	0·43.	5·5.	0·36.	0·14.
	B_r	—	14·03.	3·72.	0·95.	0·55.	0·17.	—
2.	A_r	58·04.	−21·39.	−7·16.	−3·91.	−2·41.	−2·19.	−0·96.
	B_r	—	−0·11.	0.	−0·33.	−0·14.	0·03.	—
3.	A_r	66·08.	−10·05.	−6·59.	−2·93.	−0·12.	−0·49.	−1·13.
	B_r	—	51·14.	−0·03.	1·40.	−0·86.	1·11.	—
4.	A_r	11·10.	1·75.	0·75.	0.	0.	0.	0.
	B_r	—	9·84.	0·66.	0.	0.	0.	—
5.	A_r	9·62.	4·89.	1·26.	0·29.	0·10.	0·18.	0·06
	B_r	—	3·07.	0·35.	−0·06.	2·31.	0.	—
15.	A_r	11·8.	9·95.	1·05.	0.	0.	0.	0.
	B_r	—	−1·28.	0.	0·03.	−0·03.	−0·04.	—

6. $m = 12a(\pi - 2)/\pi^3$.

7. $x = a(1 - \cos\omega t) + l - l\sqrt{1 - (a^2/l^2)\sin^2\omega t}$

$$= a(1 - \cos\omega t) + a^2(1 - \cos 2\omega t)/4l, \text{ approx.}$$

9. $x=2\sum_{r=1}(-)^{r+1}\sin rx/r, =\pi/2-4/\pi\,.\sum_{r=1}\cos(2r-1)x/(2r-1)^2.$

11. $\pi e^x/(e^{2\pi}-1)=\frac{1}{2}+\sum_{r=1}\cos rx/(r^2+1)-\sum_{r=1}\sin rx/(r^2+1).$

15. See under No. 5.

Exercises 18a. (Page 430.)

$$y_1\equiv\frac{dy}{dx},\quad y_2\equiv\frac{d^2y}{dx^2},\text{ etc.}$$

1. $y_1=y\cot x.$ **2.** $y_1=y.$ **3.** $y_1(1+x^2)=xy.$

4. $xy_1(x^3-6)+2y(x^3+6)-x^2=0.$ **5.** $(y_1-y\cot x)(1-x^2)=xy.$

6. $x^2y_1+xy=2.$ **7.** $2xyy_1=(y^4-1).$

8. $2x^3yy_1=(2x^2+y^2)y^2-x^4.$ **9.** $x(1-3x^2+3y^2)y_1+\{1+3(x^2-y^2)\}y=0.$

10. $x(y\cot y-1)y_1+(1+x\tan x)y=0.$

11. $y_2+y=0.$ **12.** $x^2y_2=2y.$ **13.** $x^2y_2+2xy_1-2y=0.$

14. $3x^2(x+1)y^2y_2+6x^2(x+1)yy_1^2-(3xy^2y_1-y^3)(x^2+4x+2)=0.$

15. $x^2y_2+xy_1+y=0.$ **16.** $3x^2y_2+2y_1^3=6xy_1.$ **17.** $y_2+y_1^2=0.$

18. $y_2(2\sin^2x-3)+3y_1\cos 2x\,.\cot x-4y\sin^2x=0.$

19. $\{4x^3\tan^{-1}x+2\log(1+x^2)+x^2\}y_2+6(1+2x\tan^{-1}x)(y-xy_1)=0.$

20. $xyy_2+(1-9xy)(xy_1-y)y_1+3x^3y_1^3-3y^3=0.$

Exercises 18b. (Page 436.)

1. $xy=A.$ **2.** $xy^a=A.$ **3.** $(y+A)(x+a)=-2a.$

4. $A(x-3)e^{\frac{1}{2}y}=x-4.$ **5.** $Ay=2x^2-4x+7.$ **6.** $Ay=\sin x.$

7. $y+A=e^x\tan x.$ **8.** $Ay^2=(x-1)(x-3)^3.$ **9.** $Ay=e^{1-x}x^x(1-x).$

10. $x=\tan\dfrac{x+y+A}{1+x^2}.$ **11.** $e^{\frac{y}{x}}=A\tan x.$

12. $y=\{e^x(\sin x-\cos x)+A\}\sin x.$

13. $(x-1)y=A(x+1)\log e^x(x+1)(x-2).$

14. $(a^2+b^2)y+Ae^{-ax}=a\cos bx+b\sin bx.$

15. $y=(x-2)^2(x+1)\log A(x-2)^2(x+1).$

16. $y\tan\dfrac{x}{2}+A=2\log\left(1+\sin^2\dfrac{x}{2}\right).$

17. $y^3(\frac{1}{2}\tan^2x+\log\cos x+A)\cos x=1.$

18. $e^x(1+x-y)=Ay(1+x).$ **19.** $y^2(1+Ae^{-\tan^{-1}x})=1.$

20. $x=y^2(x+2)\log A\,.\dfrac{\sqrt{x+2}-1}{\sqrt{x+2}+1}.$ **23.** $2y=(x-1)(x-2).$

26. $e^{\frac{y}{x}}(3-x)=4(2+x)$; $y=8\log 2=5{\cdot}5448.$ **27.** $z=0{\cdot}038$ in.

28. $(\mu^2+1)T=wr\left\{(\mu^2-1)\cos\theta-2\mu\sin\theta+2\mu e^{\mu\left(\theta-\frac{\pi}{2}\right)}\right\}$; $T=178{\cdot}4.$

30. $9e^{2x}y+7=3(3-x^2)x\sin^{-1}x+6\sqrt{1-x^2}+(1-x^2)^{\frac{3}{2}}.$

31. $4y+\sin^2x\cos x(1+\cos^2x)=0.$ **32.** $kv=g\cos\alpha(1-e^{-kt}).$

33. $y=x(x-2)\{x+2\log(x-2)\}.$

34. $(b^2+a^2p^2)y=c(b\sin pt-ap\cos pt)+Ae^{-bt/a}$; $A=acp$.

35. $aye^{bt/a}=ct+A$.

37. $\omega^2(b^2+1)=a(b\sin\theta-\cos\theta)+\frac{1}{2}a(1-b\sqrt{3})e^{b(\pi/3-\theta)}$.

38. $27x=9t^2-6t+2+Ae^{-3t}$. **39.** $Ax^2y+x^2-y^2=0$.

40. $2y+x\sin x+\sin^2 x\cos x=A\sin x$.

Exercises 18c. (Page 444.)

1. $x=a\cos\omega t$. **2.** $s=ut+\frac{1}{2}at^2$. **3.** $y+2=e^{x-2}$.

4. $y=A\cos nx+B\sin nx$. **5.** $3-u=4\cos t$.

6. $s^2=a^2+t^2$. **7.** $y=Ae^{3x}+Be^{-x}$. **8.** $y=Ae^{2x}+Be^{5x}$.

9. $y=e^{2x}(Ae^{ix}+Be^{-ix})=e^{2x}(C\cos x+D\sin x)$. **10.** $ye^{2x}=3\sin 5x$.

11. $x=(A+Bt)e^{2\cdot5t}$. **12.** $y=(A\cos x\sqrt{0\cdot11}+B\sin x\sqrt{0\cdot11})e^{-0\cdot7x}$.

13. $y=(A\cos\frac{3}{4}x\sqrt{7}+B\sin\frac{3}{4}x\sqrt{7})e^{-\frac{3}{4}x}$.

14. $z=\tan(Ae^{1\cdot5x}+Be^{0\cdot5x})$. **15.** $y=Ax^{1\cdot5}+Bx^{0\cdot8}$.

16. $y=Ax^{-1}+Bx^{-1\cdot5}$. **17.** $y=(A+B\log x)x^{1\cdot5}$.

18. $y=\{A\cos(\frac{1}{4}\sqrt{7}\log x)+B\sin(\frac{1}{4}\sqrt{7}\log x)\}x^{\frac{5}{4}}$.

19. $y=\{A\cos(\frac{1}{3}\sqrt{11}\log x)+B\sin(\frac{1}{3}\sqrt{11}\log x)\}x^{-\frac{2}{3}}$.

20. $24cy=wx(l-x)(l^2+lx-x^2)$. **21.** $y=d\sin x\sqrt{cw}$.

22. $w=\pi/4cl^2$. **23.** $ky=\frac{1}{48}w(2x^2-3l^2)x^2+\frac{5}{384}wl^4$.

24. (i) $6cy=Wx^2(3l-x)$. (ii) $24cy=wx^2(6l^2-4lx+x^2)$.

25. (i) $48cy=Wx^2(3l-4x)$. (ii) $24cy=wx^2(x-l)^2$.
(iii) $6cly=Wx(s-l)(x^2+s^2-2sl)$; $y_c=\frac{1}{3}Ws^2(l-s)^2/cl$.

26. $\mu^2s=\log\cosh\mu ct$; $t=9\cdot018$ secs.

27. $24y=cx^4+4ax^3-4l^2x(cl+3a)-5cl-16a$. **30.** $2y-3=5\sin x$; $x=\frac{\pi}{6}$.

32. $x=Ae^{-ft}\cosh(t\sqrt{f^2-n^2}+B)$; No. $x=(A+Bx)e^{-nt}$.

33. When $b^2>4ac$, $x=(A\cosh\mu t+B\sinh\mu t)e^{-bt/2a}$.
When $b^2<4ac$, $x=(A\cos i\mu t+B\sin i\mu t)e^{-bt/2a}$.
When $b^2=4ac$, $x=(A+Bt)e^{-bt/2a}$, where $\mu^2=(b^2-4ac)/4a^2$.
Frequency $=1/(2\pi\sqrt{LC})=65$.

Exercises 18d. (Page 454.)

5. $y=(14e^{5x}+e^{-5x})/336$. **6.** $y=3x^2+32x+125$.

7. $y=13(3\cos 5x-10\sin 5x)/545$. **8.** $y=-(3\cos x+\sin x)e^{4x}/10$.

9. $4y=(2x^2+1)\sinh x-2x\cosh x$. **10.** $y=(x-1)e^{3x}$. **11.** $y=x^2e^{2x}$.

12. $20y=10x\sin x+\cos x$. **13.** $y=Ae^{4x}-(x^3+3x^2+6x+B)e^{3x}$.

14. $y=(Ae^{-11x}+Be^{-x})+\frac{2}{625}(7\sin 2x-24\cos 2x)$.

15. $y=e^{-\frac{1}{2}x}\{A\sin(\frac{1}{2}x\sqrt{19})+B\cos(\frac{1}{2}x\sqrt{19})\}-\frac{1}{8}(\cos 5x+4\sin 5x)$.

16. $y=Ae^{6x}+Be^{-x}-\frac{1}{2}x+\frac{5}{12}+\frac{5}{74}(5\cos x-7\sin x)$.

17. $y=(A\sin x\sqrt{2}+B\cos x\sqrt{2}+\frac{5}{2})e^{-2x}$.